Korean J. Math. **29** (2021), No. 3, pp. 511–518
http://dx.doi.org/10.11568/kjm.2021.29.3.511

EXTENDED SPECTRUM OF THE ALUTHGE TRANSFORMATION

Asma Guemoula and Abdelouahab Mansour*

Abstract. In this paper, a relationship between the extended spectrum of the Aluthge transform and the extended spectrum of the operator T is proved. Other relationships between two different operators and other results are also given.

1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space, let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. If T is an operator in $\mathcal{B}(\mathcal{H})$, then a complex number λ is called *an extended eigenvalue* of T if there is a non zero operator X such that $TX = \lambda XT$. We denote by $\sigma_{ext}(T)$ the set of all extended eigenvalues of T. The set of all extended eigenvectors corresponding to λ will be denoted as $E_{ext}(\lambda)$.

$$\sigma_{ext}\,(\mathrm{T}) = \{\lambda \in \mathbb{C}, \quad \mathrm{X} \in \mathcal{B}(\mathcal{H})\backslash\{0\} \text{ such that } TX = \lambda XT\} \tag{1.1}$$

In [3], A. Biswas, A. Lambert and S. Petrovic introduced this notion by showing that the extended spectrum of an operator with dense image, corresponds to the point spectrum of another operator (generally unbounded). Extended eigenvalues and their corresponding extended eigenvectors were studied in several papers, for example [4], [5] and [13].

In 1990 A. Aluthge introduced in [2] a transformation called *transformation of Aluthge*, to extend some spectral properties to the p-hyponormal operators ($(T^*T)^p \geq (TT^*)^p\,, \quad p > 0$). For an operator $T \in \mathcal{B}(\mathcal{H})$, the Aluthge transformation of T denoted by $\widetilde{T}$ is defined by:

$$\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \tag{1.2}$$

where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the unitary operator associated with the polar decomposition of T ($T = U|T|$, $\ker T = \ker U$). It follows easily from the definition of $\widetilde{T}$ that $\|\widetilde{T}\| \leq \|T\|$ and $r(\widetilde{T}) = r(T)$, also $w(\widetilde{T}) \leq w(T)$ (see [11], [17]).

Received May 1, 2021. Revised June 15, 2021. Accepted June 18, 2021.
2010 Mathematics Subject Classification: 47A10, 47A25, 47A75, 47B49.
Key words and phrases: Aluthge transformation, extended spectrum, Duggal transformation, self-adjoint operator, invertible operator.
This work was supported by Laboratory of operator theory, Algeria.
$*$ Corresponding author.

In [12], K. Okubo introduced the generalized Aluthge transformation. For $\lambda \in [0,1]$, we call the λ-*Aluthge transformation* of an operator T, the operator $\widetilde{T_\lambda}$ given by:

$$\widetilde{T_\lambda} = |T|^{\lambda} U |T|^{1-\lambda}$$

Some particular cases:
In the case $\lambda = 0$, we simply have $\widetilde{T_0} = U|T| = T$. For $\lambda = \frac{1}{2}$ we find the definition of the Aluthge transformation defined in (1.2). And for $\lambda = 1$ we have $\widetilde{T_1} = \widehat{T} = |T|U$ (the *Duggal transformation*). For the details one can refer to [7].
In recent years, this transformation has attracted the attention of several mathematicians (see for example [6], [10], [14] and [15]).

The organization of the paper is as follows: Sections 2, contains some preliminaries properties that will be necessary in order to prove our main results. Sections 3, we present some results on the possible relations between the extended spectrum of AB and BA in finite dimension, and its consequences in form of several corollaries, also and we prove some equalities between the extended spectrum for operator T, for Aluthge transformation and for Duggal transformation in infinite dimension.

2. Preliminaries

We present here some lemmas and corollary which are useful for show our results.

LEMMA 2.1. *[8] Let $\sigma(T)$ be the spectrum of an operator T, and $p(t)$ be any polymomial of a complex number t. Then*

$$\sigma(p(T)) = p(\sigma(T))$$

LEMMA 2.2. *[8] Let A and B be operators in $\mathcal{B}(\mathcal{H})$. Then*

$$\sigma(AB) - \{0\} = \sigma(BA) - \{0\}.$$

LEMMA 2.3. *[9] Let A and B be operators in $\mathcal{B}(\mathcal{H})$, such that A is self adjoint . Then, AB is invertible if and only if BA is invertible.*

LEMMA 2.4. *Let A and B be operators in $\mathcal{B}(\mathcal{H})$ such that A is self adjoint . Then*

$$\sigma(AB) = \sigma(BA).$$

Proof. From Lemma 2.2, we have for any operators A and B,

$$\sigma(AB) - \{0\} = \sigma(BA) - \{0\}.$$

If A is self adjoint; using Lemma 2.3, we find:
if $\lambda = 0$

$$\begin{aligned} 0 \in \sigma(AB) &\Leftrightarrow AB \text{ is not invertible} \\ &\Leftrightarrow BA \text{ is not invertible} \\ &\Leftrightarrow 0 \in \sigma(BA). \end{aligned}$$

if $\lambda \neq 0$

$$\begin{aligned} 0 \notin \sigma(AB) &\Leftrightarrow AB \text{ is invertible} \\ &\Leftrightarrow BA \text{ is invertible} \\ &\Leftrightarrow 0 \notin \sigma(BA). \end{aligned}$$

So, $\sigma(AB) = \sigma(BA)$ $\square$

LEMMA 2.5. *[11] Let T be an operator in $\mathcal{B}(\mathcal{H})$, $\widetilde{T}$ its the Aluthge transformation. Then*

$$\sigma(T) = \sigma(\widetilde{T}).$$

LEMMA 2.6. *[1] Let T be an operator on a finite dimensional Hilbert space H. Then*

$$\sigma_{\text{ext}}(T) = \{\lambda \in \mathbb{C} : \sigma(T) \cap \sigma(\lambda T) \neq \emptyset\}.$$

COROLLARY 2.1. *[1] Let T be an operator on a finite dimensional Hilbert space H. Then*

- *If T is invertible then $\sigma_{\text{ext}}(T) = \{\alpha/\beta : \alpha, \beta \in \sigma(T)\}$.*
- *$\sigma_{\text{ext}}(T) = \{1\}$ if and only if $\sigma(T) = \{\alpha\}, \alpha \neq 0$.*
- *$\sigma_{\text{ext}}(T) = \mathbb{C}$ if and only if $0 \in \sigma(T)$.*

Moreover, this assertion remains available in infinite dimensional Hilbert spaces if $0 \in \sigma_p(T) \cap \sigma_p(T^*)$.

LEMMA 2.7. *Let T be an operator in $\mathcal{B}(\mathcal{H})$. Then $|T|^2$ is invertible if T is invertible.*

Proof.

$$\begin{aligned} T \text{ is invertible} &\Rightarrow T^* \text{ is invertible} \\ &\Rightarrow T^*T = |T|^2 \text{ is invertible.} \end{aligned}$$

$\square$

3. Main results

3.1. Extended spectrum of Aluthge transformation in finite dimension. Our main results is the relations heap between the extended spectrum of AB and BA, and its consequences in form of several corollaries.

THEOREM 3.1. *Let A and B be bounded linear operators on a finite dimensional separable complex Hilbert space H, such that A is self adjoint. Then*

$$\sigma_{ext}(AB) = \sigma_{ext}(BA)$$

Proof. For every bounded linear operators A and B defined on a finite dimensional separable complex Hilbert space, we have

$$\begin{aligned} \sigma_{\text{ext}}(AB) &= \{\lambda \in \mathbb{C} : \sigma(AB) \cap \sigma(\lambda AB) \neq \emptyset\} && \text{(by Lemma 2.6)} \\ &= \{\lambda \in \mathbb{C} : \sigma(AB) \cap \lambda\sigma(AB) \neq \emptyset\} && \text{(by Lemma 2.1)} \\ &= \{\lambda \in \mathbb{C} : \sigma(BA) \cap \lambda\sigma(BA) \neq \emptyset\} && \text{(by Lemma 2.4)} \\ &= \{\lambda \in \mathbb{C} : \sigma(BA) \cap \sigma(\lambda BA) \neq \emptyset\} && \text{(by Lemma 2.1)} \\ &= \sigma_{\text{ext}}(BA). \end{aligned}$$

$\square$

If $A = |\mathrm{T}|^{\frac{1}{2}}$ and $B = \mathrm{U}|\mathrm{T}|^{\frac{1}{2}}$, where U is the unitary with the polar decomposition of T in Theorem 3.1, then we get the following Corollary:

Corollary 3.1. *Let T be bounded linear operator defined on a finite dimensional separable complex Hilbert space* H, *such that* $\mathrm{T} = \mathrm{U}|\mathrm{T}|$ *its polar decomposition,* $\widetilde{T}$ *its the Aluthge transformation. Then*

$$\sigma_{ext}(\widetilde{T}) = \sigma_{ext}(T).$$

We can prove Corollary 3.1 in another way

Proof. For every bounded linear operator T defined on a finite dimensional separable complex Hilbert space H, such that $\mathrm{T} = \mathrm{U}|\mathrm{T}|$ its polar decomposition, we have

$$\begin{aligned}
\sigma_{\text{ext}}(T) &= \{\lambda \in \mathbb{C} : \sigma(T) \cap \sigma(\lambda T) \neq \emptyset\} && \text{(by Lemma 2.6)}\\
&= \{\lambda \in \mathbb{C} : \sigma(T) \cap \lambda\sigma(T) \neq \emptyset\} && \text{(by Lemma 2.1)}\\
&= \{\lambda \in \mathbb{C} : \sigma(\widetilde{T}) \cap \lambda\sigma(\widetilde{T}) \neq \emptyset\} && \text{(by Lemma 2.5)}\\
&= \{\lambda \in \mathbb{C} : \sigma(\widetilde{T}) \cap \sigma(\lambda\widetilde{T}) \neq \emptyset\} && \text{(by Lemma 2.1)}\\
&= \sigma_{\text{ext}}(\widetilde{T}).
\end{aligned}$$

□

If $A = |\mathrm{T}|$ and $B = \mathrm{U}$, where U is the unitary with the polar decomposition of T in Theorem 3.1, then we get the following Corollary:

Corollary 3.2. *Let T be bounded linear operator defined on a finite dimensional separable complex Hilbert space* H, *such that* $\mathrm{T} = \mathrm{U}|\mathrm{T}|$ *its polar decomposition,* $\widehat{T}$ *its the Duggal transformation . Then*

$$\sigma_{ext}(\widehat{T}) = \sigma_{ext}(T).$$

If $A = |\mathrm{T}|^{\lambda}$ and $B = \mathrm{U}|\mathrm{T}|^{1-\lambda}$, where U is the unitary with the polar decomposition of T in Theorem 3.1, then we get the following Corollary:

Corollary 3.3. *Let T be bounded linear operator defined on a finite dimensional separable complex Hilbert space* H, *such that* $\mathrm{T} = \mathrm{U}|\mathrm{T}|$ *its polar decomposition.For* $\lambda \in [0,1]$ *then*

$$\sigma_{ext}(\widetilde{T_{\lambda}}) = \sigma_{ext}(T).$$

If $A = |\mathrm{T}|^{\frac{1}{2}}$ and $B = \mathrm{T}^{n-1}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}$, where U is the unitary with the polar decomposition of T in Theorem 3.1, then we get the following Corollary:

Corollary 3.4. *Let T be bounded linear operator defined on a finite dimensional separable complex Hilbert space* H, *such that* $\mathrm{T} = \mathrm{U}|\mathrm{T}|$ *its polar decomposition. Then*

$$\sigma_{ext}(\widetilde{T^n}) = \sigma_{ext}(T^n).$$

If $A = |\mathrm{T}|$ and $B = (\mathrm{T})^{n-1}\mathrm{U}$, where U is the unitary with the polar decomposition of T in Theorem 3.1, then we get the following Corollary:

Corollary 3.5. *Let T be bounded linear operator defined on a finite dimensional separable complex Hilbert space* H, *such that* $\mathrm{T} = \mathrm{U}|\mathrm{T}|$ *its polar decomposition. Then*

$$\sigma_{ext}(\widehat{T^n}) = \sigma_{ext}(T^n).$$

Example 3.1. Let T be a 2×2 matrix defined as follows:

$$\mathrm{T} = \begin{pmatrix} 1 & 5i \\ -2i & 1 \end{pmatrix}, \quad \text{the adjoint operator of T is } \mathrm{T}^* = \begin{pmatrix} 1 & 2i \\ -5i & 1 \end{pmatrix},$$

$$\mathrm{T}^*\mathrm{T} = |\mathrm{T}|^2 = \begin{pmatrix} 5 & 7i \\ -7i & 26 \end{pmatrix}, \quad \text{also } |\mathrm{T}| = \begin{pmatrix} 2 & i \\ -i & 5 \end{pmatrix},$$

$$\text{so } |\mathrm{T}|^{\frac{1}{2}} = \begin{pmatrix} 1 & i \\ -i & -2 \end{pmatrix}, \quad \text{thus } \mathrm{U} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The Aluthge transform of operator T is $\widetilde{T} = \begin{pmatrix} -2 & -i \\ i & 4 \end{pmatrix}$.

Using Corollary 2.1, we compute the extended spectrum for the operator T and the Aluthge transform of the operator T.
$\sigma(\mathrm{T}) = \{-1-\sqrt{10}, -1+\sqrt{10}\}$, $\sigma(\widetilde{T}) = \{-1-\sqrt{10}, -1+\sqrt{10}\}$.
So, we get
$\sigma_{\text{ext}}\ (\mathrm{T}) = \{1; \frac{11+2\sqrt{10}}{-9}; \frac{11-2\sqrt{10}}{-9}\}$, $\sigma_{\text{ext}}\ (\widetilde{T}) = \{1; \frac{11+2\sqrt{10}}{-9}; \frac{11-2\sqrt{10}}{-9}\}$.
Thus $\sigma_{ext}(\widetilde{T}) = \sigma_{ext}(T)$.

3.2. Extended spectrum of Aluthge transformation in infinite dimension. In this theorem we prove some equalities between the extended spectrum for operator T, for Aluthge transformation and for Duggal transformation in infinite dimension.

Theorem 3.2. *Let T be bounded linear operator defined on separable complex Hilbert space H, such that* $\mathrm{T} = \mathrm{U}|\mathrm{T}|$ *its polar decomposition. If T is invertible and* $|\mathrm{T}|^{\frac{1}{2}}X = X|\mathrm{T}|^{\frac{1}{2}}$. *Then*

(i) $\sigma_{ext}(\widetilde{T}) = \sigma_{ext}(T) = \sigma_{ext}(\widehat{T})$.
(ii) $\sigma_{ext}(\widetilde{T^n}) = \sigma_{ext}(T^n) = \sigma_{ext}(\widehat{T^n})$ *for all $n \in \mathbb{N}$.*
(iii) $\sigma_{ext}(T\widetilde{T}) = \sigma_{ext}(\widetilde{T}\widehat{T})$.

Proof. For every bounded linear operator T defined on separable complex Hilbert space, such that $\mathrm{T} = \mathrm{U}|\mathrm{T}|$ its polar decomposition. T is invertible and $|\mathrm{T}|^{\frac{1}{2}}X = X|\mathrm{T}|^{\frac{1}{2}}$, we have

(i) $\quad \sigma_{\text{ext}}\ (\mathrm{T}) = \{\lambda \in \mathbb{C}, \mathrm{X} \in \mathcal{B}(\mathcal{H})\backslash\{0\} \text{ Such that } TX = \lambda XT\}$ (by (1.1))

$$\begin{aligned}
\mathrm{T}X = \lambda X\mathrm{T} &\Leftrightarrow \mathrm{U}|\mathrm{T}|X = \lambda X\mathrm{U}|\mathrm{T}| \\
&\Leftrightarrow \mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|^{\frac{1}{2}}X = \lambda X\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|^{\frac{1}{2}} \\
(\text{by } |\mathrm{T}|^{\frac{1}{2}}X = X|\mathrm{T}|^{\frac{1}{2}}) &\Leftrightarrow \mathrm{U}|\mathrm{T}|^{\frac{1}{2}}X|\mathrm{T}|^{\frac{1}{2}} = \lambda X\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|^{\frac{1}{2}} \\
&\Leftrightarrow |\mathrm{T}|^{\frac{1}{2}}(\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}X|\mathrm{T}|^{\frac{1}{2}})|\mathrm{T}|^{\frac{3}{2}} \\
&= |\mathrm{T}|^{\frac{1}{2}}(\lambda X\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|^{\frac{1}{2}})|\mathrm{T}|^{\frac{3}{2}} \\
(\text{by Lemma 2.7}) &\Leftrightarrow |\mathrm{T}|^{\frac{1}{2}}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}X|\mathrm{T}|^{2}(|\mathrm{T}|^{2})^{-1} \\
&= |\mathrm{T}|^{\frac{1}{2}}\lambda X\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|^{2}(|\mathrm{T}|^{2})^{-1} \\
&\Leftrightarrow |\mathrm{T}|^{\frac{1}{2}}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}X = \lambda|\mathrm{T}|^{\frac{1}{2}}X\mathrm{U}|\mathrm{T}|^{\frac{1}{2}} \\
(\text{by } |\mathrm{T}|^{\frac{1}{2}}X = X|\mathrm{T}|^{\frac{1}{2}}) &\Leftrightarrow |\mathrm{T}|^{\frac{1}{2}}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}X = \lambda X|\mathrm{T}|^{\frac{1}{2}}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}} \\
&\Leftrightarrow \widetilde{T}X = \lambda X\widetilde{T}.
\end{aligned}$$

So, σ_{ext} (T) $= \sigma_{\text{ext}}$ $(\widetilde{T})$.

$$\text{Also,} \quad \sigma_{\text{ext}}\ (\mathrm{T}) = \{\lambda \in \mathbb{C}, \mathrm{X} \in \mathcal{B}(\mathcal{H})\backslash\{0\} \text{ Such that } TX = \lambda XT\}.$$

$$\begin{aligned}
\mathrm{T}X = \lambda X\mathrm{T} &\Leftrightarrow \mathrm{U}|\mathrm{T}|X = \lambda X\mathrm{U}|\mathrm{T}| \\
(\text{by } |\mathrm{T}|^{\frac{1}{2}}X = X|\mathrm{T}|^{\frac{1}{2}}) &\Leftrightarrow \mathrm{U}X|\mathrm{T}| = \lambda X\mathrm{U}|\mathrm{T}| \\
&\Leftrightarrow |\mathrm{T}|(\mathrm{U}X|\mathrm{T}|)|\mathrm{T}| = |\mathrm{T}|(\lambda X\mathrm{U}|\mathrm{T}|)|\mathrm{T}| \\
(\text{by Lemma 2.7}) &\Leftrightarrow |\mathrm{T}|\mathrm{U}X|\mathrm{T}|^2(|\mathrm{T}|^2)^{-1} = |\mathrm{T}|\lambda X\mathrm{U}|\mathrm{T}|^2(|\mathrm{T}|^2)^{-1} \\
&\Leftrightarrow |\mathrm{T}|\mathrm{U}X = \lambda|\mathrm{T}|X\mathrm{U} \\
(\text{by } |\mathrm{T}|^{\frac{1}{2}}X = X|\mathrm{T}|^{\frac{1}{2}}) &\Leftrightarrow |\mathrm{T}|\mathrm{U}X = \lambda X|\mathrm{T}|\mathrm{U} \\
&\Leftrightarrow \widehat{T}X = \lambda X\widehat{T}.
\end{aligned}$$

So, σ_{ext} (T) $= \sigma_{\text{ext}}$ $(\widehat{T})$.

$$(ii) \quad \sigma_{\text{ext}}\ (\mathrm{T}^n) = \{\lambda \in \mathbb{C}, \mathrm{X} \in \mathcal{B}(\mathcal{H})\backslash\{0\} \text{ Such that } T^nX = \lambda XT^n\}.$$

$$\begin{aligned}
\mathrm{T}^nX = \lambda X\mathrm{T}^n &\Leftrightarrow (\mathrm{U}|\mathrm{T}|)^nX = \lambda X(\mathrm{U}|\mathrm{T}|)^n \\
&\Leftrightarrow (\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|^{\frac{1}{2}}X \\
&= \lambda X(\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|^{\frac{1}{2}} \\
(\text{by } |\mathrm{T}|^{\frac{1}{2}}X = X|\mathrm{T}|^{\frac{1}{2}}) &\Leftrightarrow (\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}X|\mathrm{T}|^{\frac{1}{2}} \\
&= \lambda X(\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|^{\frac{1}{2}} \\
&\Leftrightarrow |\mathrm{T}|^{\frac{1}{2}}((\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}X|\mathrm{T}|^{\frac{1}{2}})|\mathrm{T}|^{\frac{3}{2}} \\
&= |\mathrm{T}|^{\frac{1}{2}}(\lambda X(\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|^{\frac{1}{2}})|\mathrm{T}|^{\frac{3}{2}} \\
(\text{by Lemma 2.7}) &\Leftrightarrow |\mathrm{T}|^{\frac{1}{2}}(\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}X|\mathrm{T}|^2(|\mathrm{T}|^2)^{-1} \\
&= |\mathrm{T}|^{\frac{1}{2}}\lambda X(\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|^2(|\mathrm{T}|^2)^{-1} \\
&\Leftrightarrow |\mathrm{T}|^{\frac{1}{2}}(\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}X \\
&= \lambda|\mathrm{T}|^{\frac{1}{2}}X(\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}} \\
(\text{by } |\mathrm{T}|^{\frac{1}{2}}X = X|\mathrm{T}|^{\frac{1}{2}}) &\Leftrightarrow |\mathrm{T}|^{\frac{1}{2}}(\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}X \\
&= \lambda X|\mathrm{T}|^{\frac{1}{2}}(\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}} \\
&\Leftrightarrow (|\mathrm{T}|^{\frac{1}{2}}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}})^nX = \lambda X(|\mathrm{T}|^{\frac{1}{2}}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}})^n \\
&\Leftrightarrow \widetilde{T}^nX = \lambda X\widetilde{T}^n.
\end{aligned}$$

So, σ_{ext} $(\mathrm{T}^n) = \sigma_{\text{ext}}$ $(\widetilde{T}^n)$.

$$\text{Also,} \quad \sigma_{\text{ext}}\ (\mathrm{T}^n) = \{\lambda \in \mathbb{C}, \mathrm{X} \in \mathcal{B}(\mathcal{H})\backslash\{0\} \text{ Such that } T^nX = \lambda XT^n\}.$$

$$\begin{aligned}
\mathrm{T}^n X = \lambda X \mathrm{T}^n &\Leftrightarrow (\mathrm{U}|\mathrm{T}|)^n X = \lambda X (\mathrm{U}|\mathrm{T}|)^n \\
&\Leftrightarrow (\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}|X = \lambda X (\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}| \\
(\text{by } |\mathrm{T}|^{\frac{1}{2}}X = X|\mathrm{T}|^{\frac{1}{2}}) &\Leftrightarrow (\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}X|\mathrm{T}| = \lambda X (\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}| \\
&\Leftrightarrow |\mathrm{T}|((\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}X|\mathrm{T}|)|\mathrm{T}| \\
&= |\mathrm{T}|(\lambda X(\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}|)|\mathrm{T}| \\
(\text{by Lemma 2.7}) &\Leftrightarrow |\mathrm{T}|(\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}X|\mathrm{T}|^2(|\mathrm{T}|^2)^{-1} \\
&= \lambda|\mathrm{T}|X(\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U}|\mathrm{T}|^2(|\mathrm{T}|^2)^{-1} \\
(\text{by } |\mathrm{T}|^{\frac{1}{2}}X = X|\mathrm{T}|^{\frac{1}{2}}) &\Leftrightarrow (|\mathrm{T}|\mathrm{U})^n X = \lambda X|\mathrm{T}|(\mathrm{U}|\mathrm{T}|)^{n-1}\mathrm{U} \\
&\Leftrightarrow (|\mathrm{T}|\mathrm{U})^n X = \lambda X(|\mathrm{T}|\mathrm{U})^n \\
&\Leftrightarrow \widehat{T}^n X = \lambda X \widehat{T}^n.
\end{aligned}$$

So, $\sigma_{\text{ext}}\,(\mathrm{T}^n) = \sigma_{\text{ext}}\,(\widehat{T}^n)$.

(iii) $\sigma_{\text{ext}}\,(T\widetilde{T}) = \{\lambda \in \mathbb{C}, \mathrm{X} \in \mathcal{B}(\mathcal{H})\backslash\{0\} \text{ Such that } T\widetilde{T}X = \lambda X T\widetilde{T}\}$.

$$\begin{aligned}
T\widetilde{T}X = \lambda X T\widetilde{T} &\Leftrightarrow \mathrm{U}|\mathrm{T}||\mathrm{T}|^{\frac{1}{2}}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}X = \lambda X\mathrm{U}|\mathrm{T}||\mathrm{T}|^{\frac{1}{2}}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}} \\
(\text{by } |\mathrm{T}|^{\frac{1}{2}}X = X|\mathrm{T}|^{\frac{1}{2}}) &\Leftrightarrow \mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|\mathrm{U}X|\mathrm{T}|^{\frac{1}{2}} = \lambda X\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|\mathrm{U}|\mathrm{T}|^{\frac{1}{2}} \\
&\Leftrightarrow |\mathrm{T}|^{\frac{1}{2}}(\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|\mathrm{U}X|\mathrm{T}|^{\frac{1}{2}})|\mathrm{T}|^{\frac{3}{2}} \\
&= |\mathrm{T}|^{\frac{1}{2}}(\lambda X\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|\mathrm{U}|\mathrm{T}|^{\frac{1}{2}})|\mathrm{T}|^{\frac{3}{2}} \\
(\text{by Lemma 2.7}) &\Leftrightarrow |\mathrm{T}|^{\frac{1}{2}}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|\mathrm{U}X|\mathrm{T}|^2(|\mathrm{T}|^2)^{-1} \\
&= |\mathrm{T}|^{\frac{1}{2}}\lambda X\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|\mathrm{U}|\mathrm{T}|^2(|\mathrm{T}|^2)^{-1} \\
&\Leftrightarrow |\mathrm{T}|^{\frac{1}{2}}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|\mathrm{U}X = \lambda|\mathrm{T}|^{\frac{1}{2}}X\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|\mathrm{U} \\
(\text{by } |\mathrm{T}|^{\frac{1}{2}}X = X|\mathrm{T}|^{\frac{1}{2}}) &\Leftrightarrow |\mathrm{T}|^{\frac{1}{2}}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|\mathrm{U}X = \lambda X|\mathrm{T}|^{\frac{1}{2}}\mathrm{U}|\mathrm{T}|^{\frac{1}{2}}|\mathrm{T}|\mathrm{U} \\
&\Leftrightarrow \widetilde{T}\widehat{T}X = \lambda X\widetilde{T}\widehat{T}.
\end{aligned}$$

So, $\sigma_{ext}(T\widetilde{T}) = \sigma_{ext}(\widetilde{T}\widehat{T})$. □

References

[1] H. Alkanjo, *On extended eigenvalues and extended eigenvectors of truncated shift*, Concrete Operators **1** (2013), 19–27.

[2] A. Aluthge, *On p-hyponormal Operators for* $0 < p < 1$, Integral Equations Operator Theory **13** (1990), 307–315.

[3] A. Biswas, A. Lambert, and S. Petrovic, *Extended eigenvalues and the Volterra operator*, Glasg. Math. J. **44** (3) (2002), 521–534.

[4] A. Biswas, S. Petrovic, *On extended eigenvalues of operators*, Integral Equations Operator Theory **55** (2) (2006), 233–248.

[5] G. Cassier, H. Alkanjo, *Extended spectrum, extended eigenspaces and normal operators*, J. Math. Anal. Appl **418** (1) (2014), 305–316.

[6] K. Dykema, H. Schultz, *Brown measure and iterates of the Aluthge transform for some operators arising from measurable actions*, Trans. Amer. Math. Soc. **361** (2009), 6583–6593.

[7] C. Foias, I. B. Jung, E. Ko and C. Pearcy, *Complete Contractivity of Maps Associated with the Aluthge and Duggal Transform*, s, Pacific J. Math. **209** (2003), 249–259.

[8] T. Furuta, *Invitation to linear operators*, Taylor and Fran. Loondon, (2001).

[9] T. Huruya, *A note on p-hyponormaI operators*, Proc. Amer. Math. Soc. **125** (1997), 3617–3624.

[10] M. Ito, T. Yamazaki and M. Yanagida, *On the polar decomposition of the Aluthge transformation and related results*, J. Operator Theory **51** (2004), 303–319.
[11] I. Jung, E. Ko, and C. Pearcy, *Aluthge transforms of operators*, Integral Equations Operator Theory **37** (2000),437–448.
[12] K. Okuba, *on weakly unitarily invariant norm and the Aluthge transformation*, Linear Algebra Appl. **371** (2003), 369–375.
[13] M. Sertbas, F. Yilmaz, *On the extended spectrumof some quasinormal operators*, Turk. J. Math. **41** (2017), 1477–1481.
[14] T.Y. Tam, *λ-Aluthge iteration and spectral radius*, Integral Equations Operator Theory **60** (2008), 591–596.
[15] T. Yamazaki, *An expression of spectral radius via Aluthge transformation*, Proc. Amer. Math. Soc. **130** (2002), 1131–1137.
[16] T. Yamazaki, *On numerical range of the Aluthge transformation*, Linear Algebra Appl. **341** (2002), 111–117.
[17] K. Zaiz, A. Mansour, *On numerical range and numerical radius of convex function operations*, Korean J. Math. **27** (2019), 879–898.

Asma Guemoula
Department of Mathematics, Operators Theory and PDE Foundations and Applications Laboratory, University of El-Oued, P. O. Box 789, El-Oued 39000, Algeria.
E-mail: guemoula-asma@univ-eloued.dz

Abdelouahab Mansour
Department of Mathematics, Operators Theory and PDE Foundations and Applications Laboratory, University of El-Oued, P. O. Box 789, El-Oued 39000, Algeria.
E-mail: mansourabdelouahab@yahoo.fr