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η-RICCI SOLITONS ON PARA-KENMOTSU MANIFOLDS WITH SOME CURVATURE CONDITIONS

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Abstract. In the present paper, we study η-Ricci solitons on para-Kenmotsu manifolds with Codazzi type of the Ricci tensor. We study η-Ricci solitons on para-Kenmotsu manifolds with cyclic parallel Ricci tensor. We also study η-Ricci solitons on φ-conformally semi-symmetric, φ-Ricci symmetric and conformally Ricci semi-symmetric para-Kenmotsu manifolds. Finally, we construct an example of a three-dimensional para-Kenmotsu manifold which admits η-Ricci solitons.

1. Introduction

In 1982, Hamilton [12] introduced the notion of the Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton is a natural generalization of Einstein metric and defined on a Riemannian manifold (M, g). A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that

$$L_V g + 2S + 2\lambda g = 0,$$

where S is a Ricci tensor of M and L_V denotes the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda < 0, \lambda = 0$, or $\lambda > 0$, respectively [7]. Ricci solitons have been studied by many authors, such as [9, 10, 13] and several authors.

As a generalization of Ricci solitons, the notion of η-Ricci solitons was introduced by Cho and Kimura [6]. This notion has been studied in [4], for Hopf hypersurfaces in complex space form. A Ricci soliton is a tuple (g, V, λ, μ), where V is a vector field on M, λ and μ are real constants, and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$L_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0.$$

η-Ricci solitons on para-Kenmotsu manifolds were studied by A. M. Blaga [1] and η-Ricci solitons on Lorentzian Para-Sasakian manifolds were also studied by A. M.

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Blaga [2]. In particular, if $\mu = 0$, then the notion of η-Ricci solitons (g, V, λ, μ) reduces to the notion of Ricci solitons (g, V, λ). If $\mu \neq 0$, then the η-Ricci solitons are called proper η-Ricci solitons. Gray [11] introduced the notion of Codazzi type of the Ricci tensor. A pseudo-Riemannian manifold is said to satisfy Codazzi type of the Ricci tensor if its Ricci tensor S of type (0,2) is non-zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z),$$

which implies that *div* R=0, where *div* denotes divergence and R is the Riemannian curvature tensor of type (1,3). A Riemannian or pseudo-Riemannian manifold (M, g), $n \geq 3$, is said to be semi-symmetric if the curvature condition $R.R = 0$ holds, where R denotes the curvature tensor of the manifold. A fundamental study on Riemannian semi-symmetric manifolds was introduced by Z. I. Szabó [15]. Later E. Boeckx et al. [3] and O. Kowalski [14] and many others have studied semi-symmetric manifolds. A contact metric manifold is said to be φ-conformally semi-symmetric if $C.\varphi = 0$, where C is the conformal curvature tensor. Moreover, conformally Ricci semi-symmetric manifolds, that is $C.S = 0$, have been studied by Verstraelen [17]. Motivated by the above studies, in the present paper we consider η-Ricci solitons on para-Kenmotsu manifolds with the curvature conditions $C.\varphi = 0$ and $C.S = 0$.

The present paper is organized as follows: After the introduction, we give some required preliminaries in Section 2. Section 3 contains a brief review of Ricci and η-Ricci solitons. In Section 4, we study η-Ricci solitons on para-Kenmotsu manifolds satisfying Codazzi type of the Ricci tensor. In Section 5, we study η-Ricci solitons on para-Kenmotsu manifolds with cyclic parallel Ricci tensor. Section 6 is devoted to study η-Ricci solitons on φ-Ricci symmetric para-Kenmotsu manifolds. In the next section, we study η-Ricci solitons on φ-conformally semi-symmetric para-Kenmotsu manifolds. Section 8 deals with the study of η-Ricci solitons on conformally Ricci semi-symmetric para-Kenmotsu manifolds. In the last section we construct an example of three-dimensional para-Kenmotsu manifold which admits η-Ricci solitons.

2. Para-Kenmotsu Manifolds

Let $(M, \varphi, \eta, \xi, g)$ be a n-dimensional smooth manifold, where φ is an $(1,1)$ tensor field, ξ is a vector field, η is an 1-form and g is a pseudo-Riemannian metric on M. We say that (φ, η, ξ, g) is an almost paracontact metric structure on M, if it satisfies the conditions [1]

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{1}$$

$$\varphi\xi = 0, \quad \eta\varphi = 0, \quad rank(\varphi) = n - 1, \tag{2}$$

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{3}$$

for any vector fields X and Y on M.

If, moreover

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{4}$$

where ∇ denotes the Levi-Civita connection of g, then the almost paracontact metric structure (φ, η, ξ, g) is called para-Kenmotsu manifold.

From the definition, it follows that η is the g-dual of ξ:

$$g(X,\xi) = \eta(X), \tag{5}$$

ξ is a unitary vector field:

$$g(\xi,\xi) = 1, \tag{6}$$

and φ is a g-skew-symmetric operator. The fundamental 2-form Φ of an almost paracontact metric structure (M,φ,ξ,η,g) is defined by $\Phi(X,Y) = g(X,\varphi Y)$. If $\Phi = d\eta$, then the manifold (M,φ,ξ,η,g) is called a paracontact metric manifold and g is an associated metric. An almost paracontact metric manifold is normal if $[\varphi,\varphi](X,Y) + 2d\eta(X,Y)\xi = 0$, where $[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,\varphi Y] - \varphi[X,\varphi Y]$.

In a para-Kenmotsu manifold, we have the following formulas [18]

$$\nabla_X\xi = X - \eta(X)\xi, \tag{7}$$

$$(\nabla_X\eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{8}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{9}$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{10}$$

$$R(\xi,X)\xi = X - \eta(X)\xi, \tag{11}$$

$$S(X,\xi) = (1-n)\eta(X), \tag{12}$$

$$(L_\xi g)(X,Y) = -2\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{13}$$

where S is the Ricci tensor, R is the Riemannian curvature tensor field and ∇ is the Levi-Civita connection associated to g.

3. Ricci and η-Ricci Solitons on (M,φ,ξ,η,g)

Let (M,φ,ξ,η,g) be a paracontact metric manifold. Consider the equation

$$L_\xi g + 2S + \lambda g + 2\mu\eta\otimes\eta = 0, \tag{14}$$

where L_ξ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, and λ and μ are real constants. Writing $L_\xi g$ in terms of the Levi-Civita connection ∇, we get

$$2S(X,Y) = -g(\nabla_X\xi,Y) - g(X,\nabla_Y\xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{15}$$

for all $X,Y \in \chi(M)$, or equivalently:

$$S(X,Y) = -(\lambda+1)g(X,Y) - (\mu-1)\eta(X)\eta(Y), \tag{16}$$

for all $X,Y \in \chi(M)$.

The data (g,ξ,λ,μ) which satisfy the equation (14) is said to be an η-Ricci soliton on M [5]; in particular, if $\mu = 0$, (g,ξ,λ) is a Ricci soliton [16] and it is called shrinking, steady, or expanding according as λ is negative, zero or positive, respectively [19].

Taking $Y = \xi$ in (16), we get

$$S(X,\xi) = -(\lambda+\mu)\eta(X). \tag{17}$$

Comparing (12) and (17), we have

$$\lambda+\mu = n-1. \tag{18}$$

In this case, the Ricci operator Q defined by $g(QX,Y) = S(X,Y)$ has the expression:

$$QX = -(\lambda+1)X - (\mu-1)\eta(X)\xi. \tag{19}$$

The above equation yields that

$$r = -n(\lambda+1) - (\mu-1). \tag{20}$$

4. η-Ricci solitons on para-Kenmotsu manifolds with Ricci tensor of Coddazi type

Taking covariant differentiation of (16) with respect to Z we get

$$(\nabla_Z S)(X,Y) = -(\mu-1)[(\nabla_Z\eta)(X)\eta(Y) + (\nabla_Z\eta)(Y)\eta(X)]. \tag{21}$$

Using (8) in (21) we get

$$(\nabla_Z S)(X,Y) = -(\mu-1)[g(Z,X)\eta(Y) + g(Z,Y)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)]. \tag{22}$$

In view of (22) it follows that

$$\begin{aligned}(\nabla_Z S)(X,Y) - (\nabla_Y S)(Z,X) = -(\mu-1)[&g(X,Z)\eta(Y) + g(Y,Z)\eta(X)\\ &- g(Z,Y)\eta(X) - g(X,Y)\eta(Z)].\end{aligned} \tag{23}$$

Since, by hypothesis, the Ricci tensor is of Codazzi type, from (23) we get

$$(\mu-1)[g(X,Z)\eta(Y) + g(Y,Z)\eta(X) - g(Z,Y)\eta(X) - g(X,Y)\eta(Z)] = 0. \tag{24}$$

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Putting Z=ξ in (24), we get

$$(\mu-1)[\eta(X)\eta(Y) - g(X,Y)] = 0, \tag{25}$$

which yields

$$(\mu-1)g(\varphi X,\varphi Y) = 0.$$

From the above it follows that $\mu = 1$. Using (18) we get $\lambda = n-2$. Also from (16) we have

$$S(X,Y) = -(n-1)g(X,Y).$$

Thus we can state the following:

Theorem 4.1. *If a $(2n+1)$-dimensional para-Kenmotsu manifold $M(\varphi,\xi,\eta,g)$ admits an η-Ricci soliton whose Ricci tensor is of Coddazi type, then $\lambda = n-2, \mu = 1$ and the manifold is Einstein.*

5. η-Ricci Solitons on Para-Kenmotsu manifolds with cyclic parallel Ricci tensor

This section is devoted to study proper η-Ricci solitons on para-Kenmotsu manifolds with cyclic parallel Ricci tensor. Therefore

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0, \tag{26}$$

for all smooth vector fields $X, Y, Z \in \chi(M)$.

Using (3) and (22) in (26) we get

$$(\mu - 1)[g(\varphi X, \varphi Z)\eta(Y) + g(\varphi Y, \varphi Z)\eta(X) + g(\varphi X, \varphi Y)\eta(Z)] = 0. \tag{27}$$

Putting $X = \xi$ in (27), we get

$$(\mu - 1)[g(\varphi Y, \varphi Z)] = 0. \tag{28}$$

It follows that

$$\mu = 1. \tag{29}$$

Using (18) and above equation we get $\lambda = n - 2$. Also from (16) we have

$$S(X,Y) = -(n-1)g(X,Y).$$

Thus we are in a position to state the following:

Theorem 5.1. *If a $(2n+1)$-dimensional para-Kenmotsu manifold $M(\varphi, \xi, \eta, g)$ with cyclic parallel Ricci tensor admits η-Ricci soliton, then $\lambda = n - 2, \mu = 1$ and the manifold is Einstein.*

6. η-Ricci Solitons on φ-Ricci Symmetric Para-Kenmotsu manifolds

A para-Kenmotsu manifold is said to be φ-Ricci symmetric if

$$\varphi^2(\nabla_X Q)Y = 0, \tag{30}$$

holds for all smooth vector field X, Y. It should be mentioned that φ-Ricci symmetric Sasakian manifolds have been studied in [8].

Taking covariant derivative of (16), we get

$$\begin{aligned}(\nabla_X Q)Y &= \nabla_X QY - Q(\nabla_X Y)\\ &= -(\mu - 1)[g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)X]. \end{aligned} \tag{31}$$

Operating φ^2 on both sides of (31), we get

$$\varphi^2(\nabla_X Q)Y = -(\mu - 1)\eta(Y)\varphi^2 X. \tag{32}$$

From (30) and (32) we have

$$\mu = 1. \tag{33}$$

Also from (18) and (33) we get $\lambda = n-2$ and from (16) we have

$$S(X,Y) = -(n-1)g(X,Y).$$

Thus we are in position to state the following:

THEOREM 6.1. *If a $(2n+1)$-dimensional φ-Ricci symmetric para-Kenmotsu manifold $M(\varphi,\xi,\eta,g)$ admits η-Ricci soliton, then $\lambda = n-2, \mu = 1$ and the manifold is Einstein.*

7. η-Ricci Solitons on φ-conformally semi-symmetric Para-Kenmotsu manifolds

The conformal curvature tensor C is defined by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z &- \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{34}$$

where S is the Ricci tensor, Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$, and r is the scalar curvature of the manifold M.

This section is devoted to the study of φ-conformally semi-symmetric η-Ricci solitons on para-Kenmotsu manifolds. Then

$$C.\varphi = 0, \tag{35}$$

from which it follows that

$$C(X,Y)\varphi Z - \varphi(C(X,Y)Z) = 0. \tag{36}$$

Putting $Z = \xi$ in (36), we get

$$\varphi(C(X,Y)\xi) = 0. \tag{37}$$

Putting $Z = \xi$ in (34) and using (5), (9), (17) and (19) we get

$$\begin{aligned} C(X,Y)\xi &= \eta(X)Y - \eta(Y)X - \frac{1}{n-2}[S(Y,\xi)X - S(X,\xi)Y \\ &\quad + \eta(Y)QX - \eta(X)QY] \\ &\quad + \frac{r}{(n-1)(n-2)}[\eta(Y)X - \eta(X)Y] \\ &= \left[1 + \frac{n+\lambda}{n-2} - \frac{r}{(n-1)(n-2)}\right](\eta(X)Y - \eta(Y)X). \end{aligned} \tag{38}$$

In view of (37) and (38) we have

$$\varphi(C(X,Y)\xi) = \left[1 + \frac{n+\lambda}{n-2} - \frac{r}{(n-1)(n-2)}\right][\eta(X)Y - \eta(Y)X] = 0. \tag{39}$$

Replacing X by φX in (39) we get

$$\left[1 + \frac{n+\lambda}{n-2} - \frac{r}{(n-1)(n-2)}\right]\eta(Y)\varphi^2 X = 0. \tag{40}$$

From (40) it follows that

$$\left[1 + \frac{n+\lambda}{n-2} - \frac{r}{(n-1)(n-2)}\right] = 0.$$

By virtue of (20), we get

$$\lambda - \mu = 2n^2 - 3n + 1.$$

From (18), we get

$$\lambda = n(n-1),$$

and

$$\mu = -(n-1)^2.$$

Thus we can state the following:

THEOREM 7.1. *If a φ-conformally semisymmetric $(2n+1)$-dimensional para-Kenmotsu manifold with constant scalar curvature admits η-Ricci solitons, then $\lambda = n(n-1)$ and $\mu = -(n-1)^2$.*

8. η-Ricci solitons on conformally Ricci semi-symmetric Para-Kenmotsu manifolds

In this section we study η-Ricci solitons on conformally Ricci semi-symmetric para-Kenmotsu manifolds, that is

$$C.S = 0, \tag{41}$$

which implies

$$(C(X,Y)Z.S)(Z,W) = 0. \tag{42}$$

From (41) we get

$$S(C(X,Y)Z,W) + S(Z,C(X,Y)W) = 0. \tag{43}$$

Using (16) in (43) we get

$$(\mu - 1)[\eta(C(X,Y)Z)\eta(W) + \eta(C(X,Y)W)\eta(Z)] = 0. \tag{44}$$

Putting $X = Y = \xi$ in (44) we get

$$(\mu - 1)[\eta(C(\xi,Y)Z) + \eta(C(\xi,Y)\xi)\eta(Z)] = 0. \tag{45}$$

With the help of (38) we find

$$\begin{aligned}\eta(C(\xi,Y)Z) &= g(C(\xi,Y)Z,\xi)\\ &= -g(C(\xi,Y)\xi,Z)\\ &= -\left[1+\frac{n+\lambda}{n-2}-\frac{r}{(n-1)(n-2)}\right][g(Y,Z)-\eta(Y)\eta(Z)]=0. \qquad (46)\end{aligned}$$

Also from (46) we have

$$\eta(C(\xi,Y)\xi)=0. \qquad (47)$$

Using (46) and (47) in (45) we get

$$(\mu-1)\left[1+\frac{n+\lambda}{n-2}-\frac{r}{(n-1)(n-2)}\right][g(Y,Z)-\eta(Y)\eta(Z)]=0. \qquad (48)$$

From (48) we obtain

$$(\mu-1)\left[1+\frac{n+\lambda}{n-2}-\frac{r}{(n-1)(n-2)}\right]g(\varphi Y,\varphi Z)=0. \qquad (49)$$

Therefore, we get

$$(\mu-1)\left[1+\frac{n+\lambda}{n-2}-\frac{r}{(n-1)(n-2)}\right]=0. \qquad (50)$$

By virtue of (20), we get

$$(\mu-1)(\lambda-\mu-2n^2+3n-1)=0. \qquad (51)$$

Hence we can state the following:

THEOREM 8.1. *If a $(2n+1)$-dimensional para-Kenmotsu manifold $M(\varphi,\xi,\eta,g)$ admits η-Ricci soliton and $C.S=0$, then*

$$(\mu-1)(\lambda-\mu-2n^2+3n-1)=0.$$

9. Example of η-Ricci solitons on three-dimensional para-Kenmotsu manifold

We consider the three-dimensional manifold $M=\{(x,y,z)\in R^3, z\neq 0\}$, where (x,y,z) are the standard coordinates in R^3. The vector fields

$$e_1=\frac{\partial}{\partial x},\quad e_2=\frac{\partial}{\partial y},\quad e_3=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{\partial}{\partial z},$$

are linearly independent at each point of M and

$$[e_1,e_2]=0,[e_1,e_3]=e_1,[e_2,e_3]=e_2.$$

Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0,$$

$$g(e_1, e_1) = g(e_3, e_3) = 1, \quad g(e_2, e_2) = -1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$ and φ be the (1,1) tensor field defined by $\varphi(e_1) = e_2$, $\varphi(e_2) = e_1$, $\varphi(e_3) = 0$. Then using the linearity of φ and g we have

$$\eta(e_3) = 1, \quad \varphi^2(Z) = Z - \eta(Z)e_3,$$

$$g(\varphi Z, \varphi W) = -g(Z, W) + \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (φ, ξ, η, g) defines an almost contact metric structure on M.

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)$$
$$-g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using Koszul's formula we get the following

$$\begin{array}{lll}
\nabla_{e_1} e_3 = e_1, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = -e_3, \\
\nabla_{e_2} e_3 = e_2, & \nabla_{e_2} e_2 = e_3, & \nabla_{e_2} e_1 = 0, \\
\nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0.
\end{array}$$

From above we see that the manifold satisfies $\nabla_X \xi = X - \eta(X)\xi$, for $e_3 = \xi$. Therefore the structure $M(\varphi, \xi, \eta, g)$ is a three-dimensional para-Kenmotsu manifold.

With the help of the above results it can be verified that

$$\begin{array}{lll}
R(e_1, e_2)e_3 = 0, & R(e_2, e_3)e_3 = -e_2, & R(e_1, e_3)e_3 = -e_1, \\
R(e_1, e_2)e_2 = e_1, & R(e_2, e_3)e_2 = -e_3, & R(e_1, e_3)e_2 = 0, \\
R(e_1, e_2)e_1 = e_2, & R(e_2, e_3)e_1 = 0, & R(e_1, e_3)e_1 = e_3.
\end{array}$$

Using the expressions of the curvature tensor we find the values of the Ricci tensor as follows

$$S(e_1, e_1) = -2, \quad S(e_2, e_2) = 2, \quad S(e_3, e_3) = -2,$$

$$S(e_1, e_2) = 0, \quad S(e_1, e_3) = 0, \quad S(e_2, e_3) = 0.$$

From (16) we obtain $S(e_1, e_1) = -(\lambda + 1)$ and $S(e_3, e_3) = -(\lambda + \mu)$, therefore $\lambda = 1$ and $\mu = 1$. Hence the Theorem 4.1, Theorem 5.1 and Theorem 6.1 are verified.

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