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INEQUALITIES CONCERNING POLYNOMIAL AND ITS DERIVATIVE

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Abstract. In this paper, some sharp inequalities for ordinary derivative $P'(z)$ and polar derivative $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ are obtained by including some of the coefficients and modulus of each individual zero of a polynomial $P(z)$ of degree n not vanishing in the region $|z| > k$, $k \geq 1$. Our results also improve the bounds of Turán's and Aziz's inequalities.

1. Introduction and statement of results

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative, then concerning the estimate of $|P'(z)|$ on $|z| = 1$, we have

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1}$$

Inequality (1) is a famous result due to Bernstein [5] and equality holds if and only if $P(z)$ has all its zeros at the origin.
If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then Turán [11] proved that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{2}$$

As a generalization of inequality (2), Aziz [1] considered the modulus of each zero of the underlying polynomial in the bound and proved that, if $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu = a_n \prod_{\nu=1}^{n} (z - z_\nu)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{2}{1+k^n} \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|} \max_{|z|=1} |P(z)|. \tag{3}$$

The result is best possible and equality in (3) holds for $P(z) = z^n + k^n$.

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For a polynomial $P(z)$ of degree n, the polar derivative $D_\alpha P(z)$ of $P(z)$ with respect to a complex number α is defined as

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

$D_\alpha P(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Bernstein type inequalities on complex polynomials have been extended from ordinary derivative to polar derivative of complex polynomials. For reference, see([2], [4], [8], [9]).

Recently P. Kumar [10] proved the following generalizations of inequalities (2) and (3) by including the coefficients and considering the modulus of each individual zero. In fact, Kumar proved:

THEOREM 1.1. *If* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu = a_n \prod_{\nu=1}^{n}(z - z_\nu)$ *is a polynomial of degree* n *having all its zeros in* $|z| \le k$, $k \ge 1$, *then*

$$\max_{|z|=1}|P'(z)| \ge \left[\frac{2}{1+k^n} + \frac{(|a_n|k^n - |a_0|)(k-1)}{(1+k^n)(|a_n|k^n + k|a_0|)}\right] \sum_{\nu=1}^{n} \frac{k}{k+|z_\nu|} \max_{|z|=1}|P(z)|. \tag{4}$$

The result is best possible and equality in (4) *holds for* $P(z) = z^n + k^n$.

THEOREM 1.2. *If* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu = a_n \prod_{\nu=1}^{n}(z - z_\nu)$ *is a polynomial of degree* n *having all its zeros in* $|z| \le k$, $k \ge 1$, *then for any complex number* α *with* $|\alpha| \ge k$

$$\max_{|z|=1}|D_\alpha P(z)| \ge \left[\frac{2(|\alpha|-k)}{1+k^n} + (|\alpha|-k)\frac{(|a_n|k^n - |a_0|)(k-1)}{(1+k^n)(|a_n|k^n + k|a_0|)}\right] \sum_{\nu=1}^{n} \frac{k}{k+|z_\nu|} \max_{|z|=1}|P(z)|. \tag{5}$$

The result is best possible and equality in (5) *holds for* $P(z) = z^n + k^n$.

2. Main Results

In this section we present the main results. In the inequalities (4) and (5) the bound depends on the coefficients a_0 and a_n and modulus of each individual zero of underlying polynomial. We refine all the above results by obtaining the bound which involves modulus of each individual zero and the coefficients a_0, a_1, a_2 and a_n of the underlying polynomial. In fact, we prove:

THEOREM 2.1. *If* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu = a_n \prod_{\nu=1}^{n}(z - z_\nu)$ *is a polynomial of degree* n *having all its zeros in* $|z| \le k$, $k \ge 1$, *then*

$$\begin{aligned}\max_{|z|=1}|P'(z)| \ge{}& \left[\frac{2}{1+k^n} + \frac{(|a_n|k^n - |a_0|)(k-1)}{(1+k^n)(|a_n|k^n + k|a_0|)}\right] \sum_{\nu=1}^{n} \frac{k}{k+|z_\nu|} \max_{|z|=1}|P(z)| \\ &+ \frac{2|a_1|}{k^{n-1}}\left(\frac{k^{n-1}-1}{n+1}\right) + \frac{2|a_2|}{k^{n-1}}\left[\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}\right], \text{ for } n > 3\end{aligned} \tag{6}$$

and

(7)
$$\max_{|z|=1}|P'(z)| \geq \left[\frac{2}{1+k^3}+\frac{(|a_3|k^3-|a_0|)(k-1)}{(1+k^3)(|a_3|k^3+k|a_0|)}\right]\sum_{\nu=1}^{3}\frac{k}{k+|z_\nu|}\max_{|z|=1}|P(z)| + \frac{k-1}{2k^2}\left[(k+1)|a_1|+2(k-1)|a_2|\right], \text{ for } n=3.$$

The result is best possible and equalities in (6) *and* (7) *hold for polynomial* $P(z) = z^n + k^n$.

Remark 2.2. For $k = 1$, Theorem 2.1 reduces to inequality (2).

Remark 2.3. The bound obtained by Theorem 2.1 is sharper than the bound obtained from Theorem 1.1. In order to show the sharpness of bounds here we present the following example.

Example 2.4. For the case $n > 3$, consider $P(z) = (z + \frac{1}{2})(z + 1)^2(z + 2)$. Here we take $k = 2$, then we find all the zeros of $P(z)$ lie in $|z| \leq 2$. For this polynomial, the bound for $\max_{|z|=1}|P'(z)|$ of Theorem 1.1 comes out to be 6.12 and by inequality (6) of Theorem 2.1, it comes out to be 10.05, which is a significant improvement over the bound obtained from Theorem 1.1.

For the case $n = 3$, consider $P(z) = (z + \frac{1}{2})(z + 1)(z + 2)$. Here we take $k = 3$, then we find all the zeros of $P(z)$ lie in $|z| \leq 3$. For this polynomial, the bound for $\max_{|z|=1}|P'(z)|$ by Theorem 1.1 comes out to be 1.8 and by inequality (7) of Theorem 2.1, it comes out to be 4.91, which is also a significant improvement over the bound obtained from Theorem 1.1.

Theorem 2.5. *If* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu = a_n \prod_{\nu=1}^{n}(z - z_\nu)$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$, $k \geq 1$, *then for any complex number* α *with* $|\alpha| \geq k$

(8)
$$\max_{|z|=1}|D_\alpha P(z)| \geq \left[\frac{2(|\alpha|-k)}{1+k^n}+(|\alpha|-k)\frac{(|a_n|k^n-|a_0|)(k-1)}{(1+k^n)(|a_n|k^n+k|a_0|)}\right]\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=1}|P(z)| + \frac{2|na_0+\alpha a_1|}{k^{n-1}}\left(\frac{k^{n-1}-1}{n+1}\right)+\frac{|(n-1)a_1+2\alpha a_2|}{k^{n-1}}\left[\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right],$$

for $n > 3$ *and*

(9)
$$\max_{|z|=1}|D_\alpha P(z)| \geq \left[\frac{2(|\alpha|-k)}{1+k^3}+(|\alpha|-k)\frac{(|a_3|k^3-|a_0|)(k-1)}{(1+k^3)(|a_3|k^3+k|a_0|)}\right]\sum_{\nu=1}^{3}\frac{k}{k+|z_\nu|}\max_{|z|=1}|P(z)| + \frac{k-1}{2k^2}\left[(k+1)|3a_0+\alpha a_1|+2(k-1)|a_1+\alpha a_2|\right], \text{ for } n=3.$$

The result is best possible and equalities in (8) *and* (9) *hold for polynomial* $P(z) = z^n + k^n$.

Remark 2.6. Theorem 2.5 is an extension of Theorem 2.1 to the polar derivative. If we divide inequalities (8) and (9) both sides by $|\alpha|$ and letting $|\alpha| \to \infty$, we get Theorem 2.1.

3. Lemmas

In this section we present some lemmas which will be needed in the sequel. The first lemma is a special case of a result due to Aziz and Rather[[3], [4]].

Lemma 3.1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for $|z| = 1$*

$$|Q'(z)| \leq |P'(z)|, \tag{10}$$

where $Q(z) = z^n\overline{F(1/\overline{z})}$.

The following lemma is due to Dubinin [7](see also [10]).

Lemma 3.2. *If $P(z) = a_0 + a_1z + a_2z^2 + ... + a_nz^n$ is a polynomial of degree $n \geq 1$ having no zeros in $|z| < 1$, then for $R \geq 1$,*

$$\max_{|z|=R} |P(z)| \leq \left[\frac{(1 + R^n)(|a_0| + R|a_n|)}{(1 + R)(|a_0| + |a_n|)}\right] \max_{|z|=1} |P(z)|. \tag{11}$$

The next lemma is due to Dewan et al [6].

Lemma 3.3. *If $P(z)$ is a polynomial of degree n, then for $R \geq 1$*

$$\begin{aligned} \max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)| - 2\left(\frac{R^n - 1}{n + 2}\right) |P(0)| \\ - \left[\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2}\right] |P'(0)|, \ \textit{if } n > 2 \end{aligned} \tag{12}$$

and

(13)

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)| - \frac{(R - 1)}{2} \left[(R + 1)|P(0)| + (R - 1)|P'(0)|\right], \ \textit{if } n = 2.$$

4. Proof of theorems

Proof of Theorem 2.1. Let $F(z) = P(kz)$, then $F(z) = a_nk^n \prod_{\nu=1}^{n} (z - \frac{z_\nu}{k})$ has all its zeros in $|z| \leq 1$. Since for all z on $|z| = 1$ for which $F(z) \neq 0$,

$$\frac{zF'(z)}{F(z)} = \sum_{\nu=1}^{n} \frac{z}{z - (z_\nu/k)},$$

which gives

$$Re\left(\frac{zF'(z)}{F(z)}\right) \geq \sum_{\nu=1}^{n} \frac{1}{1 + |z_\nu/k|} = \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|}.$$

This gives for all z on $|z| = 1$ for which $F(z) \neq 0$

$$\left|\frac{zF'(z)}{F(z)}\right| \geq \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|}.$$

That is,

$$\max_{|z|=1}|F'(z)| \geq \sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=1}|F(z)|. \tag{14}$$

Replacing $F(z)$ by $P(kz)$, we obtain

$$k\max_{|z|=k}|P'(z)| \geq \sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=k}|P(z)|. \tag{15}$$

Now applying inequality (12) of Lemma 3.3 to the polynomial $P'(z)$ which is of degree $n-1$ with $R=k\geq 1$, we obtain for $n>3$,

$$\begin{aligned}\max_{|z|=k}|P'(z)| \leq k^{n-1}\max_{|z|=1}|P'(z)| - 2|a_1|\left(\frac{k^{n-1}-1}{n+1}\right)\\ -2|a_2|\left[\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right].\end{aligned} \tag{16}$$

Using inequality (16) in inequality (15), we get

$$\begin{aligned}k^n\max_{|z|=1}|P'(z)| - 2|a_1|k\left(\frac{k^{n-1}-1}{n+1}\right) - 2|a_2|k\left[\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right]\\ \geq \sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=k}|P(z)|, \text{ for } n>3.\end{aligned} \tag{17}$$

Since $F(z)=P(kz)$ has all its zeros in $|z|\leq 1$, therefore $F^*(z)=z^n\overline{F(1/\overline{z})}=z^n\overline{P(k/\overline{z})}$ has no zero in $|z|<1$. Hence by applying Lemma 3.2 to the polynomial $F^*(z)$, with $R=k\geq 1$, we have

$$\max_{|z|=k}|F^*(z)| \leq \frac{(1+k^n)(|a_n|k^n+k|a_0|)}{(1+k)(|a_n|k^n+|a_0|)}\max_{|z|=1}|F^*(z)|. \tag{18}$$

Also

$$\max_{|z|=k}|F^*(z)| = k^n\max_{|z|=1}|P(z)|,$$

$$\max_{|z|=1}|F(z)| = \max_{|z|=k}|P(z)|$$

and

$$\max_{|z|=1}|F^*(z)| = \max_{|z|=1}|F(z)|.$$

Replacing these in inequality (18), we get

$$\max_{|z|=k}|P(z)| \geq k^n\frac{(1+k)(|a_n|k^n+|a_0|)}{(1+k^n)(|a_n|k^n+k|a_0|)}\max_{|z|=1}|P(z)|. \tag{19}$$

Combining inequalities (17) and (19), we obtain for $n > 3$

$$\begin{aligned}\max_{|z|=1}|P'(z)| &\geq \left[\frac{(1+k)(|a_n|k^n+|a_0|)}{(1+k^n)(|a_n|k^n+k|a_0|)}\max_{|z|=1}|P(z)|\right]\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=1}|P(z)| \\ &+\frac{2|a_1|}{k^{n-1}}\left(\frac{k^{n-1}-1}{n+1}\right)+\frac{2|a_2|}{k^{n-1}}\left[\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right] \\ &=\left[\frac{2}{1+k^n}+\frac{(|a_n|k^n-|a_0|)(k-1)}{(1+k^n)(|a_n|k^n+k|a_0|)}\right]\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=1}|P(z)| \\ &+\frac{2|a_1|}{k^{n-1}}\left(\frac{k^{n-1}-1}{n+1}\right)+\frac{2|a_2|}{k^{n-1}}\left[\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right].\end{aligned}$$

This proves the result in case $n > 3$. For the case $n = 3$, the result follows on similar lines but instead of using inequality (12) of Lemma 3.3 , we use inequality (13) of Lemma 3.3. This proves the theorem completely. □

Proof of Theorem 2.5. Let $F(z) = P(kz)$, then all the zeros of $F(z)$ lie in $|z| \leq 1$. If $Q(z) = z^n\overline{F(1/\overline{z})}$, then one can easily obtain that

$$|Q'(z)| = |nF(z) - zF'(z)|, \text{ for } |z| = 1.$$

This gives with the help of Lemma 3.1 that

$$|F'(z)| \geq |nF(z) - zF'(z)|, \text{ for } |z| = 1. \tag{20}$$

Now for every complex number α, we have for $|z| = 1$

$$\begin{aligned}|D_{\alpha/k}F(z)| &= |nF(z) + (\alpha/k - z)F'(z)| \\ &\geq \frac{|\alpha|}{k}|F'(z)| - |nF(z) - zF'(z)|,\end{aligned}$$

which gives by using inequality (20) for $|z| = 1$ and $|\alpha| \geq k$

$$|D_{\alpha/k}F(z)| \geq \left(\frac{|\alpha|-k}{k}\right)|F'(z)|,$$

or equivalently,

$$\max_{|z|=1}|D_{\alpha/k}F(z)| \geq \left(\frac{|\alpha|-k}{k}\right)\max_{|z|=1}|F'(z)|,$$

which is nothing but

$$\max_{|z|=1}|nP(kz) + (\alpha/k - z)kP'(kz)| \geq \left(\frac{|\alpha|-k}{k}\right)\max_{|z|=1}|F'(z)|.$$

By using the definition of polar derivative and inequality (14), we obtain

$$\max_{|z|=k}|D_\alpha P(z)| \geq \left(\frac{|\alpha|-k}{k}\right)\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=1}|F(z)|.$$

This gives

$$\max_{|z|=k}|D_\alpha P(z)| \geq \left(\frac{|\alpha|-k}{k}\right)\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=k}|P(z)|. \tag{21}$$

Since $F(z) = P(kz)$ has all its zeros in $|z| \leq 1$, therefore $F^*(z) = z^n\overline{F(1/\overline{z})} = z^n\overline{P(k/\overline{z})}$ has no zero in $|z| < 1$. Hence by applying Lemma 3.2 to the polynomial $F^*(z)$, with $R = k \geq 1$, we have

$$\max_{|z|=k} |F^*(z)| \leq \frac{(1+k^n)(|a_n|k^n + k|a_0|)}{(1+k)(|a_n|k^n + |a_0|)} \max_{|z|=1} |F^*(z)|. \tag{22}$$

Also

$$\max_{|z|=k} |F^*(z)| = k^n \max_{|z|=1} |P(z)|,$$

$$\max_{|z|=1} |F(z)| = \max_{|z|=k} |P(z)|$$

and

$$\max_{|z|=1} |F^*(z)| = \max_{|z|=1} |F(z)|.$$

Replacing these in inequality (22), we get

$$\max_{|z|=k} |P(z)| \geq k^n \frac{(1+k)(|a_n|k^n + |a_0|)}{(1+k^n)(|a_n|k^n + k|a_0|)} \max_{|z|=1} |P(z)|. \tag{23}$$

Also $D_\alpha P(z)$ is a polynomial of degree $n-1$, hence by applying inequality (12) of Lemma 3.3 to the polynomial $D_\alpha P(z)$ with $R = k \geq 1$, we obtain for $n > 3$

$$\begin{aligned} \max_{|z|=k} |D_\alpha P(z)| \leq k^{n-1} \max_{|z|=1} |D_\alpha P(z)| - 2|na_0 + \alpha a_1| \left(\frac{k^{n-1}-1}{n+1}\right) \\ - |(n-1)a_1 + 2\alpha a_2| \left[\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}\right]. \end{aligned} \tag{24}$$

Inequality (21) in conjuction with inequalities (23) and (24) yields for $n > 3$

$$\begin{aligned} \max_{|z|=1} |D_\alpha P(z)| &\geq (|\alpha| - k) \left[\frac{(1+k)(|a_n|k^n + |a_0|)}{(1+k^n)(|a_n|k^n + k|a_0|)}\right] \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|} \max_{|z|=1} |P(z)| \\ &\quad + \frac{2|na_0 + \alpha a_1|}{k^{n-1}} \left(\frac{k^{n-1}-1}{n+1}\right) + \frac{|(n-1)a_1 + 2\alpha a_2|}{k^{n-1}} \left[\frac{k^{n-1}-1}{n+1} - \frac{k^{n-3}-1}{n-3}\right] \\ &= \left[\frac{2(|\alpha| - k)}{1+k^n} + (|\alpha| - k)\frac{(|a_n|k^n - |a_0|)(k-1)}{(1+k^n)(|a_n|k^n + k|a_0|)}\right] \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|} \max_{|z|=1} |P(z)| \\ &\quad + \frac{2|na_0 + \alpha a_1|}{k^{n-1}} \left(\frac{k^{n-1}-1}{n+1}\right) + \frac{|(n-1)a_1 + 2\alpha a_2|}{k^{n-1}} \left[\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}\right], \end{aligned}$$

which proves the result in case $n > 3$. For the case $n = 3$, the result follows on similar lines but instead of using inequality (12) of Lemma 3.3, we use inequality (13) of Lemma 3.3, which completely proves Theorem 2.5. □

References

[1] A. Aziz, *Inequalities for the derivative of a polynomial*, Proc. Amer. Math. Soc. **89** (1983), 259–266.

[2] A. Aziz and Q. M. Dawood, *Inequalities for a polynomial and its derivatives*, J. Approx. Theory **54** (1998), 306–313.

[3] A. Aziz and N. A. Rather, *Some Zygmund type L_q inequalities for polynomials*, J. Math. Anal. Appl. **289** (2004), 14–29.

[4] A. Aziz and N. A. Rather, *Inequalities for the polar derivative of a polynomial with restricted zeros*, Math. Bulk. **17** (2003), 15–28.

[5] S. N. Bernstein, *Sur e'ordre de la meilleure approximation des functions continues par des polynomes de degre' donne'*. Mem. Acad. R. Belg. **12** (4) (1912), 1–103.

[6] K. K. Dewan, J. Kaur and A. Mir, *Inequalities for the derivative of a polynomial* J. Math. Anal. Appl., **269** (2002), 489–499.

[7] V.N. Dubinin, *Applications of Schwarz lemma to inequalities for entire functions with constraints on zeros*, J. Math. Sci. **143** (2007), 3069–3076.

[8] N. K. Govil and P. Kumar, *On Bernstein type inequalities for polar derivative of a polynomial*, Prog. Approx. Theory Appl. Complex Anal Springer Optim Appl. **117** (2017), 41–74.

[9] N. K. Govil, P. Kumar, *On sharpening of an inequality of Turán*, Appl. Anal. Discrete Math. **13** (2019), 711–720.

[10] P. Kumar, *On the inequalities concerning polynomials*, Complex Anal. and Oper. Theory **14** (2020), 1–11 .

[11] P.Turan, *Über die Ableitung von polynomen*, Compos. Math.**7** (1939), 89–95.

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