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RADAU QUADRATURE FOR A RATIONAL ALMOST QUASI-HERMITE-FEJÉR-TYPE INTERPOLATION

Shrawan Kumar, Neha Mathur, Laxmi Rathour, Vishnu Narayan Mishra*, and Pankaj Mathur*

Abstract. The aim of this paper is to obtain a Radau type quadrature formula for a rational interpolation process satisfying the almost quasi Hermite Fejér interpolatory conditions on the zeros of Chebyshev Markov sine fraction on $[-1, 1)$.

1. Introduction

Let $\mathcal{R}_{2n-1}(a_0, a_1, a_2, \cdots, a_{2n-1})$ be a rational space defined as

$$\mathcal{R}_{2n-1}(a_0, a_1, \cdots, a_{2n-1}) := \left\{ \frac{p_{2n-1}(x)}{\prod_{k=0}^{2n-1}(1 + a_k x)} \right\} \tag{1}$$

where $p_{2n-1}(x)$ is a polynomial of degree $\leq 2n - 1$ and $\{a_k\}_{k=0}^{2n-1}$ are real and belong to $[-1, 1]$ or are paired by complex conjugation.

Study of rational interpolation processes has been a field of interest for many matheматicains. In 1962, Rusak [10] initiated the study of interpolation processes by means of rational functions on the interval $[-1, 1]$. The nodes were taken to be the zeros of Chebyshev Markov [3, 4, 11, 12] rational fractions given by

$$U_n(x) = \frac{\sin \mu_{2n}(x)}{\sqrt{1 - x^2}} \quad \mu_{2n}(x) = \frac{1}{2} \sum_{k=0}^{2n-1} \arccos \frac{x + a_k}{1 + a_k x} \tag{2}$$

where, for $n \in N$

$$\mu'_{2n}(x) = -\frac{\lambda_{2n}(x)}{\sqrt{1 - x^2}}, \quad \lambda_{2n}(x) = \frac{1}{2} \sum_{k=0}^{2n-1} \frac{\sqrt{1 - a_k^2}}{1 + a_k x} \tag{3}$$

and $a_k, k = 0, 1, \ldots, 2n - 1$ are either real with $a_k \in (-1, 1)$ or are paired by complex conjugation.

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*Corresponding author.

In [7] rational interpolation functions of Hermite-Fejér-type were constructed. Min [5] was the first to consider the rational quasi-Hermite-type interpolation. He constructed the interpolatory function and proved its uniform convergence for the continuous functions on the segment with the restriction that the poles of the approximating rational functions should not have limit points on the interval $[-1,1]$. Based on the ideas of [7] and using method that was different from that of [5], Rouba et. al. [6], [9] revisited the rational interpolation functions of Hermite-Fejér-type. They also proved the uniform convergence of the interpolation process for the function $f \in \mathcal{C}[-1,1]$ and obtained explicitly its corresponding Lobatto type quadrature formula. Recently, Shrawan Kumar et.al. [2] studied the Radau type quadrature for an almost quasi-Hermite-Fejér-type interpolation in rational spaces.

In this paper we have considered an almost quasi-Hermite-Fejér-type interpolation process on the zeros of the rational Chebyshev Markov sine fraction on the semi closed interval $[-1,1)$, that is, when the interpolatory condition is prescribed only at one of the end points "-1". A Radau type quadrature formula corresponding to the interpolation process has also been obtained explicitly.

2. Preliminaries

Let the points $a_k, k = 0,1,\dots,2n-1$ be the real and $a_k \in (-1,1)$ or be paired by complex conjugation. The rational fraction $U_n(x)$, given by (2), (3), can be expressed as

$$U_n(x) = \frac{P_{n-1}(x)}{\sqrt{\Pi_{k=0}^{2n-1}(1+a_k x)}}$$

where $P_{n-1}(x)$ is an algebraic polynomial of degree $n-1$ with real coefficient. The fraction $U_n(x)$ has $n-1$ zeros on the interval $(-1,1)$,

$$-1 < x_{n-1} < x_{n-2} < \cdots < x_2 < x_1\ , \quad \mu_{2n}(x_k) = k\pi, k = 1,2,\dots,n-1.$$

Let $\{\ell_k(x)\}_{k=1}^{n-1}$ be the fundamental polynomials of Lagrange interpolation given by

$$\ell_k(x) = \frac{U_n(x)}{(x-x_k)U_n'(x_k)}. \tag{4}$$

3. Almost Quasi-Hermite-Fejér-type interpolation

Let $x_n = -1$. Then for any function $f \in C[-1,1)$ an almost quasi type Hermite interpolation function $H_n(x,f)$ satisfying the conditions

$$H_n(x_k,f) = f(x_k), \quad k = 1,2,\dots,n$$
$$H_n'(x_k,f) = y_k, \quad k = 1,2,\dots,n-1$$

is given by

$$H_n(x,f) = \sum_{k=1}^{n} f(x_k)A_k(x) + \sum_{k=1}^{n-1} y_k B_k(x) \tag{5}$$

where $y_k, k = 1, 2, \ldots, n-1$ are arbitrarily given real numbers, $\{A_k(x)\}_{k=1}^{n}$ and $\{B_k(x)\}_{k=1}^{n-1}$ are fundamental functions of an almost quasi type Hermite interpolation are given by:

For $k = 1, 2, \cdots, n-1$

$$B_k(x) = \frac{(1+x)(1-x_k)(1-x_k^2)U_n^2(x)}{\lambda_{2n}(x)\lambda_{2n}(x_k)(x-x_k)}, \tag{6}$$

$$A_k(x) = \frac{(1-x_k)(1-x_k^2)(1+x)\{1-b_k(x-x_k)\}U_n^2(x)}{\lambda_{2n}(x_k)(x-x_k)^2\lambda_{2n}(x)} \tag{7}$$

where

$$b_k = \frac{2x_k - 1}{1 - x_k^2} \tag{8}$$

and

$$A_n(x) = \frac{U_n^2(x)}{\lambda_{2n}(x)\lambda_{2n}(-1)}. \tag{9}$$

THEOREM 3.1. *The almost quasi type Hermite interpolation function $H_n(x, f)$ is a rational function of degree atmost $2n-1$ that is*

$$H_n(f, x) \in \mathcal{R}_{2n-1}(a_1, a_2, \cdots, a_{2n-1}). \tag{10}$$

Proof. Since $U_n \in \mathcal{R}_{n-1}(a_0, a_1, \cdots, a_{2n-1})$, we can express it as

$$U_n(x) := \frac{S_{n-1}(x)}{S_n^*(x)}$$

where $S_n^*(x) := \sqrt{\prod_{k=0}^{2n-1}(1 + xa_k)}$, $S_{n-1}(x) := c_{n-1}(x-x_1)(x-x_2)\cdots(x-x_{n-1})$ and c_{n-1} depends on n and $\{a_k\}_{k=0}^{2n-1}$. So, we have

$$\ell_k(x) = \frac{S_n^*(x_k)}{S_n^*(x)} q_k(x), \quad k = 1, 2, \cdots, n-1, \tag{11}$$

where

$$q_k(x) := \frac{S_{n-1}(x)}{S'_{n-1}(x_k)(x-x_k)}, \quad k = 1, 2, \cdots, n-1. \tag{12}$$

Thus $\ell_k(x) \in \mathcal{R}_{n-2}(a_0, a_1, \cdots, a_{2n-1})$. Hence by (5), (7) and (6) the lemma follows. □

Let $y_k = 0, \; k = 1, 2, \cdots, n-1$ then (5) reduces to

$$H_n(f, x) = \sum_{k=1}^{n} f(x_k)A_k(x) \tag{13}$$

which is an almost quasi Hermite Fejér interpolation function for $f \in C[-1, 1]$.

4. Radau-type quadrature formula

For a given function f defined on $[-1, 1]$, we define the function

$$G_n(x, f) = \sum_{k=1}^{n} f(x_k)h_k(x) \tag{14}$$

where, for $k = 1, 2, \cdots, n-1$,

$$h_k(x) = \frac{1+x}{1+x_k}\left[1 - \left(\frac{U_n''(x_k)}{U_n'(x_k)} - \frac{1}{(1-x_k)}\right)(x - x_k)\right]\ell_k^2(x)$$

and

$$h_n(x) = \frac{U_n^2(x)}{U_n^2(1)}.$$

We have that $G_n(f, x) \in \mathcal{R}_{2n-1}(a_1, a_2, \cdots, a_{2n-1})$. Also the rational function $G_n(f, x)$ is an almost quasi Hermite-Fejér interpolation function. Let

$$A_k = \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} h_k(x)dx \tag{15}$$

and

$$A_n = \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \frac{U_n^2(x)}{U_n^2(-1)}dx \tag{16}$$

then the Radau-type quadrature formula corresponding to the interpolatory function (14) is given by

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} G_n(f, x)dx = \sum_{k=1}^{n-1} A_k f(x_k) + A_n f(-1). \tag{17}$$

The quadrature formula corresponding to the Almost Quasi Hermite Fejér interpolation $G_n(x, f)$ is given in the following theorem.

THEOREM 4.1. *The quadrature formula* (17) *can be expressed as*

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} G_n(x, f)dx = \sum_{k=1}^{n-1} \frac{\pi(1+x_k)}{\lambda_{2n}(x_k)} f(x_k) + \frac{2\lambda_{2n}(1)}{\lambda_{2n}^2(-1)}\pi f(-1). \tag{18}$$

To prove this theorem we shall the the following lemmas.

LEMMA 4.2. *For* $k = 1, 2, \cdots, n-1$,

$$\int_{-1}^{1} \sqrt{1-x^2}\, \ell_k^2(x)dx = \frac{\pi(1-x_k^2)}{\lambda_{2n}(x_k)}. \tag{19}$$

Proof. For $k = 1, 2, \cdots, n-1$, we have

$$\begin{aligned} \ell_k^2(x) &= \frac{U_n^2(x)}{(U_n')^2(x_k)(x-x_k)^2} \\ &= \frac{(1-x_k^2)^2 \sin^2 \mu_{2n}(x)}{\lambda_{2n}^2(x_k)(1-x^2)(x-x_k)^2}. \end{aligned} \tag{20}$$

Also,

$$U_n(1) = \lim_{x\to 1} \frac{\sin \mu_{2n}(x)}{\sqrt{1-x^2}} = \lambda_{2n}(1) \tag{21}$$

and

$$U_n(-1) = (-1)^{n+1}\lambda_{2n}(-1). \tag{22}$$

Then for $k = 1, 2, \cdots, n-1$, due to (20), we have

$$\int_{-1}^{1} \sqrt{1-x^2}\ \ell_k^2(x)dx = \frac{(1-x_k^2)^2}{\lambda_{2n}^2(x_k)} \int_{-1}^{1} \frac{\sin^2 \mu_{2n}(x)}{(x-x_k)^2\sqrt{(1-x^2)}}dx. \tag{23}$$

Consider the integrals,

$$A_k^* = \int_{-1}^{1} \frac{\sin^2 \mu_{2n}(x)}{(x-x_k)^2\sqrt{1-x^2}}dx. \tag{24}$$

Consider the transformation

$$x = \frac{1-y^2}{1+y^2} \tag{25}$$

which gives

$$dx = \frac{4y}{(1+y^2)^2}dy, \tag{26}$$

$$x - x_k = \frac{2(y^2-y_k^2)}{(1+y^2)(1+y_k^2)}, \tag{27}$$

$$1 - x = \frac{2}{1+y^2}, \tag{28}$$

$$\sqrt{1-x^2} = \frac{2y}{1+y^2}. \tag{29}$$

We know that,

$$\sin \mu_{2n}\left(\frac{y^2-1}{y^2+1}\right) = \sin \phi_{2n}(y) \tag{30}$$

where $\sin \phi_{2n}(y)$ is a Bernstein sine fraction

$$\sin \phi_{2n}(y) = \frac{1}{2i}\left(\chi_n(y) - \chi_n^{-1}(y)\right) \tag{31}$$

where $\chi_n(y) = \prod_{j=0}^{2n-1} \frac{y-z_j}{y-\bar{z}_j}$ and z_k are the roots of the equations $y^2+(a_k+1)(a_k-1)^{-1} = 0$, $\mathcal{I}z_k > 0$, $k = 0, 1, \cdots, 2n-1$. Taking into account the assumptions on the parameters a_k, $k = 0, 1, \cdots, 2n-1$, we have the following: 1) $z_0 = i$, 2) if a_k and a_l are paired by complex conjugation, then the corresponding numbers z_k and z_l are symmetric with respect to the imaginary axis. Besides, the function $\sin \phi_{2n}(y)$ has zeros at $\pm y_k$, $y_k = \sqrt{(1-x_k)/(1+x_k)}$, $k = 1, 2, \cdots, n-1$. Thus,

$$A_k^* = (1+y_k^2)^2 \int_{-\infty}^{\infty} \frac{(1+y^2)\sin^2 \phi_{2n}(y)}{4(y^2-y_k^2)^2}dy.$$

Consider the auxillary integral

$$J_k^*(z) = \int_{-\infty}^{\infty} \frac{(1+y^2)\sin^2 \phi_{2n}(y)}{(y^2-z^2)^2}dy$$

then

$$A_k^* = \frac{(1+y_k^2)^2}{4} \lim_{z\to y_k, \Im z_k>0} J_k^*(z). \tag{32}$$

From (31), we get

$$J_k^*(z) = J_{1k}^*(z) + J_{2k}^*(z) + J_{3k}^*(z) \tag{33}$$

where

$$J_{1k}^*(z) = -\frac{1}{4}\int_{-\infty}^{\infty} \frac{(1+y^2)\chi_n^2(y)}{(y^2-z^2)^2}dy.$$

$$J_{2k}^*(z) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{(1+y^2)}{(y^2-z^2)^2}dy.$$

$$J_{3k}^*(z) = -\frac{1}{4}\int_{-\infty}^{\infty} \frac{(1+y^2)\chi_n^{-2}(y)}{(y^2-z^2)^2}dy.$$

Since $z_0 = i$, thus the integrand of $J_{1k}^*(z)$ has only one singular point at $y = z$ in the upper half plane. Thus by the residue theorem we have

$$\begin{aligned} J_{1k}^*(z) &= -\frac{\pi i}{2}\lim_{y\to z}\frac{d}{dy}\left[\frac{(1+y^2)\chi_n^2(y)}{(y+z)^2}\right] \\ &= -\frac{\pi i}{2}\lim_{y\to z}\left[\chi_n^2(y)\frac{d}{dy}\frac{(y^2+1)}{(y+z)^2} + \frac{2(y^2+1)}{(y+z)^2}\chi_n(y)\frac{d}{dy}\left(\chi_n(y)\right)\right]. \end{aligned}$$

Since,

$$\chi_n(y) = \prod_{j=0}^{2n-1}\frac{y-z_j}{y-\bar{z}_j}$$

which by logarithmic differentiation gives

$$\frac{d}{dy}\chi_n(y) = \chi_n(y)\sum_{j=0}^{2n-1}\frac{z_j-\bar{z}_j}{(y-z_j)(y-\bar{z}_j)}.$$

Also

$$\frac{d}{dy}\frac{(y^2+1)}{(y+z)^2} = \frac{2(yz-1)}{(y+z)^3}.$$

Therefore,

$$J_{1k}^*(z) = -\frac{\pi i}{2}\chi_n^2(z)\left[\frac{(z^2-1)}{4z^3}\right. \tag{34}$$

$$\left.+\frac{(z^2+1)}{4z^2}\sum_{j=0}^{2n-1}\frac{z_j-\bar{z}_j}{(z-z_j)(z-\bar{z}_j)}\right]. \tag{35}$$

Proceeding similarly, we have

$$J_{3k}^*(z) = -\frac{\pi i\chi_n^{-2}(z)}{2}\left[\frac{(z^2-1)}{4z^3}\right. \tag{36}$$

$$\left.+\frac{(z^2+1)}{4z^2}\sum_{j=0}^{2n-1}\frac{z_j-\bar{z}_j}{(z-z_j)(z-\bar{z}_j)}\right]. \tag{37}$$

Again since $y = z$ is the only singular point of the integrand of $J^*_{2k}(z)$ in the upper half plane. Thus, by the residue theorem, we have

$$J^*_{2k}(z) = \frac{\pi i}{2} \lim_{y\to z} \frac{d}{dy}\left[\frac{(1+y^2)}{(y+z)^2}\right] = \frac{\pi i}{2}\frac{(z^2-1)}{4z^3}. \tag{38}$$

Taking into account that $\chi_n^2(y_k) = 1$ and putting the vales of (39), (38) and (36) in (33), it follows form (32) that

$$A^*_k = -\frac{\pi i(1+y_k^2)^3}{16y_k^2}\sum_{j=0}^{2n-1}\frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)}.$$

Since, $y_k = \sqrt{(1-x_k)/(1+x_k)}$ and $z_k = i\sqrt{(1+a_k)/(1-a_k)}$, thus by simple calculation we have,

$$\begin{aligned}\sum_{j=0}^{2n-1}\frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)} &= \sum_{j=0}^{2n-1}\left(\frac{1}{y_k - z_j} - \frac{1}{y_k - \bar{z}_j}\right)\\ &= \sum_{j=0}^{2n-1}\frac{i\sqrt{(1+a_j)}\sqrt{(1-a_j)}}{1+a_jx_k}\left(\frac{2}{1+y_k^2}\right)\\ &= \frac{4i\lambda_{2n}(x_k)}{(1+y_k^2)} \qquad (39)\end{aligned}$$

where we have used

$$1 + x_k = \frac{2}{1+y_k^2}.$$

Thus,

$$\begin{aligned}A^*_k &= -\frac{(1+y_k^2)^3}{16y_k^2}\pi i\left(-4\frac{\lambda_{2n}(x_k)}{i(1+y_k^2)}\right)\\ &= \frac{\pi\lambda_{2n}(x_k)(1+y_k^2)^2}{4y_k^2} = \frac{\pi\lambda_{2n}(x_k)}{(1-x_k^2)}.\end{aligned}$$

Therefore by (23) the lemma follows. □

Lemma 4.3. *For $k = 1, 2, \cdots, n-1$,*

$$\int_{-1}^{1}\sqrt{1-x^2}(x-x_k)\ell_k^2(x)dx = 0. \tag{40}$$

Proof. For $k = 1, 2, \cdots, n-1$, due to (20), we have

$$\begin{aligned}I_k &= \int_{-1}^{1}\sqrt{1-x^2}(x-x_k)\ell_k^2(x)dx\\ &= \frac{(1-x_k^2)^2}{\lambda_{2n}^2(x_k)}\int_{-1}^{1}\frac{\sin^2\mu_{2n}(x)}{\sqrt{1-x^2}(x-x_k)}dx \qquad (41)\end{aligned}$$

By using the transformation (25), (26), (27), (28), (29) and (30) we get

$$I_k = -\frac{(1-x_k^2)^2(1+y_k^2)}{2\lambda_{2n}^2(x_k)}\int_{-\infty}^{\infty}\frac{\sin^2\phi_{2n}(y)}{(y^2-y_k^2)}dy$$

where $\sin\phi_{2n}(y)$ is a Bernstein sine fraction given by (31).

$$I_k = \frac{(1-x_k^2)^2(1+y_k^2)}{2\lambda_{2n}^2(x_k)} \lim_{z\to y_k, \mathcal{I}z_k>0} J_k(z) \tag{42}$$

where

$$J_k(z) = \int_{-\infty}^{\infty} \frac{\sin^2\phi_{2n}(y)}{y^2-z^2}dy. \tag{43}$$

From (31) we get

$$J_k(z) = -\frac{1}{4}\int_{-\infty}^{\infty} \frac{\chi_n^2(y)-2+\chi_n^{-2}(y)}{y^2-z^2}dy. \tag{44}$$

Since $J_k(z)$ has only singular point $y = z$ in the upper half plane. Thus, by the residue theorem, we have

$$\begin{aligned} J_k(z) &= -\frac{2\pi i}{4}\lim_{y\to z}\left[\frac{\chi_n^2(y)-2+\chi_n^{-2}(y)}{(y+z)}\right] \\ &= -\frac{\pi i}{4}\left[\frac{\chi_n^2(z)-2+\chi_n^{-2}(z)}{z}\right]. \end{aligned}$$

Thus, (42) gives

$$I_k = -\frac{(1-x_k^2)^2(1+y_k^2)}{2\lambda_{2n}^2(x_k)} \lim_{z\to y_k, \mathcal{I}z_k>0}\left[\frac{\chi_n^2(z)-2+\chi_n^{-2}(z)}{z}\right]. \tag{45}$$

Since $\chi_n(y_k) = 1$, thus it follows that $I_k = 0$ which proves the lemma. □

5. Proof of the Theorem 4.1

Due to Lemma 4.2 and Lemma 4.3, the coefficients of the quadrature formula (17) $\{A_k\}_{k=1}^{n-1}$ given by (15), can be expressed as

$$\begin{aligned} A_k &= \frac{1}{(1+x_k)}\int_{-1}^{1}\sqrt{1-x^2}\ell_k^2(x)dx \\ &= \frac{\pi(1-x_k^2)}{\lambda_{2n}(x_k)}\frac{1}{(1+x_k)} = \frac{\pi(1+x_k)}{\lambda_{2n}(x_k)}. \end{aligned}$$

Proceeding on similar lines we have

$$A_n = \frac{2\lambda_{2n}(1)}{\lambda_{2n}^2(-1)}\pi \tag{46}$$

which in turn proves the theorem.

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Shrawan Kumar
Department of Mathematics and Astronomy, University of Lucknow,
Lucknow, India
E-mail: kumarshrawan996@gmail.com

Neha Mathur
Department of Mathematics, Techno Institute of Higher Studies,
University of Lucknow, Lucknow, INDIA
E-mail: neha_mathur13@yahoo.com

Laxmi Rathour
Ward number – 16, Bhagatbandh, Anuppur 484 224, Madhya Pradesh, India
E-mail: laxmirathour817@gmail.com, rathourlaxmi562@gmail.com

Vishnu Narayan Mishra
Department of Mathematics, Indira Gandhi National Tribal University,
Lalpur, Amarkantak, Anuppur, Madhya Pradesh 484 887, India
E-mail: vishnunarayanmishra@gmail.com, vnm@igntu.ac.in

Pankaj Mathur
Department of Mathematics and Astronomy, University of Lucknow,
Lucknow, INDIA
E-mail: pankaj_mathur14@yahoo.co.in