Korean J. Math. **29** (2021), No. 4, pp. 785–800
http://dx.doi.org/10.11568/kjm.2021.29.4.785

# ON BOUNDS FOR THE DERIVATIVE OF ANALYTIC FUNCTIONS AT THE BOUNDARY

Bülent Nafi Örnek and Tuğba Akyel

Abstract. In this paper, we obtain a new boundary version of the Schwarz lemma for analytic function. We give sharp upper bounds for $|f'(0)|$ and sharp lower bounds for $|f'(c)|$ with $c \in \partial D = \{z : |z| = 1\}$. Thus we present some new inequalities for analytic functions. Also, we estimate the modulus of the angular derivative of the function $f(z)$ from below according to the second Taylor coefficients of $f$ about $z = 0$ and $z = z_0 \neq 0$. Thanks to these inequalities, we see the relation between $|f'(0)|$ and $\Re f(0)$. Similarly, we see the relation between $\Re f(0)$ and $|f'(c)|$ for some $c \in \partial D$. The sharpness of these inequalities is also proved.

## 1. Introduction

We consider the Schwarz lemma which is an application of the maximum modulus principible playing an important role in problems involving mappings for analytic functions. It says that if $g : D \to D$ is analytic with $g(z) = c_1 z + c_2 z^2 + ....$, then $|g(z)| \leq |z|$, $\forall z \in D = \{z : |z| < 1\}$ and $|g'(0)| \leq 1$.

Moreover, if the equality $|g(z)| = |z|$ holds for any $z \neq 0$, or $|g'(0)| = 1$, then $f$ is a rotation; that is, $g(z) = ze^{i\theta}$, $\theta$ real ( [5], p.329). The Schwarz lemma has several applications in the field of electrical and electronics engineering. Use of positive real function and boundary analysis of these functions for circuit synthesis can be given as an exemplary application of the Schwarz lemma in electrical engineering. Furthermore, it is also used for analysis of transfer functions in control engineering and multi-notch filter design in signal processing [13].

In this study, we give a new boundary version of the Schwarz lemma for analytic function. We present some inequalities which give a relation between $|f'(0)|$ and $\Re f(0)$ and also a relation between $\Re f(0)$ and $|f'(c)|$ for some $c \in \partial D$ by obtaining sharp upper bounds for $|f'(0)|$ and sharp lower bounds for $|f'(c)|$ with $c \in \partial D = \{z : |z| = 1\}$. The sharpness of these inequalities is also proved.

Let $f(z) = f(0) + c_1 z + c_2 z^2 + ...$ be an analytic function in $D$, $\Re f(z) > 0$ for $|z| < 1$ and

$$\vartheta(z) = \frac{f(0) - f(z)}{\overline{f(0)} + f(z)}. \tag{1.1}$$

Received July 14, 2021. Revised November 28, 2021. Accepted November 30, 2021.
2010 Mathematics Subject Classification: 30C80.
Key words and phrases: Analytic function, Schwarz lemma, Angular derivative, Julia-Wolff lemma.

Here $\vartheta(z)$ is an analytic function in $D$ and $\vartheta(0)=0$. Now, let us show that $|\vartheta(z)|<1$ for $z\in D$. Since

$$\begin{aligned}|f(0)-f(z)|^2 &= (f(0)-f(z))\left(\overline{f(0)}-\overline{f(z)}\right)\\ &= |f(0)|^2 - f(0)\overline{f(z)} - f(z)\overline{f(0)} + |f(z)|^2\end{aligned}$$

and

$$\begin{aligned}\left|\overline{f(0)}+f(z)\right|^2 &= \left(\overline{f(0)}+f(z)\right)\left(f(0)+\overline{f(z)}\right)\\ &= |f(0)|^2 + \overline{f(0)}\;\overline{f(z)} + f(z)f(0) + |f(z)|^2,\end{aligned}$$

we take

$$\begin{aligned}&|f(0)-f(z)|^2 - \left|\overline{f(0)}+f(z)\right|^2\\ &\quad= -\left(f(0)\overline{f(z)} + f(z)\overline{f(0)} + \overline{f(0)}\;\overline{f(z)} + f(z)f(0)\right)\\ &\quad= -4\Re f(0)\Re f(z).\end{aligned}$$

So, $|\vartheta(z)|<1$ for $z\in D$. From the Schwarz lemma, we obtain

$$\begin{aligned}\vartheta(z) &= -\frac{c_1z+c_2z^2+\ldots}{f(0)+\overline{f(0)}+c_1z+c_2z^2+\ldots},\\ \frac{\vartheta(z)}{z} &= -\frac{c_1+c_2z+\ldots}{2\Re f(0)+c_1z+c_2z^2+\ldots},\\ \vartheta'(0) &= -\frac{c_1}{2\Re f(0)}\end{aligned}$$

and

$$|c_1| = |f'(0)| \le 2\Re f(0).$$

Now let us show that this inequality is sharp. Let

$$f(z) = \frac{f(0)-z\overline{f(0)}}{1+z}.$$

Then

$$f'(z) = -\frac{f(0)+\overline{f(0)}}{(1+z)^2}$$

and

$$|f'(0)| = 2\Re f(0).$$

In addition, we obtain

$$\begin{aligned}\Re\left(\frac{f(0)-z\overline{f(0)}}{1+z}\right) &= \frac{\frac{f(0)-z\overline{f(0)}}{1+z}+\frac{\overline{f(0)}-\overline{z}f(0)}{1+\overline{z}}}{2}\\ &= \frac{(1+\overline{z})\left(f(0)-z\overline{f(0)}\right)+(1+z)\left(\overline{f(0)}-\overline{z}f(0)\right)}{2(1+z)(1+\overline{z})}\\ &= \Re f(0)\frac{1-|z|^2}{|1+z|^2} > 0.\end{aligned}$$

We thus get the following lemma expressing the relation between $\Re f(0)$ and $|f'(0)|$.

Lemma 1.1. *Let $f(z) = f(0) + c_1 z + c_2 z^2 + ...$ be an analytic function in $D$ and $\Re f(z) > 0$ for $|z| < 1$. Then we have*

$$|f'(0)| \leq 2\Re f(0).$$

*This result is sharp with the function*

$$f(z) = \frac{f(0) - z\overline{f(0)}}{1+z}.$$

Now, consider the product

$$B(z) = \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z}.$$

The function $B(z)$ is called a finite Blaschke product, where $a_1, a_2, ..., a_n \in D$. Consider the function

$$t(z) = \frac{\vartheta(z)}{\prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}.$$

Here $a_1, a_2, ..., a_n$ are zeros of $f(z)$. Also, $t(z)$ is an analytic function in $D$, $t(0) = 0$ and $|t(z)| < 1$ for $z \in D$. So, $t(z)$ satisfies the conditions of the Schwarz lemma. Thus, from the Schwarz lemma, we obtain

$$\begin{aligned} t(z) &= \frac{\vartheta(z)}{\prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}} = \frac{f(0) - f(z)}{\overline{f(0)} + f(z)} \frac{1}{\prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}} \\ &= -\frac{c_1 z + c_2 z^2 + ...}{f(0) + \overline{f(0)} + c_1 z + c_2 z^2 + ...} \frac{1}{\prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}. \end{aligned}$$

$$\frac{t(z)}{z} = -\frac{c_1 + c_2 z + ...}{2\Re f(0) + c_1 z + c_2 z^2 + ...} \frac{1}{\prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}$$

and

$$|c_1| \leq 2\Re f(0) \prod_{k=1}^{n} |a_k|.$$

This result is sharp with equality for the function

$$f(z) = \frac{f(0) - z\overline{f(0)} \prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}{1 + z \prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}} = f(0) - \frac{\left(f(0) + \overline{f(0)}\right) z \prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}{1 + z \prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}.$$

Then

$$\frac{f(z) - f(0)}{z} = -\frac{\left(f(0) + \overline{f(0)}\right) \prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}{1 + z \prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}$$

and

$$|c_1| = 2\Re f(0) \prod_{k=1}^{n} |a_k| .$$

Also, we take

$$\Re\left(\frac{f(0) - z\overline{f(0)} \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}{1 + z \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}\right) = \frac{\frac{f(0)-z\overline{f(0)}\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}{1+z\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}} + \frac{\overline{f(0)}-\overline{z}f(0)\overline{\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}}{\overline{1+z\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}}}{2}$$

$$= \frac{\left(\overline{1 + z \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}\right)\left(f(0) - z\overline{f(0)} \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}\right)}{2\left(1 + z \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}\right)\left(\overline{1 + z \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}\right)}$$

$$+\frac{\left(1 + z \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}\right)\left(\overline{f(0)} - \overline{z}f(0)\overline{\prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}\right)}{2\left(1 + z \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}\right)\left(\overline{1 + z \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}\right)}$$

$$= \Re f(0)\frac{1 - |z|^2 \left(\prod_{k=1}^{n} \left|\frac{z-a_k}{1-\overline{a_k}z}\right|\right)^2}{\left|1 + z \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}\right|^2} > 0.$$

So, we get the following lemma expressing the relation among $Rf(0)$, $|f'(0)|$, and the zeros of the function $f$ which are different from zero.

Lemma 1.2. *Let $f(z) = f(0) + c_1 z + c_2 z^2 + ...$ be an analytic function in $D$ and $\Re f(z) > 0$ for $|z| < 1$ and $a_1, a_2, ..., a_n$ be the zeros of the function $f(z) - f(0)$ in $D$ that are different from zero. Then we have*

$$|f'(0)| \leq 2\Re f(0) \prod_{k=1}^{n} |a_k| .$$

*This result is sharp with the function*

$$f(z) = \frac{f(0) - z\overline{f(0)} \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}{1 + z \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}}.$$

Since the area of applicability of the Schwarz lemma is quite wide, there exist many studies about it. Some of these studies are called the boundary version of Schwarz lemma. The important results of the Schwarz lemma have been given by Osserman [11,15]. It is still a hot topic in the mathematics literature [1–4,6–10,12,13].

It is a consequence of Schwarz lemma that if $f$ extends continuously to some boundary point $b$ with $|c| = 1$, and if $|g(c)| = 1$ and $g'(c)$ exists, then $|g'(c)| \geq 1$, which is known as the Schwarz lemma on the boundary. In [11], R. Osserman has proposed the boundary refinement of the classical Schwarz lemma as follows:

Let $g : D \to D$ be an analytic function with $g(z) = c_1 z + c_2 z^2 + \ldots$. Assume that there is a $c \in \partial D$ so that $f$ extends continuously to $c$, $|g(c)| = 1$ and $g'(c)$ exists. Then

$$|g'(c)| \geq \frac{2}{1 + |g'(0)|}. \tag{1.2}$$

Inequality (1.2) is sharp. That is, for $c = 1$ in the inequality (1.2), equality occurs for the function $g(z) = z\frac{z+a}{1+az}$, $a \in [0, 1]$.

Mercer [8] has proved a version of the Schwarz Lemma where the images of two points are known. Also, he has considered some Schwarz and Caratheodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [9]. In addition, he has obtained a new boundary Schwarz Lemma for analytic functions mapping the unit disk to itself [10].

For our main results, we shall need the following lemma due to Julia-Wolff [14].

Lemma 1.3 (Julia-Wolff lemma). *Let $g$ be an analytic function in $D$, $g(0) = 0$ and $g(D) \subset D$. If, in addition, the function $g$ has an angular limit $g(c)$ at $c \in \partial D$, $|g(c)| = 1$, then the angular derivative $g'(c)$ exists and $1 \leq |g'(c)| \leq \infty$.*

## 2. Main results

In this section, boundary analysis results for the derivative of analytic functions are presented.

Theorem 2.1. *Let $f(z) = f(0) + c_1 z + c_2 z^2 + \ldots$ be an analytic function in $D$ and $\Re f(z) > 0$ for $|z| < 1$. Assume that, for some $c \in \partial D$, $f$ has an angular limit $f(c)$ at $c$, $f(c) = i\Im f(0)$. Then we have the inequality*

$$|f'(c)| \geq \frac{\Re f(0)}{2}. \tag{2.1}$$

*The equality in (2.1) occurs for the function*

$$f(z) = \frac{f(0) - z\overline{f(0)}}{1 + z}.$$

*Proof.* Let $\vartheta(z)$ be the same function as in (1.1). So, from the derivative of $\vartheta(z)$, we have

$$\vartheta'(z) = -\frac{2\Re f(0) f'(z)}{\left(\overline{f(0)} + f(z)\right)^2}.$$

Also, we have

$$|\vartheta(c)| = \left|\frac{f(0) - f(c)}{\overline{f(0)} + f(c)}\right| = \left|\frac{f(0) - i\Im f(0)}{\overline{f(0)} + i\Im f(0)}\right| = 1.$$

Here the function $\vartheta(z)$ satisfies the assumptions of the Schwarz lemma on the boundary. Therefore, we obtain

$$1 \leq |\vartheta'(c)| = \frac{2\Re f(0)\,|f'(c)|}{\left|\overline{f(0)} + f(c)\right|^2} = \frac{2\Re f(0)\,|f'(c)|}{\left|\overline{f(0)} + i\Im f(0)\right|^2} = \frac{2\,|f'(c)|}{\Re f(0)}$$

and

$$|f'(c)| \geq \frac{\Re f(0)}{2}.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$f(z) = \frac{f(0) - z\overline{f(0)}}{1+z}.$$

Then

$$f'(z) = -\frac{f(0) + \overline{f(0)}}{(1+z)^2}$$

and

$$|f'(1)| = \frac{\Re f(0)}{2}.$$

□

The inequality (2.1) can be strengthened as below by taking into account $c_1 = f'(0)$ which is first coefficient in the expansion of the function $f(z)$.

THEOREM 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$|f'(c)| \geq \frac{2(\Re f(0))^2}{2\Re f(0) + |f'(0)|}. \tag{2.2}$$

*The inequality* (2.2) *is sharp with extremal function*

$$f(z) = \frac{f(0) - z\overline{f(0)}}{1+z}.$$

*Proof.* Let $\vartheta(z)$ be the same function as (1.1). So, from (1.2), we obtain

$$\frac{2}{1+|\vartheta'(0)|} \leq |\vartheta'(c)| = \frac{2|f'(c)|}{\Re f(0)}.$$

Since $|\vartheta'(0)| = \frac{|c_1|}{2\Re f(0)} = \frac{|f'(0)|}{2\Re f(0)}$, we have

$$\frac{2}{1+\frac{|f'(0)|}{2\Re f(0)}} \leq \frac{2|f'(c)|}{\Re f(0)}$$

and

$$|f'(c)| \geq \frac{2(\Re f(0))^2}{2\Re f(0) + |f'(0)|}.$$

Now, we shall show that the inequality (2.2) is sharp. Let

$$f(z) = \frac{f(0) - z\overline{f(0)}}{1+z}.$$

Then

$$|f'(1)| = \frac{\Re f(0)}{2}.$$

On the other hand, we take

$$f(0) + c_1 z + c_2 z^2 + \ldots = \frac{f(0) - z\overline{f(0)}}{1+z},$$

$$c_1 z + c_2 z^2 + \ldots = \frac{f(0) - z\overline{f(0)}}{1+z} - f(0) = -\left(f(0) + \overline{f(0)}\right)\frac{z}{1+z}$$

and

$$c_1 + c_2 z + ... = -\left(f(0) + \overline{f(0)}\right) \frac{1}{1+z}.$$

Passing to limit as $z$ tends to 0 in the last equality, we get $|c_1| = |f'(0)| = 2\Re f(0)$. Therefore, we obtain

$$\frac{2\left(\Re f(0)\right)^2}{2\Re f(0) + |f'(0)|} = \frac{2\left(\Re f(0)\right)^2}{2\Re f(0) + 2\Re f(0)} = \frac{\Re f(0)}{2}.$$

□

In the following theorem, inequality (2.2) is strengthened by adding the consecutive terms $c_1$ and $c_2$ of the function $f(z)$.

THEOREM 2.3. *Let* $f(z) = f(0) + c_1 z + c_2 z^2 + ...$ *be an analytic function in* $D$ *and* $\Re f(z) > 0$ *for* $|z| < 1$. *Assume that, for some* $c \in \partial D$, $f$ *has an angular limit* $f(c)$ *at* $c$, $f(c) = i\Im f(0)$. *Then we have the inequality*

$$|f'(c)| \geq \frac{\Re f(0)}{2}\left(1 + \frac{2\left(2\Re f(0) - |c_1|\right)^2}{4\left(\Re f(0)\right)^2 - |c_1|^2 + |2\Re f(0) c_2 - c_1^2|}\right). \tag{2.3}$$

*The inequality* (2.3) *is sharp with equality for the function*

$$f(z) = \frac{f(0) - z^2\overline{f(0)}}{1+z^2}.$$

*Proof.* Let $\vartheta(z)$ be the same function as (1.1) and $k(z) = z$. The maximum principle implies that for each $z \in D$, we have $|\vartheta(z)| \leq |k(z)|$. So, $\varphi(z) = \frac{\vartheta(z)}{k(z)}$ is analytic function in $D$ and $|\varphi(z)| < 1$ for $z \in D$. In particular, we obtain

$$|\varphi(0)| = \frac{|c_1|}{2\Re f(0)}$$

and

$$|\varphi'(0)| = \frac{|2\Re f(0) c_2 - c_1^2|}{4\left(\Re f(0)\right)^2}.$$

In addition, with the simple calculations, we take

$$\frac{c\vartheta'(c)}{\vartheta(c)} = |\vartheta'(c)| \geq |k'(c)| = \frac{ck'(c)}{k(c)}.$$

The composite function

$$d(z) = \frac{\varphi(z) - \varphi(0)}{1 - \overline{\varphi(0)}\varphi(z)}$$

is analytic in the unit disc $D$, $d(0) = 0$, $|d(z)| < 1$ for $z \in D$ and $|d(c)| = 1$. From (1.2), we obtain

$$\begin{aligned} \frac{2}{1+|d'(0)|} &\leq |d'(c)| = \frac{1 - |\varphi(0)|^2}{\left|1 - \overline{\varphi(0)}\varphi(c)\right|^2} |\varphi'(c)| \\ &\leq \frac{1+|\varphi(0)|}{1-|\varphi(0)|}\left(|\vartheta'(c)| - |k'(c)|\right). \end{aligned}$$

Since

$$|d'(0)| = \frac{|\varphi'(0)|}{1-|\varphi(0)|^2} = \frac{\frac{\left|2\Re f(0)c_2-c_1^2\right|}{4(\Re f(0))^2}}{1-\left(\frac{|c_1|}{2\Re f(0)}\right)^2} = \frac{|2\Re f(0)c_2-c_1^2|}{4\left(\Re f(0)\right)^2-|c_1|^2},$$

we take

$$\frac{2}{1+\frac{\left|2\Re f(0)c_2-c_1^2\right|}{4(\Re f(0))^2-|c_1|^2}} \leq \frac{2\Re f(0)+|c_1|}{2\Re f(0)-|c_1|}\left(\frac{2\left|f'(c)\right|}{\Re f(0)}-1\right)$$

and

$$|f'(c)| \geq \frac{\Re f(0)}{2}\left(1+\frac{2\left(2\Re f(0)-|c_1|\right)^2}{4\left(\Re f(0)\right)^2-|c_1|^2+|2\Re f(0)c_2-c_1^2|}\right).$$

Now, let us show that the inequality (2.3) is sharp. Let

$$f(z) = \frac{f(0)-z^2\overline{f(0)}}{1+z^2}.$$

Then

$$|f'(1)| = \Re f(0).$$

On the other hand, we obtain

$$f(0)+c_1z+c_2z^2+... = \frac{f(0)-z^2\overline{f(0)}}{1+z^2},$$

$$c_1z+c_2z^2+... = \frac{f(0)-z^2\overline{f(0)}}{1+z^2}-f(0) = \frac{f(0)-z^2\overline{f(0)}-f(0)-z^2f(0)}{1+z^2}$$

and

$$c_1z+c_2z^2+... = -\frac{z^2\left(f(0)+\overline{f(0)}\right)}{1+z^2}.$$

Passing to limit as $z$ tends to 0 in the last equality, we get $c_1 = 0$. Similarly, using straightforward calculations, we take $c_2 = -2\Re f(0)$. Therefore, we take

$$\frac{\Re f(0)}{2}\left(1+\frac{2\left(2\Re f(0)-|c_1|\right)^2}{4\left(\Re f(0)\right)^2-|c_1|^2+|2\Re f(0)c_2-c_1^2|}\right)$$
$$= \frac{\Re f(0)}{2}\left(1+\frac{8\left(\Re f(0)\right)^2}{4\left(\Re f(0)\right)^2+4\left(\Re f(0)\right)^2}\right) = \Re f(0).$$

In addition, we obtain

$$\Re\left(\frac{f(0)-z^2\overline{f(0)}}{1+z^2}\right) = \Re f(0)\frac{1-|z|^4}{|1+z^2|^2} > 0.$$

□

If $f(z)-f(0)$ has zeros different from $z=0$, taking into account these zeros, the inequality (2.3) can be strengthened in another way. This is given by the following theorem.

Theorem 2.4. *Let $f(z)=f(0)+c_1z+c_2z^2+...$ be an analytic function in $D$ and $\Re f(z)>0$ for $|z|<1$. Assume that, for some $c\in\partial D$, $f$ has an angular limit $f(c)$ at $c$, $f(c)=i\Im f(0)$. Let $a_1,a_2,...,a_n$ be zeros of the function $f(z)-f(0)$ in $D$ that are different from zero. Then we have the inequality*

$$|f'(c)|\geq\frac{\Re f(0)}{2}\left(1+\sum_{k=1}^{n}\frac{1-|a_k|^2}{|c-a_k|^2}\right.$$
$$\left.+\frac{2\left(2\Re f(0)\prod_{k=1}^{n}|a_k|-|c_1|\right)^2}{\left(2\Re f(0)\prod_{k=1}^{n}|a_k|\right)^2-|c_1|^2+\prod_{k=1}^{n}|a_k|\left|2\Re f(0)c_2-c_1^2+2\Re f(0)c_1\sum_{k=1}^{n}\frac{1-|a_k|^2}{a_k}\right|}\right). \tag{2.4}$$

*The results* (2.4) *is sharp for the function given by*

$$f(z)=\frac{f(0)-\overline{f(0)}z^2\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}{1+z^2\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}},$$

*where $a_1,a_2,...,a_n$ are positive real numbers.*

*Proof.* Let $\vartheta(z)$ be the same function as (1.1) and $a_1,a_2,...,a_n$ be the zeros of the function $f(z)$ in $D$ that are different from zero. The function

$$B(z)=z\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}$$

is analytic in $D$, and $|B(z)|<1$ for $z\in D$. By the maximum principle, for each $z\in D$, we have $|\vartheta(z)|\leq|B(z)|$. Let

$$m(z)=\frac{\vartheta(z)}{B(z)}.$$

This function is analytic in $D$ and $|m(z)|\leq1$ for $z\in D$. Therefore, we have

$$\begin{aligned}m(z)&=\frac{\vartheta(z)}{z\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}=\frac{f(0)-f(z)}{\overline{f(0)}+f(z)}\frac{1}{z\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}\\&=-\frac{c_1z+c_2z^2+...}{f(0)+\overline{f(0)}+c_1z+c_2z^2+...}\frac{1}{z\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}},\\&=-\frac{c_1+c_2z+...}{2\Re f(0)+c_1z+c_2z^2+...}\frac{1}{\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}\end{aligned}$$

$$|m(0)|=\frac{|c_1|}{2\Re f(0)\prod_{k=1}^{n}|a_k|}$$

and

$$|m'(0)|=\frac{\left|2\Re f(0)c_2-c_1^2+2\Re f(0)c_1\sum_{k=1}^{n}\frac{1-|a_k|^2}{a_k}\right|}{4\left(\Re f(0)\right)^2\prod_{k=1}^{n}|a_k|}.$$

Moreover, it is obvious that

$$|B'(c)| = \frac{cB'(c)}{B(c)} = 1 + \sum_{k=1}^{n} \frac{1-|a_k|^2}{|c-a_k|^2}$$

and

$$\frac{c\vartheta'(c)}{\vartheta(c)} = |\vartheta'(c)| \geq |B'(c)| = \frac{cB'(c)}{B(c)},\ c \in \partial D.$$

Consider the auxiliary function

$$\Theta(z) = \frac{m(z)-m(0)}{1-\overline{m(0)}m(z)}.$$

This function is analytic in $D$, $\Theta(0)=0$, $|\Theta(z)|<1$ for $|z|<1$ and $|\Theta(c)|=1$ for $c \in \partial D$. Therefore, from the Schwarz lemma on the boundary, we obtain

$$\begin{aligned} \frac{2}{1+|\Theta'(0)|} &\leq |\Theta'(c)| = \frac{1-|m(0)|^2}{\left|1-\overline{m(0)}m(c)\right|^2}|m'(c)| \\ &\leq \frac{1+|m(0)|}{1-|m(0)|}\left\{|\vartheta'(c)|-|B'(c)|\right\}. \end{aligned}$$

Also, it can be seen that

$$\Theta'(z) = \frac{1-|m(0)|^2}{\left(1-\overline{m(0)}m(z)\right)^2}m'(z)$$

and

$$\begin{aligned} |\Theta'(0)| &= \frac{|m'(0)|}{1-|m(0)|^2} = \frac{\dfrac{\left|2\Re f(0)c_2-c_1^2+2\Re f(0)c_1\sum\limits_{k=1}^{n}\frac{1-|a_k|^2}{a_k}\right|}{4(\Re f(0))^2\prod\limits_{k=1}^{n}|a_k|}}{1-\left(\dfrac{|c_1|}{2\Re f(0)\prod\limits_{k=1}^{n}|a_k|}\right)^2} \\ &= \prod_{k=1}^{n}|a_k|\frac{\left|2\Re f(0)c_2-c_1^2+2\Re f(0)c_1\sum\limits_{k=1}^{n}\frac{1-|a_k|^2}{a_k}\right|}{\left(2\Re f(0)\prod\limits_{k=1}^{n}|a_k|\right)^2-|c_1|^2}. \end{aligned}$$

Thus, we take

$$\begin{aligned} &\frac{2}{1+\prod\limits_{k=1}^{n}|a_k|\dfrac{\left|2\Re f(0)c_2-c_1^2+2\Re f(0)c_1\sum\limits_{k=1}^{n}\frac{1-|a_k|^2}{a_k}\right|}{\left(2\Re f(0)\prod\limits_{k=1}^{n}|a_k|\right)^2-|c_1|^2}} \\ &\qquad \leq \frac{2\Re f(0)\prod\limits_{k=1}^{n}|a_k|+|c_1|}{2\Re f(0)\prod\limits_{k=1}^{n}|a_k|-|c_1|}\left\{\frac{2|f'(c)|}{\Re f(0)}-1-\sum_{k=1}^{n}\frac{1-|a_k|^2}{|c-a_k|^2}\right\} \end{aligned}$$

and

$$\frac{2\left(2\Re f(0)\prod\limits_{k=1}^{n}|a_k|-|c_1|\right)^2}{\left(2\Re f(0)\prod\limits_{k=1}^{n}|a_k|\right)^2-|c_1|^2+\prod\limits_{k=1}^{n}|a_k|\left|2\Re f(0)c_2-c_1^2+2\Re f(0)c_1\sum\limits_{k=1}^{n}\frac{1-|a_k|^2}{a_k}\right|}$$
$$\leq\frac{2|f'(c)|}{\Re f(0)}-1-\sum_{k=1}^{n}\frac{1-|a_k|^2}{|c-a_k|^2}.$$

Hence, we get inequality (2.4). Now, let us show that inequality (2.4) is sharp. Let

$$f(z)=\frac{f(0)-\overline{f(0)}z^2\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}{1+z^2\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}=-\overline{f(0)}+\frac{2\Re f(0)}{1+z^2\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}.$$

Then

$$\begin{aligned} f'(z) &= -2\Re f(0)\frac{\left(2z\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}+z^2\sum\limits_{k=1}^{n}\frac{1-|a_k|^2}{(1-\overline{a_k}z)^2}\prod\limits_{\substack{m=1\\k\neq m}}^{n}\frac{z-a_m}{1-\overline{a_m}z}\right)}{\left(1+z^2\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}\right)^2} \\ &= -2\Re f(0)\frac{z\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}\left(2+z\sum\limits_{k=1}^{n}\frac{1-|a_k|^2}{(1-\overline{a_k}z)}\frac{1}{z-a_k}\right)}{\left(1+z^2\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}\right)^2} \end{aligned}$$

and for $z=1$

$$f'(1)=-2\Re f(0)\frac{\prod\limits_{k=1}^{n}\frac{1-a_k}{1-\overline{a_k}}\left(2+\sum\limits_{k=1}^{n}\frac{1-|a_k|^2}{(1-\overline{a_k})}\frac{1}{1-a_k}\right)}{\left(1+\prod\limits_{k=1}^{n}\frac{1-a_k}{1-\overline{a_k}}\right)^2}.$$

So, we have

$$f'(1)=-2\Re f(0)\frac{\left(2+\sum\limits_{k=1}^{n}\frac{1-a_k^2}{(1-a_k)^2}\right)}{4}=-\frac{\Re f(0)}{2}\left(2+\sum_{k=1}^{n}\frac{1+a_k}{1-a_k}\right)$$

since $a_1,a_2,...,a_n$ are positive real numbers. On the other hand, we obtain

$$f(0)+c_1z+c_2z^2+...=f(0)-\frac{2\Re f(0)z^2\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}{1+z^2\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}},$$

$$c_1z+c_2z^2+...=-\frac{2\Re f(0)z^2\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}{1+z^2\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}$$

and

$$c_1 + c_2 z + \ldots = -\frac{2\Re f(0) z \prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k} z}}{1 + z^2 \prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k} z}}.$$

Passing to limit as $z$ tends to 0 in the last equality, we get $|c_1| = 0$. Similarly, using straightforward calculations, we get $|c_2| = 2\Re f(0) \prod\limits_{k=1}^{n} |a_k|$. Therefore, (2.4) holds.

Also, we obtain

$$\Re\left( \frac{f(0) - \overline{f(0)} z^2 \prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k} z}}{1 + z^2 \prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k} z}} \right) = \Re f(0) \frac{1 - |z|^4 \left( \prod\limits_{k=1}^{n} \left| \frac{z-a_k}{1-\overline{a_k} z} \right| \right)^2}{\left| 1 + z^2 \prod\limits_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k} z} \right|^2} > 0.$$

□

Now we shall estimate the modulus of the angular derivative of the function $f(z)$ from below according to the second Taylor coefficients of $f$ about $z = 0$ and $z = z_0 \neq 0$. Motivated by the results of the work presented in [2], the following result is obtained.

THEOREM 2.5. *Let $f(z) = f(0) + c_1 z + c_2 z^2 + \ldots$ be an analytic function in $D$, $\Re f(z) > 0$ for $|z| < 1$, and $f(z_0) = f(0)$ for $0 < |z_0| < 1$. Assume that, for some $c \in \partial D$, $f$ has an angular limit $f(c)$ at $c$, $f(c) = i\Im f(0)$. Then we have the inequality*

$$|f'(c)| \geq \frac{\Re f(0)}{2} \left( 1 + \frac{1-|z_0|^2}{|c-z_0|^2} + \frac{2\Re f(0)|z_0| - |f'(0)|}{2\Re f(0)|z_0| + |f'(0)|} \right. \tag{2.5}$$
$$\times \left[ 1 + \frac{4(\Re f(0))^2|z_0|^2 + |f'(z_0)|\left(1-|z_0|^2\right)|f'(0)| - 2\Re f(0)|f'(z_0)|\left(1-|z_0|^2\right) - 2\Re f(0)|f'(0)|}{4(\Re f(0))^2|z_0|^2 + |f'(z_0)|\left(1-|z_0|^2\right)|f'(0)| + 2\Re f(0)|f'(z_0)|\left(1-|z_0|^2\right) + 2\Re f(0)|f'(0)|} \frac{1-|z_0|^2}{|c-z_0|^2} \right].$$

*The inequality (2.5) is sharp, with equality for each possible value of $|f'(0)|$ and $|f'(z_0)|$.*

*Proof.* Let

$$\rho(z) = \frac{z - z_0}{1 - \overline{z_0} z}.$$

Also, let $G : D \to D$ be an analytic function and $z_0 \in D$ such that

$$\left| \frac{G(z) - G(z_0)}{1 - \overline{G(z_0)} G(z)} \right| \leq \left| \frac{z - z_0}{1 - \overline{z_0} z} \right| = |\rho(z)|$$

and

$$|G(z)| \leq \frac{|G(z_0)| + |\rho(z)|}{1 + |G(z_0)|\,|\rho(z)|} \tag{2.6}$$

by Schwarz-Pick Lemma [5]. For an analytic function $\omega : D \to D$ and $0 < |z_0| < 1$, if we consider the following function

$$G(z) = \frac{\omega(z) - \omega(0)}{z\left(1 - \overline{\omega(0)} \omega(z)\right)}$$

in (2.6), we get

$$\left|\frac{\omega(z)-\omega(0)}{z\left(1-\overline{\omega(0)}\omega(z)\right)}\right| \leq \frac{\left|\frac{\omega(z_0)-\omega(0)}{z_0\left(1-\overline{\omega(0)}\omega(z_0)\right)}\right| + |\rho(z)|}{1+\left|\frac{\omega(z_0)-\omega(0)}{z_0\left(1-\overline{\omega(0)}\omega(z_0)\right)}\right| |\rho(z)|}$$

and

$$|\omega(z)| \leq \frac{|\omega(0)| + |z| \frac{|C|+|\rho(z)|}{1+|C||\rho(z)|}}{1+|\omega(0)| |z| \frac{|C|+|\rho(z)|}{1+|C||\rho(z)|}}, \tag{2.7}$$

where

$$C = \frac{\omega(z_0)-\omega(0)}{z_0\left(1-\overline{\omega(0)}\omega(z_0)\right)}.$$

Without loss of generality, we can take $c = 1$. If we take

$$\omega(z) = \frac{\vartheta(z)}{z\left(\frac{z-z_0}{1-\overline{z_0}z}\right)},$$

then

$$\omega(0) = \frac{\vartheta'(0)}{-z_0},\ \omega(z_0) = \frac{\vartheta'(z_0)\left(1-|z_0|^2\right)}{z_0}$$

and

$$C = \frac{\frac{\vartheta'(z_0)\left(1-|z_0|^2\right)}{z_0} + \frac{\vartheta'(0)}{z_0}}{z_0\left(1+\frac{\overline{\vartheta'(0)}}{\overline{z_0}}\frac{\vartheta'(z_0)\left(1-|z_0|^2\right)}{z_0}\right)},$$

where $|C| \leq 1$. Let $|\omega(0)| = \beta$ and

$$M = \frac{\left|\frac{\vartheta'(z_0)\left(1-|z_0|^2\right)}{z_0}\right| + \left|\frac{\vartheta'(0)}{z_0}\right|}{|z_0|\left(1+\left|\frac{\vartheta'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|\left|\frac{\vartheta'(0)}{z_0}\right|\right)}.$$

By (2.7), we get

$$|\vartheta(z)| \leq |z| |\rho(z)| \frac{\beta + |z| \frac{M+|\rho(z)|}{1+M|\rho(z)|}}{1+\beta |z| \frac{M+|\rho(z)|}{1+M|\rho(z)|}}$$

and

$$\frac{1-|\vartheta(z)|}{1-|z|} \geq \frac{1+\beta |z| \frac{M+|\rho(z)|}{1+M|\rho(z)|} - \beta |z| |\rho(z)| - |\rho(z)| |z|^2 \frac{M+|\rho(z)|}{1+M|\rho(z)|}}{(1-|z|)\left(1+\beta |z| \frac{M+|\rho(z)|}{1+M|\rho(z)|}\right)}.$$

Let $\kappa(z) = 1 + \beta |z| \frac{M+|\rho(z)|}{1+M|\rho(z)|}$ and $\tau(z) = 1 + M |\rho(z)|$. Considering the functions $\kappa$ and $\tau$ in the above inequality, we have

$$\frac{1-|\vartheta(z)|}{1-|z|} \geq \frac{1}{\kappa(z)\tau(z)}\left\{\frac{1-|z|^2 |\rho(z)|^2}{1-|z|} + M |\rho(z)| \frac{1-|z|^2}{1-|z|} + \beta |z| M \frac{1-|\rho(z)|^2}{1-|z|}\right\}. \tag{2.8}$$

Since

$$\lim_{z\to 1}\kappa(z)=\lim_{z\to 1}\left(1+\beta\left|z\right|\frac{M+\left|\rho(z)\right|}{1+M\left|\rho(z)\right|}\right)=1+\beta,$$

$$\lim_{z\to 1}\tau(z)=\lim_{z\to 1}\left(1+M\left|\rho(z)\right|\right)=1+M,$$

$$\lim_{z\to 1}\frac{1-\left|z\right|^{i}\left|\frac{z-z_0}{1-\overline{z_0}z}\right|^{j}}{1-\left|z\right|}=i+j\frac{1-\left|z_0\right|^2}{\left|1-z_0\right|^2}$$

for non-negative integers $i$ and $j$ and

$$1-\left|\rho(z)\right|^2=1-\left|\frac{z-z_0}{1-\overline{z_0}z}\right|^2=\frac{\left(1-\left|z_0\right|^2\right)\left(1-\left|z\right|^2\right)}{\left|1-\overline{z_0}z\right|^2},$$

passing to the angular limit in (2.8) gives

$$\left|\vartheta'(1)\right|\geq 1+\frac{1-\left|z_0\right|^2}{\left|1-z_0\right|^2}+\frac{1-\beta}{1+\beta}\left[1+\frac{1-M}{1+M}\frac{1-\left|z_0\right|^2}{\left|1-z_0\right|^2}\right].$$

Moreover, since

$$\frac{1-\beta}{1+\beta}=\frac{1-\left|\omega(0)\right|}{1+\left|\omega(0)\right|}=\frac{1-\left|\frac{\vartheta'(0)}{z_0}\right|}{1+\left|\frac{\vartheta'(0)}{z_0}\right|}=\frac{\left|z_0\right|-\left|\vartheta'(0)\right|}{\left|z_0\right|+\left|\vartheta'(0)\right|}=\frac{2\Re f(0)\left|z_0\right|-\left|f'(0)\right|}{2\Re f(0)\left|z_0\right|+\left|f'(0)\right|},$$

$$\frac{1-M}{1+M}=\frac{1-\frac{\left|\frac{\vartheta'(z_0)\left(1-\left|z_0\right|^2\right)}{z_0}\right|+\left|\frac{\vartheta'(0)}{z_0}\right|}{\left|z_0\right|\left(1+\left|\frac{\vartheta'(z_0)(1-\left|z_0\right|^2)}{z_0}\right|\left|\frac{\vartheta'(0)}{z_0}\right|\right)}}{1+\frac{\left|\frac{\vartheta'(z_0)(1-\left|z_0\right|^2)}{z_0}\right|+\left|\frac{\vartheta'(0)}{z_0}\right|}{\left|z_0\right|\left(1+\left|\frac{\vartheta'(z_0)(1-\left|z_0\right|^2)}{z_0}\right|\left|\frac{\vartheta'(0)}{z_0}\right|\right)}},$$

$$\frac{1-M}{1+M}=\frac{\left|z_0\right|^2+\left|\vartheta'(z_0)\right|\left(1-\left|z_0\right|^2\right)\left|\vartheta'(0)\right|-\left|\vartheta'(z_0)\right|\left(1-\left|z_0\right|^2\right)-\left|\vartheta'(0)\right|}{\left|z_0\right|^2+\left|\vartheta'(z_0)\right|\left(1-\left|z_0\right|^2\right)\left|\vartheta'(0)\right|+\left|\vartheta'(z_0)\right|\left(1-\left|z_0\right|^2\right)+\left|\vartheta'(0)\right|}$$

and,

$$\frac{1-M}{1+M}=\frac{4(\Re f(0))^2\left|z_0\right|^2+\left|f'(z_0)\right|\left(1-\left|z_0\right|^2\right)\left|f'(0)\right|-2\Re f(0)\left|f'(z_0)\right|\left(1-\left|z_0\right|^2\right)-2\Re f(0)\left|f'(0)\right|}{4(\Re f(0))^2\left|z_0\right|^2+\left|f'(z_0)\right|\left(1-\left|z_0\right|^2\right)\left|f'(0)\right|+2\Re f(0)\left|f'(z_0)\right|\left(1-\left|z_0\right|^2\right)+2\Re f(0)\left|f'(0)\right|},$$

we obtain

$$\left|\vartheta'(1)\right|\geq 1+\frac{1-\left|z_0\right|^2}{\left|1-z_0\right|^2}+\frac{2\Re f(0)\left|z_0\right|-\left|f'(0)\right|}{2\Re f(0)\left|z_0\right|+\left|f'(0)\right|}$$
$$\times\left[1+\frac{4(\Re f(0))^2\left|z_0\right|^2+\left|f'(z_0)\right|\left(1-\left|z_0\right|^2\right)\left|f'(0)\right|-2\Re f(0)\left|f'(z_0)\right|\left(1-\left|z_0\right|^2\right)-2\Re f(0)\left|f'(0)\right|}{4(\Re f(0))^2\left|z_0\right|^2+\left|f'(z_0)\right|\left(1-\left|z_0\right|^2\right)\left|f'(0)\right|+2\Re f(0)\left|f'(z_0)\right|\left(1-\left|z_0\right|^2\right)+2\Re f(0)\left|f'(0)\right|}\frac{1-\left|z_0\right|^2}{\left|1-z_0\right|^2}\right].$$

From definition of $\vartheta(z)$, we have

$$\left|\vartheta'(1)\right|=\frac{2\left|f'(1)\right|}{\Re f(0)}.$$

We thus take the inequality (2.5). Let us choose arbitrary real numbers $z_0$, $x$ and $y$ such that $0 < x = |\vartheta'(0)| < |z_0|^2$, $0 < y = |\vartheta'(z_0)| < \frac{|z_0|^2}{\left(1-|z_0|^2\right)^2}$ to show that the inequality (2.5) is sharp. Let

$$\vartheta(z) = z\left(\frac{z-z_0}{1-\overline{z_0}z}\right)\frac{-\frac{x}{z_0}+z\frac{\mathtt{N}+\frac{z-z_0}{1-\overline{z_0}z}}{1+\mathtt{N}\frac{z-z_0}{1-\overline{z_0}z}}}{1-\frac{x}{z_0}z\frac{\mathtt{N}+\frac{z-z_0}{1-\overline{z_0}z}}{1+\mathtt{N}\frac{z-z_0}{1-\overline{z_0}z}}}, \tag{2.9}$$

where

$$\mathtt{N} = \frac{1}{z_0^2}\frac{y\left(1-|z_0|^2\right)+x}{1+xy\frac{1-|z_0|^2}{z_0^2}}.$$

From (2.9), with the simple calculations, we obtain

$$\vartheta'(0) = x,\ \vartheta'(z_0) = y$$

and

$$\vartheta'(1) = 1+\frac{1-z_0^2}{(1-z_0)^2}+\frac{z_0+x}{z_0-x}\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{z_0^2+xy\left(1-|z_0|^2\right)-y\left(1-|z_0|^2\right)-x}{z_0^2+xy\left(1-|z_0|^2\right)+y\left(1-|z_0|^2\right)+x}\right).$$

Choosing suitable signs of the numbers $z_0$, $x$ and $y$, we conclude from the last equality that the inequality (2.5) is sharp. Also, we take

$$\Re\left(f(z)\right) = \Re\left(\frac{f(0)-\overline{f(0)}\vartheta(z)}{1+\vartheta(z)}\right) = \Re f(0)\frac{1-|\vartheta(z)|^2}{|1+\vartheta(z)|^2} > 0.$$

□

## References


[1] T. Akyel and B. N. Örnek, *Some remarks on Schwarz lemma at the boundary*, Filomat, **31** (13) (2017), 4139–4151.

[2] T. A. Azeroğlu and B. N. Örnek, *A refined Schwarz inequality on the boundary*, Complex Variables and Elliptic Equations, **58** (2013), 571–577.

[3] H. P. Boas, *Julius and Julia: Mastering the art of the Schwarz lemma,* American Mathematical Monthly, **117** (2010), 770–785.

[4] V. N. Dubinin, *The Schwarz inequality on the boundary for functions regular in the disc*, J. Math. Sci., **122** (2004), 3623–3629.

[5] G. M. Golusin, Geometric Theory of Functions of Complex Variable Translations of Mathematical Monographs 26, American Mathematical Society, 1969.

[6] Z. Huang, D. Zhao and H. Li, *A boundary Schwarz lemma for pluriharmonic mappings between the unit polydiscs of any dimensions*, Filomat, **34** (2020), 3151–3160.

[7] M. Mateljević, N. Mutavdzc, B. N. Örnek, *Note on some classes of holomorphic functions related to Jack's and Schwarz's Lemma*,
DOI: 10.13140/RG.2.2.25744.15369, ResearchGate.

[8] P. R. Mercer, *Sharpened versions of the Schwarz Lemma*, Journal of Mathematical Analysis and Applications, **205** (1997), 508–511.

[9] P. R. Mercer, *Boundary Schwarz inequalities arising from Rogosinski's lemma*, Journal of Classical Analysis, **12** (2018), 93–97.

[10] P. R. Mercer, *An improved Schwarz Lemma at the boundary* , Open Mathematics, **16** (2018), 1140–1144.

[11] R. Osserman, *A sharp Schwarz inequality on the boundary*, Proc. Amer. Math. Soc., **128** (2000), 3513–3517.

[12] B. N. Örnek, *Bounds of Hankel determinants for analytic function*, Korean J. Math., **28** (4) (2020), 699–715.

[13] B. N. Ornek and T. Düzenli, *Boundary analysis for the derivative of driving point impedance functions*, IEEE Transactions on Circuits and Systems II: Express Briefs, **65** (9) (2018), 1149–1153.

[14] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin, 1992.

[15] H. Unkelbach, *Über die Randverzerrung bei konformer Abbildung*, Mathematische Zeitschrift, **43** (1938), 739–742.

**Bülent Nafi Örnek**
Department of Computer Engineering, Amasya University
Merkez-Amasya, 05100, Turkey
*E-mail*: nafiornek@gmail.com, nafi.ornek@amasya.edu.tr

**Tuğba Akyel**
The Faculty of Engineering and Natural Sciences, Maltepe University
Maltepe, Istanbul, Turkey
*E-mail*: tugbaakyel@maltepe.edu.tr