Korean J. Math. **29** (2021), No. 4, pp. 775–784
http://dx.doi.org/10.11568/kjm.2021.29.4.775

L_P-TYPE INEQUALITIES FOR DERIVATIVE OF A POLYNOMIAL

Irfan Ahmad Wani*, Mohammad Ibrahim Mir, and Ishfaq Nazir

Abstract. For the polynomial $P(z)$ of degree n and having all its zeros in $|z| \leq k$, $k \geq 1$, Jain [6] proved that

$$\max_{|z|=1} |P'(z)| \geq n \frac{|c_0| + |c_n|k^{n+1}}{|c_0|(1 + k^{n+1}) + |c_n|(k^{n+1} + k^{2n})} \max_{|z|=1} |P(z)|.$$

In this paper, we extend above inequality to its integral analogous and there by obtain more results which extended the already proved results to integral analogous.

1. Introduction and statement of results

Let $\mathcal{P}_n$ denote the space of all algebraic polynomials of the form $P(z) = \sum_{j=0}^{n} c_j z^j$ of degree n and let $P'(z)$ be its derivative. The study of inequalities for different norms of derivatives of a univariate complex polynomial in terms of the polynomial norm is a classical topic in analysis. A classical inequality that provides an estimate to the size of the derivative of a given polynomial on the unit disk, relative to size of the polynomial itself on the same disk is the famous Bernstein inequality [2]. It states that: if $P(z)$ is a polynomial of degree n, then on $|z| = 1$,

$$\max_{|z|=1} \left|P'(z)\right| \leq n \max_{|z|=1} \left|P(z)\right| \tag{1}$$

Equality holds in (1) if and only if $P(z)$ has all its zeros at the origin. Concerning the maximum of $|P'(z)|$ in terms of maximum of $|P(z)|$ on $|z| = 1$, Turán [11] showed that, if $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{2}$$

Equality in inequality (2) holds for those polynomials $P \in \mathcal{P}_n$ which have all their zeros on $|z| = 1$. As a generalization of the inequality (2) Malik [7] proved that, if $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{3}$$

Received July 4, 2021. Revised December 6, 2021. Accepted December 7, 2021.
2010 Mathematics Subject Classification: 26D10, 30C15, 41A17.
Key words and phrases: polynomial, derivative and Integral mean inequalities.
∗ Corresponding author.

The result is sharp and best possible for $P(z) = (z + k)^n$. For $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k$, $k \geq 1$, Govil [3] proved that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1 + k^n} \max_{|z|=1} |P(z)|. \tag{4}$$

The result is sharp and best possible for $P(z) = z^n + k^n$. In 1991, Govil [4] further improves the bound in (4) and proved under the same hypothesis that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1 + k^n} \Big\{ \max_{|z|=1} |P(z)| + \min_{|z|=k} |P(z)| \Big\}. \tag{5}$$

The result is sharp and best possible for $P(z) = z^n + k^n$. Furthermore results in this direction, one can consult the books of Milovanovic *et al.* [9] and Rehman and Schmeisser [10] . Recently, Jain [6] obtained the following refinement of (4) by using the classical generalization of Schwarz Lemma.

THEOREM 1.1. *If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k$, $k \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq n \frac{|c_0| + |c_n| k^{n+1}}{|c_0|(1 + k^{n+1}) + |c_n|(k^{n+1} + k^{2n})} \max_{|z|=1} |P(z)|. \tag{6}$$

The result is sharp and best possible for $P(z) = z^n + k^n$.

We know that from the analysis that if $P \in \mathcal{P}_n$ then for each $p > 0$

$$\lim_{p \to \infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^p d\theta \right\}^{\frac{1}{p}} = \max_{|z|=1} |P(z)|.$$

Malik [8] obtained a generalization of (2) in the sense that the left - hand side of (2) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$. In fact, he proved that, if $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq 1$, then for each $p > 0$

$$n \left\{ \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^p d\theta \right\}^{\frac{1}{p}} \leq \left\{ \int_0^{2\pi} \left| 1 + e^{i\theta} \right|^p d\theta \right\}^{\frac{1}{p}} \max_{|z|=1} \left| P'(z) \right|. \tag{7}$$

If we let $p \to \infty$ in (7), we get inequality (1.2). Aziz [1] obtained the generalization of inequality (4) which is similar to (7). In fact he proved that if $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k$, $k \geq 1$, then for each $p > 0$

$$n \Bigg\{ \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^p d\theta \Bigg\}^{\frac{1}{p}} \leq \Bigg\{ \int_0^{2\pi} \left| 1 + k^n e^{i\theta} \right|^p d\theta \Bigg\}^{\frac{1}{p}} \max_{|z|=1} \left| P'(z) \right|. \tag{8}$$

If we let $p \to \infty$ in (8), we get inequality (4).

2. Main Results

In this paper, we shall prove some L_p- type inequalities for polynomials. We shall first prove a result that generalizes Theorem 1.1 as well as extends to its integral analogous and there by obtain more results which extended the already proved results to integral analogous. First, we prove the following result, which is L_p extension of Theorem 1.1.

THEOREM 2.1. *if $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \le k$, $k \ge 1$, then for $p > 0$*

$$\left\{\int_0^{2\pi} \left|P'(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \ge n\frac{|c_0| + |c_n|k^{n+1}}{|c_0|(1+k^{n+1}) + |c_n|(k^{n+1}+k^{2n})} \times \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \tag{9}$$

REMARK 2.2. If we let $p \to \infty$ in (9), Theorem 2.1 reduces to Theorem 1.1.

Next, we prove the result which generalizes the Theorem 2.1. In fact we prove

THEOREM 2.3. *If $P(z) = \sum_{j=m}^{n} c_j z^j$, $(0 \le m < n)$ of degree n, having all its zeros in $|z| \le k$, $k \ge 1$, then*

$$\left\{\int_0^{2\pi} \left|P'(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \ge n\frac{(n-m)|c_m|k^m + n|c_n|k^{n+1}}{(n-m)|c_m|(k^m+k^{n+1}) + n|c_n|(k^{n+1}+k^{2n-m})} \times \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \tag{10}$$

REMARK 2.4. For $m = 0$ in (10), we get Theorem 2.1.

Further, we prove the following theorem as an application of Theorem 2.1.

THEOREM 2.5. *If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \le k$, $k \ge 1$, then for $0 \le t \le 1$ and $p > 0$*

$$\left\{\int_0^{2\pi} \left(\left|P'(e^{i\theta})\right| - tmn\right)^p d\theta\right\}^{\frac{1}{p}} \ge n\frac{|c_0| + (|c_n| - tm)k^{n+1}}{|c_0|(1+k^{n+1}) + (|c_n| - tm)(k^{n+1}+k^{2n})} \times \left\{\int_0^{2\pi} \left|P(e^{i\theta}) - tm\right|^p d\theta\right\}^{\frac{1}{p}}. \tag{11}$$

Where $m = min_{|z|=k} P(z)$.

REMARK 2.6. For $t = 0$, Theorem 2.5 reduces to Theorem 2.1.

REMARK 2.7. If we let $p \to \infty$ in (11) we get the refinement of Theorem 1.1.

3. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 3.1. *Let $T(z)$ be polynomial of degree n, having all its zeros in $|z| \leq 1$ and let $R(z)$ be a polynomial with its degree $\leq n$, if*

$$|R(z)| \leq |T(z)|\,, |z| = 1$$

then for $0 \leq s \leq n$

$$|R^s(z)| \leq |T^s(z)|\,, |z| \geq 1 \tag{12}$$

where $R^s(z)$ and $T^s(z)$ be the s^{th} derivative of $R(z)$ and $T(z)$.

The above lemma is due to Jain [6].

Lemma 3.2. *If $P(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$ then for $|z| = 1$*

$$|Q'(z)| \leq |P'(z)| \tag{13}$$

Where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Proof of Lemma 3.2: It can be easily verified with the help of Lemma 1.

Lemma 3.3. *Let $f(z)$ be analytic in $|z| < 1$, with $f(0) = a$ and $|f(z)| \leq M, \quad |z| < 1$. Then*

$$|f(z)| \leq M \frac{M|z| + |a|}{|a||z| + M}, \quad |z| < 1. \tag{14}$$

Lemma 3.3 is a well - known generalization of Schwarz's lemma (See [[12], p. 212])

Lemma 3.4. *Let $f(z)$ be analytic in $|z| \leq 1$, with $f(0) = a$ and $|f(z)| \leq M, \quad |z| \leq 1$. Then*

$$|f(z)| \leq M \frac{M|z| + |a|}{|a||z| + M}, \quad |z| \leq 1. \tag{15}$$

Proof of Lemma 3.4: It easily follows from lemma 3.3.

Lemma 3.5. *If $P \in \mathcal{P}_n$, then for $p > 0, \ R > 1$,*

$$\left\{ \int_0^{2\pi} \left| P(Re^{i\theta}) \right|^p d\theta \right\}^{\frac{1}{p}} \leq R^n \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta \right\}^{\frac{1}{p}}. \tag{16}$$

Lemma 3.5 is a simple deduction from a well known result of G. H. Hardy [5].

4. Proof of Theorems

Proof of Theorem 2.1: Let $Q(z) = z^n \overline{P(1/\overline{z})}$, then $P(z) = z^n \overline{Q(1/\overline{z})}$ and it can be easily verified that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)|$$

which implies by triangles inequality

$$|P'(z)| + |Q'(z)| \geq n|P(z)| \tag{17}$$

Let $F(z) = P(kz)$ be a polynomial of degree n, having all its zeros in $|z| \leq 1$ and

$$\begin{aligned} G(z) &= \overline{z^n F\left(1/\overline{z}\right)} \\ &= k^n \overline{\left(\frac{\overline{z}}{k}\right)^n F\left(1/\overline{z}\right)} \\ &= k^n Q\left(\frac{z}{k}\right) \end{aligned}$$

is a polynomial of degree $\leq n$, with $|G(z)| = |F(z)|$, $|z| = 1$. Therefore by Lemma 3.2 we can say that

$$|G'(z)| \leq |F'(z)|, \quad |z| = 1. \tag{18}$$

Using (18) we can say that a zero z_j { with $|z_j| \leq 1$ and multiplicity m_j }, of $F'(z)$ will also be a zero, with multiplicity $(\geq m_j)$, of $G'(z)$, thereby helping us to write

$$F'(z) = \Phi(z) F_1(z) \tag{19}$$

$$G'(z) = \Phi(z) G_1(z) \tag{20}$$

where

(21)

$$\Phi(z) = \begin{cases} 1, \quad F'(z) \neq 0 \quad on \quad |z| = 1 \\ \prod_{j=1}^{p} (z - z_j)^{m_j}, \quad |z_j| = 1 \quad \forall j, \quad F'(z) \ has \ certain \ zero, on \ |z| = 1 \end{cases}$$

$$F_1(z) \neq 0, \quad |z| = 1 \tag{22}$$

and by (18), (19) and (20)

$$|G_1(z)| \leq |F_1(z)|, \quad |z| = 1. \tag{23}$$

Since $F(z)$ has all its zeros in $|z| \leq 1$, we can say by Gauss- Lucas theorem that $F'(z)$ has all its zeros in $|z| \leq 1$. Therefore by (18),(21) and (22) we can say that

$$\Psi(z) = \frac{G_1(z)}{F_1(z)}$$

is analytic in $|z| > r'$, (for certain r', with $(0 < r' < 1)$), including ∞ and accordingly.

$$f(z) = \Psi\left(\frac{1}{z}\right), \tag{24}$$

with

$$\begin{aligned} f(0) = \Psi(\infty) &= \lim_{z \to \infty} \Psi(z) \\ &= \lim_{z \to \infty} \frac{G'(z)}{F'(z)} \\ &= \frac{\overline{c_0}}{c_n k^n}. \end{aligned} \tag{25}$$

Further $|\Psi|(z) \leq 1, \quad |z| = 1$ by (23) and therefore

$$|f(z)| \leq 1, \quad |z| = 1 \tag{26}$$

which by (25) and Lemma 3.4 helps us to write

$$|f(z)| \leq \frac{|z| + |\frac{c_0}{c_n k^n}|}{|\frac{c_0}{c_n k^n}||z| + 1}, \quad |z| \leq 1.$$

that is, for $z = re^{i\theta},\ r \leq 1, \quad 0 \leq \theta \leq 2\pi$

$$\left|f\left(re^{i\theta}\right)\right| \leq \frac{|c_n|k^n r + |c_0|}{|c_0|r + |c_n|k^n}, \tag{27}$$

by (24) and for $0 < r \leq 1$ *and* $0 \leq \theta \leq 2\pi$

$$\left|\Psi\left(\frac{1}{r}e^{-i\theta}\right)\right| \leq \frac{|c_n|k^n r + |c_0|}{|c_0|r + |c_n|k^n}$$

which implies, for $R \geq 1$ *and* $0 \leq \theta \leq 2\pi$

$$\left|\Psi\left(Re^{-i\theta}\right)\right| \leq \frac{|c_n|k^n + |c_0|R}{|c_0| + |c_n|k^n R}$$

$$\Rightarrow \left|G_1\left(Re^{-i\theta}\right)\right| \leq \frac{|c_n|k^n + |c_0|R}{|c_0| + |c_n|k^n R}\left|F_1\left(Re^{-i\theta}\right)\right|, \quad R \geq 1 \text{ and } 0 \leq \theta \leq 2\pi$$

$$\Rightarrow \left|G'\left(Re^{-i\theta}\right)\right| \leq \frac{|c_n|k^n + |c_0|R}{|c_0| + |c_n|k^n R}\left|F'\left(Re^{-i\theta}\right)\right|, \quad R \geq 1 \text{ and } 0 \leq \theta \leq 2\pi$$

$$\Rightarrow \left|G'\left(z\right)\right| \leq \frac{|c_n|k^n + |c_0||z|}{|c_0| + |c_n|k^n|z|}\left|F'\left(z\right)\right|, \quad |z| \geq 1$$

that is

$$k^{n-2}\left|Q'\left(\frac{z}{k}\right)\right| \leq \frac{|c_n|k^n + |c_0||z|}{|c_0| + |c_n|k^n|z|}\left|P'\left(kz\right)\right|, \quad |z| \geq 1.$$

By taking $z = ke^{i\theta}$, we get

$$\Rightarrow k^{n-3}\left|Q'\left(e^{i\theta}\right)\right| \leq \frac{|c_n|k^{n-1} + |c_0|}{|c_0| + |c_n|k^{n+1}}\left|P'\left(k^2e^{i\theta}\right)\right|, \quad 0 \leq \theta \leq 2\pi$$

which implies

$$k^{n-3}\left\{\int_0^{2\pi}\left|Q'\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \leq \frac{|c_n|k^{n-1} + |c_0|}{|c_0| + |c_n|k^{n+1}}\left\{\int_0^{2\pi}\left|P'\left(k^2e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}, \quad 0 \leq \theta \leq 2\pi$$

and therefore by lemma 3.5,

$$\left\{\int_0^{2\pi}\left|Q'\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \leq \frac{|c_n|k^{n-1} + |c_0|}{|c_0| + |c_n|k^{n+1}}k^{n+1}\left\{\int_0^{2\pi}\left|P'\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}. \quad 0 \leq \theta \leq 2\pi \tag{28}$$

Also by (28) and for $z = e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$,

$$|P'(e^{i\theta})| + |Q'(e^{i\theta})| \geq n|P(e^{i\theta})|$$

which implies

$$\left\{\int_0^{2\pi}\left||P'(e^{i\theta})| + \left|Q'\left(e^{i\theta}\right)\right|\right|^p d\theta\right\}^{\frac{1}{p}} \geq n\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}, \quad 0 \leq \theta \leq 2\pi$$

by minkowski inequality

$$\left\{\int_0^{2\pi} \left|P'(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} + \left\{\int_0^{2\pi} \left|Q'\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \geq n\left\{\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}, \quad 0 \leq \theta \leq 2\pi$$

using (28), we get

$$\left\{\int_0^{2\pi} \left|P'(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \geq \frac{n(|c_0| + |c_n|k^{n+1})}{|c_0|(1 + k^{n+1}) + |c_n|(k^{n+1} + k^{2n})} \times \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}}. \tag{29}$$

This proves Theorem 2.1 completely. □

Proof of Theorem 2.3: Since the $P(z) = \sum_{j=m}^{n} c_j z^j,\ (0 \leq m < n)$ of degree n, having all its zeros in $|z| \leq k,\ k \geq 1$, then the function

$$\lambda(z) = \frac{G'(z)}{F'(z)} = \frac{(n-m)\overline{c_m}k^m z^{n-m-1} + \cdots + \overline{c_{n-1}}k^{n-1}}{nc_n k^n z^{n-1} + \cdots + c_m k^m m z^{m-1}} \tag{30}$$

where $F(z)$ and $G(z)$ are defined in Theorem 2.1, are analytic in $1 < |z| < \infty$ as well as in $r' < |z| < 1$, for certain r', with $(0 < r' < 1)$

Let

$$\begin{aligned} f(z) &= \lambda(\frac{1}{z}),\ (0 < |z| < 1,\ 0 < |z| < \frac{1}{r'}) \\ &= z^m \frac{(n-m)\overline{c_m}k^m + \cdots + \overline{c_{n-1}}k^{n-1}z^{n-m-1}}{nc_n k^n + \cdots + c_m k^m m z^{n-m}} \\ &= z^m S(z),\ (say), \end{aligned} \tag{31}$$

with

$$S(0) = \frac{(n-m)\overline{c_m}k^m}{nc_n k^n} = d. \tag{32}$$

And

$$S(z) = \frac{f(z)}{z^m},\ (0 < |z| < 1,\ 0 < |z| < \frac{1}{r'}) \tag{33}$$

$$S(z) = \lim_{\zeta \to z} S(\zeta),\ |z| = 1, \tag{34}$$

Since $S(z)$ is analytic in $|z| < 1$, by (30), (31) and (32) and $S(z)$ is continuous in$|z| \leq 1$ by (33) and (34), which implies $S(z)$ is analytic in $|z| < 1$. Hence

$$|S(z)| \leq 1,\ |z| \leq 1$$

Applying Lemma 3.3 to $S(z)$, we get

$$|S(z)| \leq \frac{|z| + |d|}{1 + |z||d|},\ |z| < 1,$$

which by (33), implies that

$$|f(z)| \leq |z|^m \frac{|z|+|d|}{1+|z||d|},\ 0<|z|<1,$$

and therefore,

$$|f(z)| \leq |z|^m \frac{|z|+|d|}{1+|z||d|},\ |z|<1,$$

as well as by (26)

$$|f(z)| \leq |z|^m \frac{|z|+|d|}{1+|z||d|},\ |z|\leq 1,$$

i.e.

$$|f(re^{i\theta})| \leq \frac{n|c_n|k^n r^{m+1} + (n-m)|c_m|k^m r^m}{n|c_n|k^n + (n-m)|c_m|k^m r}, \quad r \leq 1,$$

on repeating the steps from (28) to (29), which immediately leads to desired result

$$\left\{\int_0^{2\pi} \left|P'(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \geq n\frac{(n-m)|c_m|k^m + n|c_n|k^{n+1}}{(n-m)|c_m|(k^m+k^{n+1}) + n|c_n|(k^{n+1}+k^{2n-m})} \times \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}}. \tag{35}$$

This completes the proof of Theorem 2.3. completely. □

Proof of Theorem 2.5: Let $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k,\ k \geq 1$. If $P(z)$ has a zero on $|z| = k$, then $m = min_{|z|=k} P(z) = 0$ and the result follows from Theorem 2.1 in this case. Henceforth, we suppose that all the zeros of $P(z)$ lie in $|z| < k$ so that $m > 0$.

Since $m \leq |P(z)|$ for $|z| = 1$, therefore if β is any complex number with $|\beta| \leq 1$, then for $|z| = 1$ we have

$$|m\beta z^n| < |P(z)| \tag{36}$$

Since all the zeros of $P(z)$ are $|z| \leq k,\ k \geq 1$, it follows by Rouche's Theorem all the zeros of $P(z) - m\beta z^n$ are $|z| \leq k,\ k \geq 1$. Hence, by Theorem 2.1, we have for any $p > 0$,

$$\left\{\int_0^{2\pi} \left|P'(e^{i\theta}) - \beta mn\left(e^{i(n-1)\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \geq n\frac{|c_0| + |c_n - \beta m|k^{n+1}}{|c_0|(1+k^{n+1}) + |c_n - \beta m|(k^{n+1}+k^{2n})} \times \left\{\int_0^{2\pi} \left|P(e^{i\theta}) - \beta m\left(e^{in\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}. \tag{37}$$

Since the function $\dfrac{|c_0| + xk^{n+1}}{|c_0|(1+k^{n+1}) + x(k^{n+1}+k^{2n})}$ is non decreasing function of x, we have

$$\frac{|c_0| + |c_n - \beta m|k^{n+1}}{|c_0|(1+k^{n+1}) + |c_n - \beta m|(k^{n+1}+k^{2n})} \geq \frac{|c_0| + (|c_n| - |\beta|m)k^{n+1}}{|c_0|(1+k^{n+1}) + (|c_n| - |\beta|m)(k^{n+1}+k^{2n})}. \tag{38}$$

Also by triangles inequality , for $|z|=1$, we have

$$\Big|P(z)-\beta m z^n\Big| \geq \big|P(z)\big|-|\beta| m \tag{39}$$

Applying the argument of (38) and (39) to (37) respectively , we have

$$\left\{\int_0^{2\pi}\left|P'(e^{i\theta})-\beta mn\left(e^{i(n-1)\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \geq n\frac{|c_0|+(|c_n|-|\beta|m)k^{n+1}}{|c_0|(1+k^{n+1})+(|c_n|-|\beta|m)(k^{n+1}+k^{2n})}$$

$$\times\left\{\int_0^{2\pi}\Big|P(e^{i\theta})-|\beta|m\Big|^p d\theta\right\}^{\frac{1}{p}}. \tag{40}$$

Now choosing the argument of β suitably on the left hand side of (40) such that for $|z|=1$

$$\left|P'(e^{i\theta})-\beta mn\left(e^{i(n-1)\theta}\right)\right| = \big|P'(e^{i\theta})\big|-|\beta|mn \tag{41}$$

which is possible by (41), we get

$$\left\{\int_0^{2\pi}\left(\Big|P'(e^{i\theta})\Big|-|\beta|mn\right)^p d\theta\right\}^{\frac{1}{p}} \geq n\frac{|c_0|+(|c_n|-|\beta|m)k^{n+1}}{|c_0|(1+k^{n+1})+(|c_n|-|\beta|m)(k^{n+1}+k^{2n})}$$

$$\times\left\{\int_0^{2\pi}\Big|P(e^{i\theta})-|\beta|m\Big|^p d\theta\right\}^{\frac{1}{p}}. \tag{42}$$

Put $|\beta|=t$ in (42), we get

$$\left\{\int_0^{2\pi}\left(\Big|P'(e^{i\theta})\Big|-tmn\right)^p d\theta\right\}^{\frac{1}{p}} \geq n\frac{|c_0|+(|c_n|-tm)k^{n+1}}{|c_0|(1+k^{n+1})+(|c_n|-tm)(k^{n+1}+k^{2n})}$$

$$\times\left\{\int_0^{2\pi}\Big|P(e^{i\theta})-tm\Big|^p d\theta\right\}^{\frac{1}{p}}. \tag{43}$$

Where $0\leq t<1$ and this completes the proof of Theorem 2.5 completely.

□

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Irfan Ahmad Wani
Department of Mathematics, University of Kashmir, South Campus, Anantnag 192101, Jammu and Kashmir, India
E-mail: irfanmushtaq62@gmail.com

Mohammad Ibrahim Mir
Department of Mathematics, University of Kashmir, South Campus, Anantnag 192101, Jammu and Kashmir, India
E-mail: ibrahimmir@uok.edu.in

Ishfaq Nazir
Department of Mathematics, University of Kashmir, South Campus, Anantnag 192101, Jammu and Kashmir, India
E-mail: ishfaqnazir02@gmail.com