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ON COUNTABLY g-COMPACTNESS AND SEQUENTIALLY GO-COMPACTNESS

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Abstract. In this paper, we investigate some properties of countably g-compact and sequentially GO-compact spaces. Also, we discuss the relation between countably g-compact and sequentially GO-compact. Next, we introduce the definition of g-subspace and study the characterization of g-subspace.

1. Preliminaries

Let (X, τ) be a topological space. A subset A of X is called *g-closed* [4] if $cl(A) \subset G$ holds whenever $A \subset G$ and G is open in X.

A is called *g-open* of X if its complement A^c is g-closed in X. Every open set is g-open [8]. A topological space X is said to be $T_{1/2}$ [2] if every g-closed set in X is closed in X. A is called *sequentially closed* [5] if for every sequence (x_n) in A with $(x_n) \to x$, then $x \in A$.

A sequence (x_n) in a space X *g-converges* to a point $x \in X$ [4] if (x_n) is eventually in every g-open set containing x and is denoted by $(x_n) \xrightarrow{g} x$ and x is called the g-limit of the sequence (x_n), denoted by *glim* x_n.

A is called *sequentially g-closed* [4] if every sequence in A g-converges to a point in A. $S[A]$ denote the set of all sequences in A and $c_g(X)$ denote the set of all g-convergent sequences in X. A *sequentially g-open* subset U (which is the complement of a sequentially g-closed set) is one in which every sequence in X which g-converges to a point in U is eventually in U. A space X is said to be *GO-compact* [7] if every g-open cover of X has a finite subcover. A space X is said to be *g-Lindelöf* [7] if every g-open cover of X has a countable subcover. A subset A of X is said to be *sequentially GO-compact* [4] if every sequence in A has a subsequence which g-converges to a point in A. A space X is *countably g-compact* [7] if every countable cover of X by g-open sets of X has a finite subcover.

A map $f : X \to Y$ from a topological space (X, τ) into a topological space (Y, σ) is called *g-continuous* [2] if the inverse image of every closed set in Y is g-closed in X. A map $f : X \to Y$ is said to be *strongly g-continuous* [2] if the inverse image of every g-closed set in Y is closed in X.

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Let (X, τ) and (Y, σ) be any two topological spaces. Then a map $f : (X, \tau) \to (Y, \sigma)$ is said to be *sequentially g-continuous at $x \in X$* [4] if the sequence $(f(x_n)) \xrightarrow{g} f(x)$ whenever the sequence $(x_n) \xrightarrow{g} x$. If f is sequentially g-continuous at each $x \in X$, then it is said to be a sequentially g-continuous function.

LEMMA 1.1. *[1] Suppose X is a topological space and $A \subset X$. The sequential closure of A is defined as the set $\{\lim x \mid x \in s(A) \cap c(X)\}$ where $s(A)$ denotes the set of all sequences in A, $c(X)$ denote the set of all g-convergent sequences in X and it is denoted by $[A]_{seq}$. Then $A \subset [A]_{seq}$.*

2. Sequentially GO-compact

DEFINITION 2.1. A subset A of a topological space (X, τ) is called a *g-neighborhood* of a point $x \in X$ if there exists a g-open set U with $x \in U \subset A$.

DEFINITION 2.2. Let (X, τ) be a topological space, $A \subset X$ and let $S[A]$ be the set of all sequences in A. Then the sequential g-closure of A, denoted by $[A]_{g_{seq}}$, is defined as

$$[A]_{g_{seq}} = \{x \in X \mid x = glim\ x_n \text{ and } (x_n) \in S[A] \cap c_g(X)\}$$

$c_g(X)$ denote the set of all g-convergent sequences in X.

LEMMA 2.3. [16, Lemma 3.3] *Let (X, τ) be a topological space. Then the following hold.*

(a) *Every g-convergence sequence is convergence sequence.*
(b) *If (X, τ) is a $T_{1/2}$ space, then the concept of convergence and g-convergence coincide.*

The following Example 2.4 shows that Every g-convergence sequence is convergence sequence. But converse of Lemma 2.3 (a) need not be true.

EXAMPLE 2.4. Consider the topological space (X, τ) where $X = [0, 2)$, $\tau = \{\emptyset, (0, 1), X\}$. Suppose that $(x_n) = (\frac{1}{n})$ for $n \in \mathbb{N}$. Then (x_n) converges to 0. If $A = (0, 1]$, then A is g-closed and so $X \setminus A$ is g-open. That is, $\{0\} \cup (1, 2)$ is a g-open subset of X. But $\frac{1}{n} \notin \{0\} \cup (1, 2)$ for any n. Hence (x_n) does not g-convergent to 0.

THEOREM 2.5. *Let (X, τ) be a topological space and $A \subset X$. Then the following hold.*

(a) Every sequentially closed set is a sequentially g-closed set.
(b) A is sequentially g-closed if and only if $[A]_{g_{seq}} \subset A$.
(c) Every sequentially g-closed set is g-closed hence every sequentially closed set is g-closed.

Proof. (a) Let $A \subset X$. Suppose A is sequentially closed. Let (x_n) be a sequence in A such that $(x_n) \xrightarrow{g} x$. By Theorem 2.3 (a), $(x_n) \to x$ in A and so $x \in A$. Thus, A is sequentially g-closed.
(b) Suppose $x \in [A]_{g_{seq}}$. Then there exists $(x_n) \in S[A] \cap c_g(X)$ such that $x = glim\ x_n$. Since A is a sequentially g-closed subset of X, $x \in A$. Hence $[A]_{g_{seq}} \subset A$. Conversely, let (x_n) be a sequence in A such that $(x_n) \xrightarrow{g} x$. Then $x \in [A]_{g_{seq}}$. By assumption, $[A]_{g_{seq}} \subset A$ and so $x \in A$. Hence A is sequentially g-closed.

(c) Suppose that A is sequentially g-closed. Then $[A]_{g_{seq}} \subset A$, by (b). Let $(x_n) \in S[A] \cap c_g(X)$. Then $glim\ x_n \in [A]_{g_{seq}}$. Since $[A]_{g_{seq}} \subset A$, A is closed. Thus, A is g-closed. By (a), every sequentially closed set is sequentially g-closed. Therefore, every sequentially closed set is g-closed.

THEOREM 2.6. *Let (X, τ) be a topological space and A be a subset of X. If A is open then A is sequentially g-open.*

Proof. Let A be open and (x_n) be a sequence in $X \setminus A$. Let $y \in A$. Then there is a g-neighborhood U of y which contained in A. Hence U does not contain any term of (x_n). So y is not a limit of the sequence (x_n). Since every g-convergent sequence is convergent (By Theorem 2.3(a)), y is not a g-limit of the sequence. Therefore, A is sequentially g-open.

THEOREM 2.7. *Every sequentially GO-compact space is a sequentially compact space.*

Proof. Suppose that (X, τ) is a sequentially GO-compact space and (x_n) is a sequence in X. Then by the definition of sequentially GO-compactness, there exists a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) g-converges to x. By Lemma 2.3 (a), $(x_{n_k}) \to x$. Therefore, X is sequentially compact.

In general, the converse of the Theorem 2.7 need not be true by Example 2.4.

DEFINITION 2.8. A topological space (X, τ) is said to be *g-sequential* if any subset A of X with $[A]_{g_{seq}} \subset A$ is closed in X, that is, every sequentially g-closed set in X is a closed set.

Next, we have to show that Theorem 2.9 every sequentially GO-compact space is countably g-compact space but converse true in g-sequential.

THEOREM 2.9. *Every sequentially GO-compact space is countably g-compact space.*

Proof. Suppose that (X, τ) is not countably g-compact. Let $\mathcal{C}$ be a countable g-open cover that does not have a finite subcover. We choose $x_j \in X$, for each $j > 1$. Let $U_j \in \mathcal{C}$ that contains a point x_j but not in $\bigcup_{i=1}^{j-1} U_i$. We enough to show that the sequence (x_n) does not have a subsequence that g-converges.

Let $x \in X$. Then there exists k such that for every g-neighborhood U_k of x, $x_j \in U_k$ for every $j > k$. Thus, no subsequence of (x_n) g-converges to x. Since x is any arbitrary point, no subsequence of (x_n) g-converges to x. Therefore, (X, τ) is not a sequentially GO-compact space.

THEOREM 2.10. *Let (X, τ) be a g-sequential space. Every countably g-compact space is sequentially GO-compact space.*

Proof. Suppose that (X, τ) is a countably g-compact space. It suffices to show that any sequence (x_n) of points of a countably g-compact g-sequential space X has a g-convergent subsequence.

Suppose that $x_i \neq x_j$ if $i \neq j$. Let x be a g-limit point of the infinite set A. Since $x \in cl(A \setminus \{x\})$, the set $A \setminus \{x\}$ is not closed. So that, X being a g-sequential space, the set $A \setminus \{x\}$ contains a sequence g-converging to a point in the complement of $A \setminus \{x\}$. Rearranging the sequence (y_n), we get a g-convergent subsequence of (x_n).

Theorem 2.11. *If the topological space (X, τ) is countably g-compact, then every sequence (x_n) has a g-limit point.*

Proof. Let (x_n) be a sequence in X and let $A = \{x_n \mid n \in \mathbb{N}\}$. Suppose that A is an infinite set. Then A has a set of g-limit point of x. Let U be a g-neighborhood of x. Then there is a sequence (y_n) in $A \setminus \{x\}$ such that $g \lim y_n = x$. This implies that $x_n \in A \setminus \{x\}$, $x_n \in U$. Therefore, x is a g-limit point of A. If A is finite, then there exists $x \in X$ such that $x_n = x$ for infinitely many $n \in \mathbb{N}$. Then for every g-open set U containing x. Hence x is a g-limit point of A.

Theorem 2.12. *The Cartesian product $X \times Y$ of a countably g-compact space X and a sequentially GO-compact space Y is countably g-compact.*

Proof. Consider a countably infinite set $A = \{m_1, m_2, ...\} \subset X \times Y$, where $m_i = (x_i, y_i)$ for $i = 1, 2, ...$ and $m_i \neq m_j$ whenever $i \neq j$. Let $y_{k_1}, y_{k_2}, ...$ be a subsequence of $y_1, y_2, ...$ that g-converges to a point $y \in Y$. If the set $\{x_{k_1}, x_{k_2}, ...\}$ is finite, then there exists a point $x \in X$ and a subsequence $k_{l_1}, k_{l_2}, ...$ of the sequence $k_1, k_2, ...$ such that $x_{k_{l_i}} = x$ for $i = 1, 2, ...$ If the set $\{x_{k_1}, x_{k_2}, ...\}$ is infinite, then it has g-limit point $x \in X$. Therefore, $(x, y) \in X \times Y$ is a g-limit point of the set A.

Theorem 2.13. *If X is a countably g-compact space and Y is a g-sequential space, then the projection $P : X \times Y \to Y$ is closed.*

Proof. Let A be a closed subset of $X \times Y$. Consider a sequence (y_n) of points of $P(A)$ and U be g-open neighborhood of y in Y, $y_i \in U$ and a point $y \in g \lim y_i$. We Choose a point $x_i \in X$ such that $(x_i, y_i) \in A$ for $i = 1, 2, ...$ Suppose the set $A = \{x_1, x_2, ...\}$ is finite, then there exists $x \in X$ such that $x_{k_i} = x$ for infinite sequence $k_1 < k_2 < ...$ of integers. So that $(x, y) \in g \lim(x_{k_i}, y_{k_i})$ implies that $(x, y) \in [A]_{g_{seq}} = A$, since A is closed, that is, $y \in P(A)$. Suppose the set A is infinite, then it has g-limit point $x \in A$ so that $(x, y) \in [A]_{g_{seq}} = A$ implies that $y \in P(A)$. Since Y is a g-sequential space, the set $P(A)$ is closed in Y.

Proposition 2.14. *Every g-closed subset of countably g-compact space is countably g-compact relative to X.*

Proof. Let A be a g-closed subset of a countably g-compact space X. Then A^c is g-open in X. Let B be a countable cover of A by g-open sets in X. Then $\{B, A^c\}$ is a g-open cover of X. Since X is countably g-compact it has a finite subcover say $\{C_1, C_2, ..., C_n\}$. If this subcover contains A^c, we remove it. Otherwise leave the subcover as it is. Thus, we have obtained finite g-open subcover of A and so A is countably g-compact relative to X.

Theorem 2.15. *Let X be countably g-compact and Y be any space. If $f : X \to Y$ is g-continuous, then $f(X)$ is countably g-compact.*

Proof. Let A be an infinite subset of $f(X)$. Then $A = \{f(x) \mid x \in B\}$ where $B \subseteq X$ is infinite. Since X is countably g-compact. B has a g-limit point k. Let V_k be a g-neighborhood of $f(k)$. Since f is g-continuous, there exists some g-neighborhood U_k of k such that $f(U_k) \subseteq V_k$.

Since k is a g-limit point of B, there exists some $y_n \in B$ such that $y_n \neq k, y_n \in U_k$. Thus, $f(y_n) \in f(U_k) \subseteq V_k$. Since $f(y_n) \in A \setminus f(k)$, $f(y_n) \xrightarrow{g} f(k)$. Since every g-neighborhood V_k of $f(k)$, $f(y_n) \in V_k$, that is $f(k)$ is a g-limit point of A. By Theorem 2.11, $f(X)$ has a g-limit point. Therefore, $f(X)$ is countably g-compact.

THEOREM 2.16. *If X is g-Lindelöf, then countably g-compactness implies GO-compactness.*

Proof. Suppose X is not GO-compact. Suppose that X has an g-open cover which has no finite subcover. We assume that the g-open cover to be countable, since X is g-Lindelöf. So, $X = \bigcup_{k\in\mathbb{N}} U_k$ where each U_k is g-open. Assume that if $U_m \subset \bigcup_{k=1}^{m-1} U_k$, then U_m is not a part of the cover. Now, for each m, let $x_m \in U_m - (\bigcup_{k=1}^{m-1} U_k)$. So (x_m) is an infinite set which has a g-limit point x. Because $\{U_k\}_{k\in\mathbb{N}}$ covers X, $x \in U_n$ for some n and $x_i \in U_n$ for $i > n$. But this is impossible, since the x_i's were chosen to be disjoint from $\bigcup_{k=1}^{m-1} U_k$.

PROPOSITION 2.17. *A g-sequential space has unique g-limit if and only if each countably g-compact subset is closed.*

Proof. Suppose that $A = \{x\} \bigcup \{x_n \mid n \in \mathbb{N}\}$ is an infinite subset of X which is g-converging to two distinct points x and y, then A has a countably g-compact subset of X which is not closed.

Conversely, let A be a countably g-compact subset of X. Suppose that (x_n) is a sequence in A and $(x_n) \xrightarrow{g} x$. Then $\{x\} \bigcup \{x_n \mid n \in \mathbb{N}\}$ is sequentially g-closed and closed. Thus, x is the only possible g-limit of $\{x_n \mid n \in \mathbb{N}\}$. If $\{x_n \mid n \in \mathbb{N}\}$ is infinite, then $x \in A$. If $\{x_n \mid n \in \mathbb{N}\}$ is finite, then $x_n = x$ for all n and $x_n \in A$. Hence A is closed.

COROLLARY 2.18. *A g-sequential space has unique g-limit if and only if each sequentially GO-compact subset is closed.*

Proof. Suppose that A is a sequentially GO-compact subset of a g-sequential space X with unique g-limit. Then A is countably g-compact. By Proposition 2.17, A is closed. The converse part of the proof follows from Theorem 2.9.

3. g-subspace

Let (X, τ) be a topological space, Y be a subspace of X and $A \subset Y$.

$$[A]_{g|_{Y_{seq}}} = \{x \in Y \mid x = glim x_n \text{ and } x_n \in S[A] \cap c_{g|_Y}(Y)\} = [Y]_{g_{seq}} \cap Y$$

where $c_{g|_Y}(Y) = \{x_n \in S[Y] \cap c_g(X) \mid x \in Y\}$

PROPOSITION 3.1. *Let (X, τ) be a topological space and $A \subset Y \subset X$. Then $[A]_{g|_{Y_{seq}}} = [A]_{g_{seq}} \cap Y$.*

Proof. If $x \in [A]_{g|_{Y_{seq}}}$, then there exists a sequence $(x_n) \in c_{g|_Y}(Y) \cap S[A]$ with $(x_n) \xrightarrow{g|_Y} x$. Thus, $x \in [A]$. Next, suppose that $[A]_{g_{seq}} \cap Y$, then there exists a $(x_n) \in S[A] \cap c_g(X)$ with $(x_n) \xrightarrow{g} x \in Y$. Therefore, $(x_n) \in c_{g|_Y}(Y)$ and $x \in [A]_{g|_{Y_{seq}}}$. Thus, $[A]_{g|_{Y_{seq}}} = [A]_{g_{seq}} \cap Y$.

COROLLARY 3.2. *Let (X, τ) be a topological space and Y be a subspace of X. If A is sequentially g-closed in X, then the set $A \cap Y$ is sequentially $g|_Y$-closed in Y.*

Proof. Since A is sequentially g-closed in X, $[A]_{g_{seq}} \subset A$, by Theorem 2.5 (b). By Proposition 3.1, $[A \cap Y]_{g|_{Y_{seq}}} = [A \cap Y]_{g_{seq}} \cap Y \subset [A]_{g_{seq}} \cap Y \subset A \cap Y$. Thus, $A \cap Y$ is sequentially $g|_Y$-closed in Y.

COROLLARY 3.3. *Let (X, τ) be a topological space and $A \subset Y \subset X$. If A is sequentially $g|_Y$-closed in Y and Y is sequentially g-closed in X, then A is sequentially g-closed in X*

Proof. Since Y is sequentially g-closed in X, $[A]_{g_{seq}} \subset [Y]_{g_{seq}} \subset Y$. Since A is sequentially $g|_Y$-closed in Y, $[A]_{g|_{Y_{seq}}} \subset A$. By Proposition 3.1, $[A]_{g_{seq}} = [A]_{g_{seq}} \cap Y = [A]_{g|_{Y_{seq}}} \subset A$. Therefore, A is sequentially g-closed in X.

THEOREM 3.4. *Every g-closed subset of a g-sequential space is g-sequential.*

Proof. Suppose that X is a g-sequential space and Y is a g-closed set of X. We have show that the subspace Y is a $g|_Y$-sequential space.

Let A be a subset of Y with $[A]_{g_{seq}} \subset A$, that is, A is sequentially $g|_Y$-closed in Y. Since $[A]_{g_{seq}} \subset [Y]_{g_{seq}}$, $[A]_{g_{seq}} = [A]_{g_{seq}} \cap Y = [A]_{g|_{Y_{seq}}} \subset A$ and so A is closed in X. Therefore, A is closed in Y. Hence Y is a g-sequential space.

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