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# A NEW ALGORITHM FOR VARIATIONAL INCLUSION PROBLEM

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Abstract. The target of this article is to modify the algorithm given by Fang and Huang [6]. The rate of convergence of our algorithm is faster than that of Fang and Huang [6]. A numerical example is given to justify our statement.

## 1. Introduction and preliminaries

As variational inequalities have lot of applications, these have been extended and generalized in different directions. Variational inclusion is one of the important generalizations of variational inequalities. Many authors see [1–3, 5–7, 10, 11, 15] have developed different types of algorithms for different types of variational inclusions. The resolvent operator technique is interesting and essential to study the existence of solution and develop iterative algorithms for different types of variational inclusions. In 2003, Fang and Huang [6] developed $\mathbb{H}$-monotone operators and extended the concept of resolvent operators associated with maximal monotone operators to new $\mathbb{H}$-monotone operators. Using this new resolvent operator technique, they have studied the approximate solutions of a new class of variational inclusions associated with $\mathbb{H}$-monotone operators.

In this paper, we modify the algorithm given by Fang and Huang [6] and show that the rate of convergence of our algorithm is faster than that of Fang and Huang [6].

Everywhere in the paper, we have assumed H as a real Hilbert space associated with norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. The collection of all nonempty subsets of H is denoted by $2^{\mathrm{H}}$. Let us first recall some definitions and results which have been utilized in the paper.

Definition 1.1. Let $B, \mathbb{H} : \mathrm{H} \to \mathrm{H}$ be two single valued operators. B is said to be

(i) *monotone* if

$$\langle B(s) - B(t), s - t\rangle \geq 0 \ \forall s, t \in \mathrm{H};$$

(ii) *strictly monotone*, if $B$ is monotone and

$$\langle B(s) - B(t), s - t\rangle = 0 \ \textit{iff} \ s = t;$$

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(iii) *strongly monotone* if for some positive constant $\alpha$

$$\langle B(s) - B(t), s - t\rangle \geq \alpha\|s - t\|^2 \ \forall s, t \in \mathrm{H};$$

(iv) *strongly monotone with respect to* $\mathbb{H}$ if for some positive constant $\beta$

$$\langle B(s) - B(t), \mathbb{H}(s) - \mathbb{H}(t)\rangle \geq \beta\|s - t\|^2 \ \forall s, t \in \mathrm{H};$$

(v) *Lipschitz continuous* if for some positive constant $\gamma$

$$\|B(s) - B(t)\| \leq \gamma\|s - t\| \ \forall s, t \in \mathrm{H}.$$

Definition 1.2. Let $\mathbb{H} : \mathrm{H} \to \mathrm{H}$ be a single valued operator, then a multi-valued mapping $M : \mathrm{H} \to 2^{\mathrm{H}}$ is said to be

(i) *monotone* if

$$\langle s - t, x - y\rangle \geq 0 \ \forall \ x, y \in \mathrm{H}; \ s \in M(x), \ t \in M(y);$$

(ii) *strongly monotone* if there exists some positive constant $\eta$ such that

$$\langle s - t, x - y\rangle \geq \eta\|x - y\|^2 \ \forall \ x, y \in \mathrm{H}; \ s \in M(x), \ t \in M(y);$$

(iii) *maximal monotone* if $M$ is monotone and $(I + \lambda M)(\mathrm{H}) = \mathrm{H}$ for all $\lambda > 0$, where $I$ is the identity mapping on H;

(iv) $\mathbb{H}$-*monotone* if M is monotone and $(\mathbb{H} + \lambda M)(\mathrm{H}) = \mathrm{H}$ holds for every $\lambda > 0$;

(v) *strongly* $\mathbb{H}$-*monotone*, if M is strongly monotone and $(\mathbb{H} + \lambda M)(\mathrm{H}) = \mathrm{H}$ holds for every $\lambda > 0$.

The relation between strongly $\mathbb{H}$-monotone mapping and $\mathbb{H}$-monotone mapping is given as follows:
$\{\mathit{strongly}\ \mathbb{H} - \mathit{monotone\ mapping}\} \subset \{\mathbb{H} - \mathit{monotone\ mapping}\}$.

Lemma 1.1. [4] *Let $M : \mathrm{H} \to 2^{\mathrm{H}}$ be a maximal monotone mapping and $B : \mathrm{H} \to \mathrm{H}$ be a Lipschitz continuous mapping. Then a mapping $B + M : \mathrm{H} \to 2^{\mathrm{H}}$ is a maximal monotone mapping.*

Using Lemma 1.1, we define a new resolvent operator in the following way:

Definition 1.3. Let $\mathbb{H} : \mathrm{H} \to \mathrm{H}$ be a strictly monotone operator, $B : \mathrm{H} \to \mathrm{H}$ be a Lipschitz continuous mapping and $M : \mathrm{H} \to 2^{\mathrm{H}}$ be a maximal monotone operator, so that $B + M : \mathrm{H} \to 2^{\mathrm{H}}$ is a strongly $\mathbb{H}$-monotone operator. Then the resolvent operator is defined as:

$$\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) = [\mathbb{H} + \lambda(B + M)]^{-1}(s) \ \forall s \in \mathrm{H}. \tag{1.1}$$

Lemma 1.2. *Let $\mathbb{H} : \mathrm{H} \to \mathrm{H}$ be a strictly monotone operator, $B : \mathrm{H} \to \mathrm{H}$ be Lipschitz continuous and $M : \mathrm{H} \to 2^{\mathrm{H}}$ be a maximal monotone operator, so that $B + M : \mathrm{H} \to 2^{\mathrm{H}}$ is a strongly $\mathbb{H}$-monotone operator. Then the resolvent operator $\mathbb{R}^{\mathbb{H}}_{B+M,\lambda} = [\mathbb{H} + \lambda(B + M)]^{-1}$ is single valued.*

*Proof.* Suppose for $s \in \mathrm{H}$, and $x, y \in [\mathbb{H}+\lambda(B+M)]^{-1}(s)$. Then from the definition of resolvent operator (1.1), it follows that $-\mathbb{H}x + s \in \lambda(B + M)x$ and $-\mathbb{H}y + s \in \lambda(B + M)y$. Using monotonocity of $B + M$, we get

$$\langle(-\mathbb{H}x + s) - (-\mathbb{H}y + s), x - y\rangle = \langle\mathbb{H}y - \mathbb{H}x, x - y\rangle \geq 0.$$

As $\mathbb{H}$ is strictly monotone operator, so we get $x = y$. Thus $[\mathbb{H}+\lambda(B+M)]^{-1}$ is single valued. $\square$

Lemma 1.3. *Let* $\mathbb{H} : \mathrm{H} \to \mathrm{H}$ *be a continuous and strongly monotone operator with constant* $\alpha > 0$, $B : \mathrm{H} \to \mathrm{H}$ *be Lipschitz continuous mapping and* $M : \mathrm{H} \to 2^{\mathrm{H}}$ *be a maximal monotone operator, so that* $B + M : \mathrm{H} \to 2^{\mathrm{H}}$ *is a strongly* $\mathbb{H}$*-monotone operator with positive constant* $\eta$. *Then the resolvent operator* $\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) = [\mathbb{H} + \lambda(B+M)]^{-1}(s)$ $\forall s \in \mathrm{H}$ *and* $\lambda > 0$ *is* $\left(\frac{1}{\alpha+\lambda\eta}\right)$ *Lipschitizian continuous, i.e.,*

$$\left\|\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) - \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t)\right\| \leq \left(\frac{1}{\alpha + \lambda\eta}\right)\|s - t\| \ \forall\ s, t \in \mathrm{H}.$$

*Proof.* For two given points $s,\ t \in \mathrm{H}$, we have

$$\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) = [\mathbb{H} + \lambda(B+M)]^{-1}(s)$$

and

$$\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t) = [\mathbb{H} + \lambda(B+M)]^{-1}(t).$$

This means that

$$\frac{1}{\lambda}\big(s - \mathbb{H}(\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s))\big) \in (B+M)(\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s))$$

and

$$\frac{1}{\lambda}\big(t - \mathbb{H}(\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t))\big) \in (B+M)(\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t)).$$

Since $B + M$ is $\eta$ strongly monotone, we have

$$\begin{aligned}
&\eta\|\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) - \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t)\| \\
&\leq \big\langle s - \mathbb{H}(\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) - (t - \mathbb{H}(\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t)), \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) - \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t)\big\rangle \\
&= \frac{1}{\lambda}\big\langle s - t - \big(\mathbb{H}(\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s)) - \mathbb{H}(\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t))\big), \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) - \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t)\big\rangle.
\end{aligned}$$

Using Cauchy Schwartz Inequality, we get

$$\begin{aligned}
&\|s - t\|\|\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) - \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t)\| \\
&\geq \big\langle s - t, \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) - \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t)\big\rangle \\
&\geq \big\langle \mathbb{H}(\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s)) - \mathbb{H}(\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t), \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) - \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t)\big\rangle \\
&+\lambda\eta\|\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) - \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t)\| \\
&\geq \alpha\|\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) - \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t)\|^2 \\
&+\lambda\eta\|\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) - \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t)\|^2 \\
&= (\alpha + \lambda\eta)\|\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) - \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t)\|^2.
\end{aligned}$$

So, we get

$$\|\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(s) - \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}(t)\| \leq \left(\frac{1}{\alpha + \lambda\eta}\right)\|s - t\| \ \forall\ s, t \in \mathrm{H}.$$

□

## 2. Algorithm for variational inclusion problem

In this section, we study variational inclusion problem which has been studied by many authors in different settings. We develop an algorithm for the inclusion problem.

Let $B, \mathbb{H} : \mathrm{H} \to \mathrm{H}$ be two single valued operators and $M : \mathrm{H} \to 2^{\mathrm{H}}$ be a muti-valued operator. Then variational inclusion problem consists in finding $s \in \mathrm{H}$ such that

$$0 \in B(s) + M(s). \tag{2.1}$$

Note that

(1) When $\mathrm{H} = \mathbb{R}^n$, $B(s) = 0(\forall s \in \mathrm{H})$ and M is maximal monotone, such problem was considered by Rockafellar [15] and he proved the convergence of the proximal point algorithm for solving the problem.
(2) If $M = \partial\psi$, where $\psi$ is the sub-differential of a proper, convex and lower semi-continuous functional $\psi : \mathrm{H} \to \mathbb{R} \cup \{+\infty\}$, then problem (2.1) becomes nonlinear variational inequality problem, which is defined as:

$$find\ s \in \mathrm{H} :\ \langle B(s), t - s\rangle + \psi(t) - \psi(s) \geq 0,\ \forall t \in \mathrm{H}. \tag{2.2}$$

The above problem (2.2) was examined by many authors like [8, 9, 12, 13].
(3) When $\mathrm{H} = \mathbb{R}^n$, B is continuously differentiable and M is maximal monotone, the problem was examined by Robinson [14]. He proved the existence of solution of the problem (2.1).
(4) When M is maximal monotone and B is Lipschitz continuous and strongly monotone, the problem has been examined by Fang and Huang [6]. They developed an iterative algorithm for approximating the solution of the problem (2.1). Their algorithm is given below:

**Algorithm 2.1.** For any $s_0 \in \mathrm{H}$, the iterative sequence $\{s_n\} \subset \mathrm{H}$ is defined by

$$s_{n+1} = \mathbb{R}^{\mathbb{H}}_{M,\lambda}[\mathbb{H}(s_n) - \lambda B(s_n)],\ \forall\ n \geq 0.$$

They also proved that the sequence generated by their algorithm converges strongly to unique solution of problem (2.1).
Motivated by their work, we construct a new algorithm for the variational inclusion problem (2.1). Before giving our algorithm, we first state the fixed point formulation of our problem in the form of a lemma.

Lemma 2.1. *Let $\mathbb{H} : \mathrm{H} \to \mathrm{H}$ be a strictly monotone operator $B : \mathrm{H} \to \mathrm{H}$ be Lipschitz continuous and $M : \mathrm{H} \to 2^{\mathrm{H}}$ be a maximal monotone operator, so that $B + M : \mathrm{H} \to 2^{\mathrm{H}}$ is a strongly $\mathbb{H}$-monotone operator. Then $s \in \mathrm{H}$ is a solution of problem* (2.1) *if and only if*

$$s = \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}[\mathbb{H}(s)].$$

Utilizing Lemma (2.1), we set up our algorithm for the problem (2.1).
**Algorithm 2.2.** For any $s_0 \in \mathrm{H}$, the iterative sequence $\{s_n\} \subset \mathrm{H}$ is given by

$$\begin{cases} s_{n+1} = \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}[\mathbb{H}(t_n)], \\ t_n = \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}[\mathbb{H}(s_n)],\ n = 0, 1, 2, \cdots . \end{cases}$$

## 3. Main result

In this section, we prove that the sequence generated by our algorithm converges strongly to the unique solution of problem (2.1). We also show by numerical example that the rate of convergence of our algorithm is faster than that of Fang and Huang.

THEOREM 3.1. *Let $\mathbb{H} : \mathrm{H} \to \mathrm{H}$ be a strongly monotone and Lipschitz continuous operator with positive constants $\alpha$ and $\beta$, respectively. Let $B : \mathrm{H} \to \mathrm{H}$ be Lipschitz continuous and strongly monotone with respect to $\mathbb{H}$ with positive constants $\gamma$ and $\delta$, respectively. Let $M : \mathrm{H} \to 2^{\mathrm{H}}$ be a maximal monotone operator and $B+M : \mathrm{H} \to 2^{\mathrm{H}}$, be strongly $\mathbb{H}$-monotone with positive constant $\eta$ and assume that there exists some positive constant $\lambda$ such that $\left(\frac{\beta}{\alpha+\lambda\eta}\right) < 1$. Then the iterative sequence $\{s_n\}$ generated by algorithm 2.2 converges strongly to the unique solution of problem* (2.1).

*Proof.* Let $s^*$ be the solution of the problem (2.1), then it follows that

$$\begin{aligned} \|s_{n+1} - s^*\| &= \|\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}[\mathbb{H}(t_n)] - \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}[\mathbb{H}(s^*)]\| \\ &\leq \frac{1}{\alpha+\lambda\eta}\|\mathbb{H}(t_n) - \mathbb{H}(s^*)\| \\ &\leq \frac{\beta}{\alpha+\lambda\eta}\|t_n - s^*\| \\ &= \frac{\beta}{\alpha+\lambda\eta}\|\mathbb{R}^{\mathbb{H}}_{B+M,\lambda}[\mathbb{H}(s_n)] - \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}[\mathbb{H}(s^*)]\| \\ &\leq \left(\frac{\beta}{\alpha+\lambda\eta}\right)^2 \|s_n - s^*\|. \end{aligned}$$

Continuing in this way, we obtain

$$\|s_{n+1} - s^*\| \leq \left(\frac{\beta}{\alpha+\lambda\eta}\right)^{2(n+1)} \|s_0 - s^*\|.$$

Taking the limit $n \to \infty$, we get

$$\lim_{n\to\infty} \|s_{n+1} - s^*\| = 0.$$

This implies that $s_n$ converges to $s^*$ strongly. □

Now we show that the convergence is unique.
Let $s$ be another solution of problem (2.1). Then, by Lemma 2.1, $s = \mathbb{R}^{\mathbb{H}}_{B+M,\lambda}[\mathbb{H}(s)]$. Now with similar arguments as earlier, we obtain

$$\|s^* - s\| \leq \left(\frac{\beta}{\alpha+\lambda\eta}\right)\|s^* - s\|.$$

Since $0 \leq \left(\frac{\beta}{\alpha+\lambda\eta}\right) < 1$, we get $s^* = s$. This means that $s^*$ is a unique solution of problem (2.1).

EXAMPLE 3.1. Let $H = \mathbb{R}$, the set of reals and let $\mathbb{H}, B : \mathbb{R} \to \mathbb{R}$ and $M : \mathbb{R} \to 2^{\mathbb{R}}$ be defined as $\mathbb{H}(s) = 3s \ \forall \ s \in \mathbb{R}$, $B(s) = \frac{s}{2} \ \forall \ s \in \mathbb{R}$ and $M(s) = \{2s\} \ \forall \ s \in \mathbb{R}$. Then $\mathbb{H}$ is strongly monotone with constant $\alpha = 2.9$ and Lipschitz continuous with constant $\beta = 3.1$, $B$ is Lipschitz continuous with constant $\gamma = 0.6$ and is strongly monotone with respect to $\mathbb{H}$ with constant $\delta = 1.4$ and $B+M$ is strongly $\mathbb{H}$- monotone operator

with constant $\eta = 7.4$.
Under these conditions the above example satisfies the conditions of both the theorems, Theorem 3.1 of Fang and Huang [6] as well as Theorem 3.1.

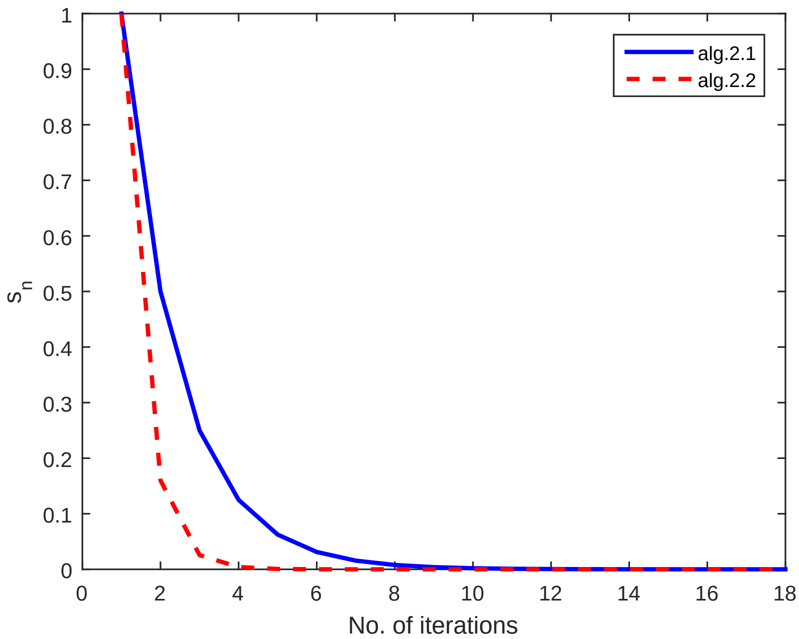


FIGURE 1. The convergence of $\{s_n\}$ with initial value $s_1 = 1$.

TABLE 1. The values of $\{s_n\}$ with initial value $s_1 = 1$.

| No. of Iterations | Old Algorithm | Our Algorithm |
|---|---|---|
| 1 | 1.0000 | 1.0000 |
| 2 | 0.5000 | 0.1600 |
| 3 | 0.2500 | 0.0256 |
| 4 | 0.1250 | 0.0041 |
| 5 | 0.0625 | 0.0007 |
| 6 | 0.0156 | 0.0001 |
| 7 | 0.0078 | 0.0000 |
| 8 | 0.0039 | 0.0000 |
| 9 | 0.0020 | 0.0000 |
| 10 | 0.0010 | 0.0000 |
| 11 | 0.0005 | 0.0000 |
| 12 | 0.0002 | 0.0000 |
| 13 | 0.0001 | 0.0000 |
| 14 | 0.0001 | 0.0000 |
| 15 | 0.0000 | 0.0000 |

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