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LINE GRAPHS OF UNIT GRAPHS ASSOCIATED WITH THE DIRECT PRODUCT OF RINGS

S. Pirzada* and Aaqib Altaf

Abstract. Let R be a finite commutative ring with non zero identity. The unit graph of R denoted by $G(R)$ is the graph obtained by setting all the elements of R to be the vertices of a graph and two distinct vertices x and y are adjacent if and only if $x + y \in U(R)$, where $U(R)$ denotes the set of units of R. In this paper, we find the commutative rings R for which $G(R)$ is a line graph. Also, we find the rings for which the complements of unit graphs are line graphs.

1. Introduction

Let $G = (V, E)$ be a simple finite graph with vertex set $V(G)$ and edge set $E(G)$. The order and size of the graph are respectively $|V(G)|$ and $|E(G)|$. We denote by P_n the path, by C_n the cycle both on n vertices. A complete graph is denoted by K_n and a complete bipartite graph by $K_{m,n}$. Also, the complement of G, denoted by $\bar{G}$, is the graph having the same vertex set as G and two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G. The line graph $L(G)$ of a graph G is the graph with vertex set as $E(G)$ and two edges in $L(G)$ are adjacent if and only if their corresponding edges are adjacent in G. For standard notations and definitions, we refer to [12].

We have the following well known characterization of line graphs, due to Beineke [6].

Theorem 1.1. *A graph G is the line graph of some graph if and only if none of the nine graphs in Fig. 1 is an induced subgraph of G.*

All rings are finite commutative with non zero identity. An element $u \in R$ is said to be a unit if there exists an element $v \in R$ such that $uv = vu = 1$. The group of units is denoted by $U(R)$. For a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n. Further, $\mathbb{F}_p$ denotes the finite field of p elements and $J(R)$ denotes the Jacobian radical. A ring is Artinian if it satisfies the descending chain condition on ideals. It is clear that finite rings are Artinian rings. However, the converse is not true in general. For example, any infinite field is an Artinian ring. A ring R is a local ring if $R/J(R)$ is a field. More on rings can be seen in [9].

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*Corresponding author.

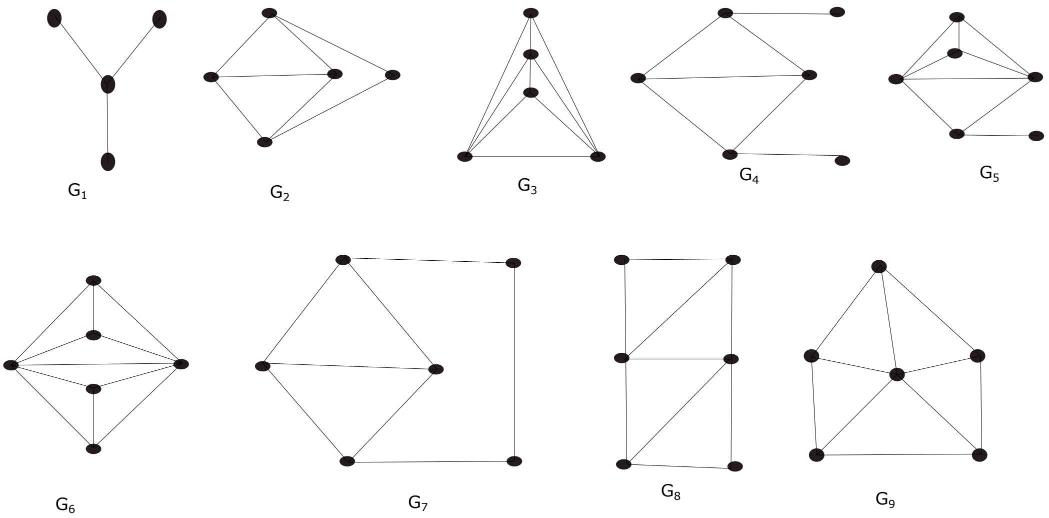

FIGURE 1. Forbidden induced sub-graphs of line graphs

The unit graph of a ring R, denoted $G(R)$, is the simple graph defined on the elements of R with an edge between two distinct vertices x and y if and only if $x + y$ is a unit of R. The unit graph was first investigated by Grimaldi [7] for $\mathbb{Z}_n$, the ring of integers modulo n. Ashrafi et al. [2] generalized the unit graph $G(\mathbb{Z}_n)$ to $G(R)$ for an arbitrary ring R. Some work on this topic can be seen in [1, 8, 10, 11, 20–23].

The zero divisor graph of a ring R, denoted by $\Gamma(R)$, is a simple graph whose vertex set is $\mathbb{Z}^*(R) = \mathbb{Z}(R) \setminus \{0\}$ and two distinct vertices a and b are adjacent if and only if $ab = 0$. Recently, Barati [3] investigated line graphs in zero divisor graphs. More recent work on zero divisor graphs can be seen in [4, 16–19]. Further, the spectrum of different matrices of the zero divisor graph associated to $\mathbb{Z}_n$ can be found in [5, 13–15].

In this paper, we investigate line graphs of unit graphs. Also, we find the rings for which the complements of unit graphs are line graphs. Throughout we consider the direct product of the integer modulo rings unless otherwise stated.

2. Unit graphs which are line graphs

The direct product of two rings R_1 and R_2, denoted by $R_1 \times R_2$, consists of all ordered pairs (a, b) with $a \in R_1$ and $b \in R_2$. The addition rule for such pairs is $(a, b) + (c, d) = (a + c, b + d)$ and the multiplication rule is $(a, b)(c, d) = (ac, bd)$.

If R is an Artinian ring, by the structure theorem of Artinian rings, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is a local ring for all $1 \le i \le n$.

THEOREM 2.1. *For $n = 2$, the unit graph $G(R)$ is a line graph if and only if R is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_6$.*

Proof. Let $R \cong R_1 \times R_2$. We have the following cases.
Case 1. Let $|R_i| = n_i \ge 3$, for all $1 \le i \le 2$. We know that the Euler's totient function $\phi(n)$ is even, for $n > 2$ ($\phi(n)$ is equal to the number of positive integers less than n which are relatively prime to n). Therefore, $|U(R_1)| \ge 2$ and $|U(R_2)| \ge 2$ which implies that $|U(R_1 \times R_2)| \ge 4$. That is, the ring R contains at least four units.

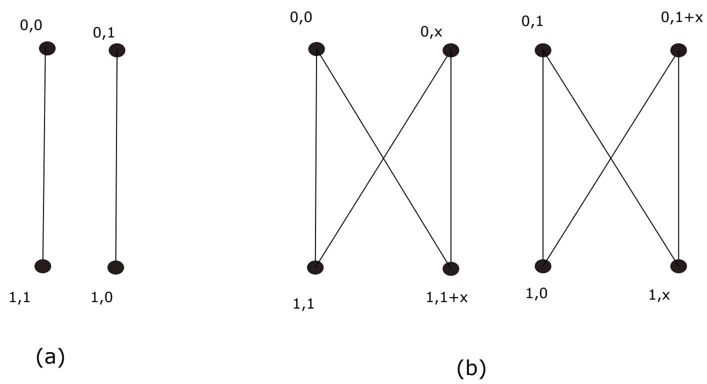

FIGURE 2. (a) $G(\mathbb{Z}_2 \times \mathbb{Z}_2)$ (b) $G\left(\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}\right)$

In the unit graph $G(R)$, consider the subset S of the vertex set as follows.

$$S = \{(0,0), (1,1), (n_1 - 1, 1), (1, n_2 - 1)\}$$

where $(1,1)$, $(n_1 - 1, 1)$ and $(1, n_2 - 1)$ are the units of R, since $n_1 - 1$ is coprime with n_1 and $n_2 - 1$ is coprime with n_2. Now, by definition of the unit graph, no two vertices corresponding to the three units are adjacent. Also, each vertex corresponding to these units is adjacent to the vertex corresponding to $(0,0)$. Clearly, the induced subgraph with vertex set as S is $K_{1,3}$. Therefore, in this case, $G(R)$ is not a line graph.

Case 2. Either $|R_1| = n_1 = 5$ or ≥ 7; or $|R_2| = n_2 = 5$ or ≥ 7. As $\phi(n_i) \geq 3$ for $i = 1, 2$, so $|U(R_1 \times R_2)| \geq 3$. Therefore, in this case, R contains at least three units. By the same argument as in Case 1, $G(R)$ contains $K_{1,3}$ as an induced subgraph. So $G(R)$ is not a line graph.

Case 3. We have the following possibilities.

Case 3 (a). $|R_1| = 2$, $|R_2| = 2$. Then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Clearly, in this case, the unit graph $G(R)$ contains only one unit, namely $(1,1)$. We observe that $G(R)$ consists of two components, each being a path of length one, see Figure 2(a). Thus $G(R)$ is a line graph.

Case 3 (b). $|R_1| = 2$, $|R_2| = 3$. Then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, since there exists only one ring of order 3 up to isomorphism. Clearly, in this case, the unit graph $G(R)$ is isomorphic to C_6. Thus, $G(R)$ is a line graph, as the line graph of a cycle is a cycle.

Case 3 (c). $|R_1| = 2$, $|R_2| = 4$. Therefore, in this case, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. Thus R has only two units, namely $(1,1)$ and $(1,3)$. So the unit graph contains two components, each being cycle C_4, see Figure 2(b). Thus the unit graph in this case is a line graph.

Case 3 (d). $|R_1| = 2$, $|R_2| = 6$. So $R \cong G(\mathbb{Z}_2 \times \mathbb{Z}_6)$. Clearly, in this case, $G(R)$ contains four components, each being a path. Thus $G(R)$ is a line graph.

Hence, for n=2, we conclude that the unit graph $G(R)$ is a line graph if and only if R is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_6$. □

The arguments of Theorem 2.1 can be extended to the case when $n = 3$ in the following way.

THEOREM 2.2. *For $n = 3$, the unit graph $G(R)$ is a line graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_k$, where $k = 2, 3, 4$ or 6.*

Proof. Let $R \cong R_1 \times R_2 \times R_3$. We have the following cases.
Case 1. Let $|R_i| = n_i \geq 3$, for all $1 \leq i \leq 3$. By the same argument as in Theorem 2.1, the ring R contains at least three units. In the unit graph $G(R)$, consider the subset S of the vertex set as follows.

$$S = \{(0,0,0), (1,1,1), (1, n_2 - 1, 1), (1, 1, n_3 - 1)\}$$

Clearly, $G(R)$ contains $K_{1,3}$ as an induced subgraph, so is not a line graph.
Case 2. When two of the rings have at least three elements. Assume that $|R_2| \geq 3$ and $|R_3| \geq 3$. Then the ring R contains at least three units. So by the same argument as in Case 1, the unit graph $G(R)$ is not a line graph.
Case 3. When exactly one ring has greater or equal to three elements, say $|R_3| = n_3 \geq 3$. Also, $|R_1| = |R_2| = 2$. We have the following possibilities.
Case 3 (a). $|R_3| = n_3 = 5$ or ≥ 7. We have $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times R_3$. Now, for $n_3 = 5$ or ≥ 7, the ring R_3 has always three or more units. Consider the subset S of the vertex set of $G(R)$ with $S = \{(0,0,0), (1,1,a), (1,1,b), (1,1,c)\}$, where $a, b, c \neq 0, 1$ are the units of R_3. Every element except $(0,0,0)$ of S is a unit of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times R_3$. So S induces $K_{1,3}$ in $G(R)$ and thus $G(R)$ is not a line graph.
Case 3 (b). $|R_3| = 3$, *or* 4, *or* 6. In case $|R_3| = 3$, we have $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, the unit graph $G(R)$ consists of two disjoint copies of C_6, so is a line graph. For $|R_3| = 4$, we have $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$. Therefore, $G(R)$ consists of four disjoint copies of C_6. Thus $G(R)$ is a line graph. Finally, if $|R_3| = 6$, then $G(R)$ consists of four disjoint copies of C_6, thus is a line graph.
Case 4. $|R_i| = 2$ for all $i = 1, 2, 3$. So, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Obviously, $G(R)$ consists of four components, each being K_2. Therefore, $G(R)$ is a line graph.

Hence, for n = 3, we see that the unit graph is a line graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_k$, where $k = 2, 3, 4, 6$. □

Using the similar arguments as in Theorems 2.1 and 2.2, we can prove the following.

Theorem 2.3. *For $n = 4$, the unit graph $G(R)$ is a line graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$.*

Now, we conclude with the following result.

Theorem 2.4. *Let $R \cong R_1 \times R_2 \times R_3 \times \cdots \times R_n$, $n \geq 5$. Then the unit graph $G(R)$ is a line graph if and only if*
(a) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_2$, n copies of $\mathbb{Z}_2$
(b) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_3$, $n-1$ copies of $\mathbb{Z}_2$
(c) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_4$, $n-1$ copies of $\mathbb{Z}_2$
(d) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_6$, $n-1$ copies of $\mathbb{Z}_2$

Proof. **(a).** Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, with $\mathbb{Z}_2$ being n times. This ring has only one unit, namely $(1,1,1,\ldots,1)$. So corresponding to every vertex $v_\alpha = (a_1, a_2, a_3, \ldots, a_n)$ of $G(R)$, their exists a vertex $v_{\alpha'} = (b_1, b_2, b_3, \ldots, b_n)$ in $G(R)$ such that $v_\alpha \sim v_{\alpha'}$. Thus $G(R)$ is a bipartite graph in which partite sets are of cardinality 2^{n-1}, that is, the unit graph $G(R)$ contains 2^{n-1} disjoint copies of K_2. So $G(R)$ is a line graph.
(b). Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_3$, with $\mathbb{Z}_2$ being $n-1$ times. Then the unit graph contains 2^{n-2} disjoints copies of C_6. Clearly, the unit graph is a line graph.
(c). Now, let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_4$ with $\mathbb{Z}_2$ being $n-1$ times. Then the unit graph contains 2^{n-2} disjoint copies of C_4. So the unit graph is a line graph.

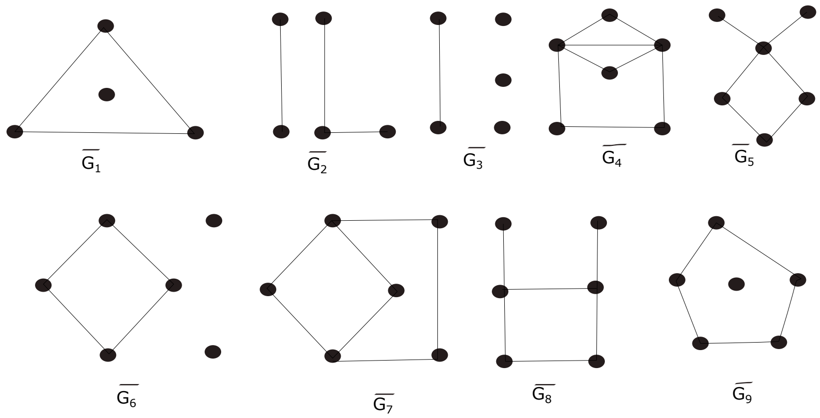

FIGURE 3. Induced sub-graphs of compliment of line graphs

(d). Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_6$ with $\mathbb{Z}_2$ being $n-1$ times. In this case, the unit graph contains 2^{n-2} disjoints copies of C_6. So the unit graph is a line graph.

Finally, let $R \cong R_1 \times R_2 \times R_3 \times \cdots \times R_n$, $n \geq 5$, where R is not any one of the rings considered in above cases. So assume that $|R_i| \neq 2$, for $i = 1, 2, 3, \ldots, n-1$ and $|R_n| \neq 2, 3, 4, 6$. Assume that $|R_2| = n_2$ and $|R_3| = n_3$. Here the ring R always contains at least three units. Let $S = \{(0,0,0,\ldots,0), (1,1,1,\ldots,1), (1, n_2 - 1, 1, \ldots, 1), (1, 1, n_3 - 1, \ldots, 1)\}$ be the subset of the vertex set of $G(R)$. Therefore, except $(0,0,0,\ldots,0)$, all elements of S are units in $R \cong R_1 \times R_2 \times R_3 \times \cdots \times R_n$, $n \geq 5$. Clearly, the subgraph induced by the vertices of S is $K_{1,3}$. Hence the unit graph is not a line graph. □

3. Unit graphs which are the complement of line graphs

We have the following characterization of graphs which are the complements of line graphs.

THEOREM 3.1. *A graph G is the complement of a line graph if and only if none of the nine graphs $\bar{G}_i$ of Figure 3 is an induced subgraph of G.*

THEOREM 3.2. *If $R = R_1 \times R_2 \times \cdots \times R_n$, $n \geq 3$, then the unit graph $G(R)$ is not the complement of a line graph.*

Proof. Assume that $|R_i| = n_i$, for all $1 \leq i \leq n$. Let $G(R)$ be the unit graph of R. Consider the subset S of the vertex set of $G(R)$ given by

$$S = \{(0,0,0,0,0,\ldots,0), (n_1 - 1, n_2 - 1, n_3 - 1, \ldots, n_n - 1),$$
$$(1,0,0,\ldots,0), (0,1,0,\ldots,0), (0,0,1,\ldots,0)\}.$$

Also, $n_i - 1$ is a unit of the ring R_i. The subgraph induced by the vertices of S is given in Figure 4, which is clearly $\bar{G}_3$. Thus $G(R)$ is not the complement of the line graph. □

Now, we find those unit graphs which are the complements of line graphs.

THEOREM 3.3. *Let $R = R_1 \times R_2$ be a finite commutative ring, where R_i is a local ring for $i = 1, 2$. The unit graph $G(R)$ is the complement of a line graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_3$.*

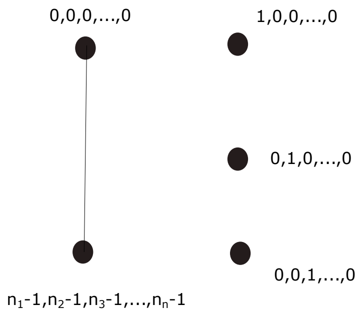

FIGURE 4. $\bar{G_3}$

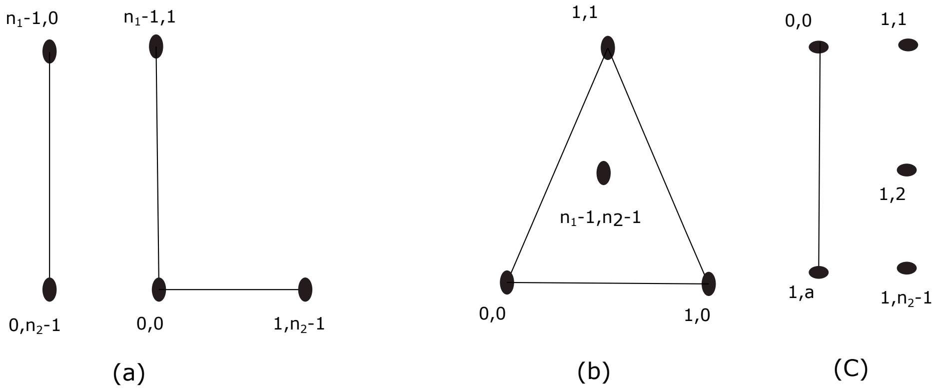

FIGURE 5. (a) $\bar{G_2}$ (b) $\bar{G_1}$ (c) $\bar{G_3}$

Proof. We consider the following cases.

Case 1. $|R_1| = n_1 \geq 3$ and $|R_2| = n_2 \geq 3$. We have the following possibilities.

(a) For $2 \notin U(R_1)$ and $2 \notin U(R_2)$, we have $(2,2) \notin U(R_1 \times R_2)$. Consider the subset of the vertex set of $G(R)$ as

$$S_1 = \{(0,0), (n_1 - 1, 0), (n_1 - 1, 1), (0, n_2 - 1), (1, n_2 - 1)\}.$$

Clearly, the induced sub graph formed by S_1 is $\bar{G_2}$, shown in Figure 5(a). Therefore the unit graph is not the complement of the line graph.

(b) If 2 belongs to both $U(R_1)$ and $U(R_2)$, then $(2,2) \in U(R_1 \times R_2)$. Consider the subset S_2 of the vertex set of $G(R)$ as

$$S_2 = \{(1,1), (1,0), (0,1), (n_1 - 1, n_2 - 1)\}.$$

Obviously the induced sub graph by formed by S_2 is $\bar{G_1}$, as shown in Figure 5(b). Thus the unit graph $G(R)$ is not the complement of a line graph.

(c) Let $2 \in U(R_1)$ or $2 \in U(R_2)$. Without loss of generality, assume that $2 \in U(R_2)$. Consider the subset S_3 of the vertex set of $G(R)$ given by $S_3 = \{(0,0), (n_1-1, 0), (n_1-1, 1), (0, n_2 - 1), (1, n_2 - 1)\}$. This induces the subgraph $\bar{G_2}$ and so $G(R)$ is not the complement of the line graph.

Case 2. $|R_1| \geq 4$ or $|R_2| \geq 4$. Without loss of generality, let $|R_2| = n_2 \geq 4$. We have the following two possibilities.

(d) Let $2 \notin U(R_2)$. Then the unit $(1,2) \notin U(\mathbb{Z}_2 \times R_2)$. Consider the subset S_4 of the vertex set of $G(\mathbb{Z}_2 \times R_2)$ given by

$$S_4 = \{(0,0), (1,0), (1,1), (0, n_2 - 1), (1, n_2 - 1)\}.$$

Here S_4 induces $\bar{G}_2$. So the unit graph $G(R)$ is not the complement of the line graph.
(e) Let $2 \in U(R_2)$. Then the unit $(1,2) \in U(\mathbb{Z}_2 \times R_2)$. If $2 \in U(R_2)$ and $|R_2| \geq 4$, then their exists at least one unit in R_2, say a with $a \neq 1, 2, n_2 - 1$. Consider the subset S_5 of the vertex set of $G(R)$ given by $S_5 = \{(0,0), (1,a), (1,1), (1,2), (1, n_2 - 1)\}$. Clearly, the induced sub graph formed by S_5 is $\bar{G}_3$, as shown in Figure $5(c)$. Therefore, $G(R)$ is not the complement of a line graph.
Case 3. $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_3$. Now, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then the unit graph associated to this ring is a bipartite graph with partite sets $V_1 = \{(0,0), (1,1)\}$ and $V_2 = \{(0,1), (1,0)\}$. This does not contain an induced subgraph isomorphic to any one of the nine graphs. Thus $G(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is the complement of the line graph. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, then the unit graph is a cycle of length six and does not have an induced subgraph isomorphic to any one of the nine graphs. So the unit graph $G(\mathbb{Z}_2 \times \mathbb{Z}_3)$ is the complement of the line graph.

□

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S. Pirzada
Department of Mathematics, University of Kashmir, Srinagar, 190006,
Srinagar, Kashmir, India
E-mail: pirzadasd@kashmiruniversity.ac.in

Aaqib Altaf
Department of Mathematics, University of Kashmir, Srinagar, 190006,
Kashmir, India
E-mail: aaqibwani777@gmail.com