Korean J. Math. **29** (2021), No. 4, pp. 741–747
http://dx.doi.org/10.11568/kjm.2021.29.4.741

THE CAYLEY-BACHARACH THEOREM VIA TRUNCATED MOMENT PROBLEMS

Seonguk Yoo

Abstract. The Cayley–Bacharach theorem says that every cubic curve on an algebraically closed field that passes through a given 8 points must contain a fixed ninth point, counting multiplicities. Ren et al. introduced a concrete formula for the ninth point in terms of the 8 points [4]. We would like to consider a different approach to find the ninth point via the theory of truncated moment problems. Various connections between algebraic geometry and truncated moment problems have been discussed recently; thus, the main result of this note aims to observe an interplay between linear algebra, operator theory, and real algebraic geometry.

1. Introduction

We begin with the following classical results from algebraic geometry.

Theorem 1.1. [1, Chasles] *Let C_1, C_2 be cubic plane curves meeting in nine points $P_1, \ldots, P_9$ in $\mathbb{P}^2$. If C is any cubic containing $P_1, \ldots, P_8$, then C also passes through P_9.*

Theorem 1.2 (Cayley–Bacharach). *Let $P_1, \ldots, P_8$ be eight distinct points in the plane, no three on a line and no six on a conic.There exists a unique ninth point P_9 such that every cubic curve through $P_1, \ldots, P_8$ also contains P_9.*

The point P_9 in Theorem 1.2 is called the *Cayley–Bacharach point* and an explicit formulas for it was presented in terms of algebraic invariants of the other eight points in [4]. In this paper, we would like to review the formula briefly and introduce a more intuitive way to find the formula via truncated moment problems. The formula was given in the projection plane $\mathbb{P}^2$; if $P_9 = (x_9 : y_9 : z_9)$ is the ninth intersecting point of $P_1 = (x_1 : y_1 : z_1)$, $P_2 = (x_2 : y_2 : z_2)$, ..., $P_8 = (x_8 : y_8 : z_8)$, then by Theorem 1.2, P_9 only depends on $P_1, \ldots, P_8$. This means that each coordinate of the ninth point $(x_9/z_9, y_9/z_9)$ in $\mathbb{R}^2$ can be written as rational functions in the 24 unknowns of $x_1, y_1, z_1, \ldots, x_8, y_8, z_8$. Observe that the two rational functions consist of polynomials with integer coefficients in their numerators and denominators.

Received August 2, 2021. Revised November 8, 2021. Accepted December 16, 2021.
2010 Mathematics Subject Classification: 44A60, 47B35, 15A83, 15A60, 47A30, 15-04.
Key words and phrases: truncated moment problems, moment matrix extensions, rank-one decomposition, consistency.
The author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (2020R1F1A1A01070552).

We now need to define some polynomials that will be ingredients in the formula. If six points lie on a conic, then the cubic polynomial derived from the determinant in (1.1) would be equal to zero. For $P_i = (x_i, y_i, z_i)$, $i = 1, \ldots, 8$, define

$$C\left(P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}, P_{i_5}, P_{i_6}\right) = \det \begin{pmatrix} x_{i_1}^2 & x_{i_1}y_{i_1} & x_{i_1}z_{i_1} & y_{i_1}^2 & y_{i_1}z_{i_1} & z_{i_1}^2 \\ x_{i_2}^2 & x_{i_2}y_{i_2} & x_{i_2}z_{i_2} & y_{i_2}^2 & y_{i_2}z_{i_2} & z_{i_2}^2 \\ x_{i_3}^2 & x_{i_3}y_{i_3} & x_{i_3}z_{i_3} & y_{i_3}^2 & y_{i_3}z_{i_3} & z_{i_3}^2 \\ x_{i_4}^2 & x_{i_4}y_{i_4} & x_{i_4}z_{i_4} & y_{i_4}^2 & y_{i_4}z_{i_4} & z_{i_4}^2 \\ x_{i_5}^2 & x_{i_5}y_{i_5} & x_{i_5}z_{i_5} & y_{i_5}^2 & y_{i_5}z_{i_5} & z_{i_5}^2 \\ x_{i_6}^2 & x_{i_6}y_{i_6} & x_{i_6}z_{i_6} & y_{i_6}^2 & y_{i_6}z_{i_6} & z_{i_6}^2 \end{pmatrix}. \tag{1.1}$$

If eight points lie on a cubic curve, then the matrix in (1.2) is singular at the first point P_k, and define

$$D\left(P_k; P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}, P_{i_5}, P_{i_6}, P_{i_7}\right) :=$$

$$\det \begin{pmatrix} x_{i_1}^3 & x_{i_1}^2y_{i_1} & x_{i_1}^2z_{i_1} & x_{i_1}y_{i_1}^2 & x_{i_1}y_{i_1}z_{i_1} & x_{i_1}z_{i_1}^2 & y_{i_1}^3 & y_{i_1}^2z_{i_1} & y_{i_1}z_{i_1}^2 & z_{i_1}^3 \\ x_{i_2}^3 & x_{i_2}^2y_{i_2} & x_{i_2}^2z_{i_2} & x_{i_2}y_{i_2}^2 & x_{i_2}y_{i_2}z_{i_2} & x_{i_2}z_{i_2}^2 & y_{i_2}^3 & y_{i_2}^2z_{i_2} & y_{i_2}z_{i_2}^2 & z_{i_2}^3 \\ x_{i_3}^3 & x_{i_3}^2y_{i_3} & x_{i_3}^2z_{i_3} & x_{i_3}y_{i_3}^2 & x_{i_3}y_{i_3}z_{i_3} & x_{i_3}z_{i_3}^2 & y_{i_3}^3 & y_{i_3}^2z_{i_3} & y_{i_3}z_{i_3}^2 & z_{i_3}^3 \\ x_{i_4}^3 & x_{i_4}^2y_{i_4} & x_{i_4}^2z_{i_4} & x_{i_4}y_{i_4}^2 & x_{i_4}y_{i_4}z_{i_4} & x_{i_4}z_{i_4}^2 & y_{i_4}^3 & y_{i_4}^2z_{i_4} & y_{i_4}z_{i_4}^2 & z_{i_4}^3 \\ x_{i_5}^3 & x_{i_5}^2y_{i_5} & x_{i_5}^2z_{i_5} & x_{i_5}y_{i_5}^2 & x_{i_5}y_{i_5}z_{i_5} & x_{i_5}z_{i_5}^2 & y_{i_5}^3 & y_{i_5}^2z_{i_5} & y_{i_5}z_{i_5}^2 & z_{i_5}^3 \\ x_{i_6}^3 & x_{i_6}^2y_{i_6} & x_{i_6}^2z_{i_6} & x_{i_6}y_{i_6}^2 & x_{i_6}y_{i_6}z_{i_6} & x_{i_6}z_{i_6}^2 & y_{i_6}^3 & y_{i_6}^2z_{i_6} & y_{i_6}z_{i_6}^2 & z_{i_6}^3 \\ x_{i_7}^3 & x_{i_7}^2y_{i_7} & x_{i_7}^2z_{i_7} & x_{i_7}y_{i_7}^2 & x_{i_7}y_{i_7}z_{i_7} & x_{i_7}z_{i_7}^2 & y_{i_7}^3 & y_{i_7}^2z_{i_7} & y_{i_7}z_{i_7}^2 & z_{i_7}^3 \\ 3x_k^2 & 2x_ky_k & 2x_kz_k & y_k^2 & y_kz_k & z_k^2 & 0 & 0 & 0 & 0 \\ 0 & x_k^2 & 0 & 2x_ky_k & x_kz_k & 0 & 3y_k^2 & 2y_kz_k & z_k^2 & 0 \\ 0 & 0 & x_k^2 & 0 & x_ky_k & 2x_kz_k & 0 & y_k^2 & 2y_kz_k & 3z_k^2 \end{pmatrix}. \tag{1.2}$$

To state the formula for the Cayley-Bacharach point, we also need to define:

$$\begin{aligned} C_x &= C(P_1, P_4, P_5, P_6, P_7, P_8), & C_y &= C(P_2, P_4, P_5, P_6, P_7, P_8), \\ C_z &= C(P_3, P_4, P_5, P_6, P_7, P_8), & D_x &= D(P_1; P_2, P_3, P_4, P_5, P_6, P_7, P_8), \\ D_y &= D(P_2; P_3, P_1, P_4, P_5, P_6, P_7, P_8), & D_z &= D(P_3; P_1, P_2, P_4, P_5, P_6, P_7, P_8). \end{aligned}$$

Theorem 1.3. *[4] For the given 8 points $P_i = (x_i, y_i, z_i)$, $i = 1, \ldots, 8$ as in Theorem 1.2, the Cayley–Bacharach point $P_9 = (x_9, y_9, z_9)$ is given by the formula*

$$\begin{aligned} x_9 &= C_xD_yD_zx_1 + D_xC_yD_zx_2 + D_xD_yC_zx_3, \\ y_9 &= C_xD_yD_zy_1 + D_xC_yD_zy_2 + D_xD_yC_zy_3, \\ z_9 &= C_xD_yD_zz_1 + D_xC_yD_zz_2 + D_xD_yC_zz_3. \end{aligned} \tag{1.3}$$

Example 1.4. Consider 8 points $P_i = (i, i^3)$ on $y = x^3$ for $i = 1, \ldots, 8$. Note that no three of the eight points lie on a line and no six of the eight lie on a conic. We may use custom $(x, y) \approx (x : y : 1)$ for $(x, y) \in \mathbb{R}^2$ and $(x : y : 1) \in \mathbb{P}^2$. A calculation shows

$$\begin{aligned} C_x &= -22498560, & C_y &= -6635520, & C_z &= -1140480, \\ D_x &= 70160113336320000, & D_y &= -10293761802240000, & D_z &= 3521550090240000, \end{aligned}$$

which gives

$$x_9 = 766896147865211049499793817600000000,$$

$$y_9 = 99389740763331352015173278760960000000000,$$

$$z_9 = -213026707740336402638831616000000000.$$

We see finally that the Cayley-Bacharach point $P_9 = (u, v)$ is obtained as

$$u = \frac{x_9}{z_9} = -36, \qquad v = \frac{y_9}{z_9} = -46656.$$

Thus, any cubic passing through $P_1 = (1, 1^3), \ldots, P_8 = (8, 8^3)$ must contain the ninth point $(-36, (-36)^3)$.

2. Truncated Moment Problems

We now want to discuss the Cayley-Barach point through the truncated moment problem, so we need a quick review about basic moment theories. Let $\beta \equiv \beta^{(m)} = \{\beta_{00}, \beta_{10}, \beta_{01}, \cdots, \beta_{m,0}, \beta_{m-1,1}, \cdots, \beta_{1,m-1}, \beta_{0,m}\}$ with $\beta_{00} > 0$ denote a real 2-dimensional multisequence of order m. The *truncated moment problem* entails finding necessary and sufficient conditions for the existence of a positive Borel measure μ such that supp $\mu \subseteq \mathbb{R}^2$ and

$$\beta_{ij} = \int x^i y^j \; d\mu \;\; (i, j \in \mathbb{Z}_+, \; 0 \leq i + j \leq m).$$

In this case, we call μ a *representing measure* for β or the moment matrix $M(n)$ that will be defined right below. When the order of a moment sequence is even like $m = 2n$ for some $n \in \mathbb{N}$, it is possible to define the *moment matrix* $M(n) \equiv M(n)(\beta^{(2n)})$ of β as

$$M(n) \equiv M(n)(\beta^{(2n)}) := (\beta_{\mathbf{i}+\mathbf{j}})_{\mathbf{i},\mathbf{j} \in \mathbb{Z}_+^2 : |\mathbf{i}|, |\mathbf{j}| \leq n}.$$

It is well-known that for the existence of a representing measure for β, $M(n)$ must be positive semidefinite (in symbol, $M(n) \geq 0$). There are more necessary conditions; to introduce them, let $\mathcal{P}_n$ denote the set of bivariate polynomials in $\mathbb{R}[x, y]$ whose degree is at most n. We label the columns in $M(n)$ with monomials *1*$, X, Y, X^2, XY, Y^2, \ldots, X^n, \ldots, Y^n$ in the degree-lexicographic order. For example, a sextic moment matrix $M(3)$ looks like

$$\begin{array}{rc}
 & \begin{array}{cccccccccc} 1 & X & Y & X^2 & XY & Y^2 & X^3 & X^2Y & XY^2 & Y^3 \end{array} \\
\begin{array}{r} 1 \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \\ X^3 \\ X^2Y \\ XY^2 \\ Y^3 \end{array} &
\left(\begin{array}{c|cc|ccc|cccc}
\beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{03} \\ \hline
\beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{13} \\
\beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{04} \\ \hline
\beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} & \beta_{50} & \beta_{41} & \beta_{32} & \beta_{23} \\
\beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{32} \\
\beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} & \beta_{32} & \beta_{23} & \beta_{32} & \beta_{23} \\ \hline
\beta_{30} & \beta_{40} & \beta_{31} & \beta_{50} & \beta_{41} & \beta_{32} & \beta_{60} & \beta_{51} & \beta_{42} & \beta_{33} \\
\beta_{21} & \beta_{31} & \beta_{22} & \beta_{41} & \beta_{32} & \beta_{23} & \beta_{51} & \beta_{42} & \beta_{33} & \beta_{24} \\
\beta_{12} & \beta_{22} & \beta_{13} & \beta_{32} & \beta_{23} & \beta_{13} & \beta_{42} & \beta_{33} & \beta_{24} & \beta_{15} \\
\beta_{03} & \beta_{13} & \beta_{04} & \beta_{23} & \beta_{13} & \beta_{05} & \beta_{33} & \beta_{24} & \beta_{15} & \beta_{06}
\end{array}\right)
\end{array}.$$

When $M(n)$ has a column relation, it can be written as $p(X, Y) = \mathbf{0}$ for some polynomial $p(x, y) = \sum_{ij} a_{ij} x^i y^j \in \mathcal{P}_n$; this is so-called functional calculus. Column relations in $M(n)$ are not a polynomial but they can be regarded as polynomials and give crucial information about a representing measure for $M(n)$. For instance, if β has a representing measure, then the following condition must hold:

$$p(X, Y) = \mathbf{0} \Longrightarrow (p\,q)(X, Y) = \mathbf{0} \text{ for each polynomial } q, \text{ with } \deg(p\,q) \leq n.$$

In this case, $M(n)$ is said to be *recursively generated.*

Let $\mathcal{Z}(p)$ be the zero set of p and define the *algebraic variety* of β by

$$\mathcal{V} \equiv \mathcal{V}\left(M(n)\right) \equiv \mathcal{V}(\beta) := \bigcap\nolimits_{p\,(X,Y)=\mathbf{0},\,\deg\, p \le n} \mathcal{Z}(p). \tag{2.1}$$

As in [2], if $M(n)$ admits a representing measure μ, then

$$\text{supp } \mu \subseteq \mathcal{V}(\beta) \quad \text{ and } \quad \text{rank } M(n) \le \text{card supp } \mu \le \text{card}\, \mathcal{V}.$$

The second inequality in the above is called the *variety condition.*

The most important result in the truncated moment theory is the Flat Extension Theorem that says if $M(n)$ admits a rank-preserving positive extension $M(n+1)$, then β has a rank $M(n)$-atomic representing measure. The extension $M(n+1)$ is called a *flat extension.* Another contribution of this result enables us to obtain a closed form of a representing measure as appeared in the coming Example 2.2. A special case is when rank $M(n) =$ rank $M(n-1)$ in which $M(n)$ is said to be *flat*; in this situation, β has a unique rank $M(n)$-atomic representing measure.

We now summarize a way to build a flat extension. Observe that each rectangular block with the same order moments of $M(n)$ is Hankel, and that an extension $M(n+1)$ can be written as $M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix}$, for some matrices B and C. To make sure a prospective moment matrix $M(n+1)$ is positive semidefinite, we use the following classical result:

Theorem 2.1. *[5, Smul'jan's Theorem] Let A, B, C be matrices of complex numbers, with A and C square matrices. Then*

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \ge 0 \iff \begin{cases} A \ge 0 \\ B = AW \ (\text{for some } W). \\ C \ge W^*AW \end{cases}$$

Moreover, rank $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \text{rank } A \iff C = W^*AW$

Incidentally, solving truncated moment problems can be thought of as finding roots of a system of multivariate polynomial equations. C.Bayer and J. Teichmann showed if a moment sequence $\beta^{(2n)}$ admits one or more representing measures, one of them must be finitely atomic. Thus, if a real sequence $\beta^{(2n)}$ has a finitely atomic representing measure μ, then it can be written as

$$\mu := \sum_{k=1}^{\ell} \rho_k \delta_{w_k},$$

where $\ell \le \dim \mathcal{P}_{2n}$. We then try to find positive numbers $\rho_1, \ldots, \ldots, \rho_k$ called *densities* and points $(x_1, y_1), \ldots, (x_k, y_k)$ called *atoms* of the measure such that for $i, j \in \mathbb{Z}_+$ and $0 \le i + j \le n$

$$\beta_{ij} = \rho_1 x_1^i y_1^j + \cdots + \rho_\ell x_\ell^i y_\ell^j = \int x^i y^j d\mu.$$

Next, let us illustrate how the flat extension theorem is applied:

EXAMPLE 2.2. Consider a quadratic moment sequence $\beta^{(4)} \equiv \{\beta_{ij}\} = \{5, 5, 14, 5, 14, 50\}$ and write its moment matrix as

$$M(1) = \begin{pmatrix} 5 & 5 & 14 \\ 5 & 5 & 14 \\ 14 & 14 & 50 \end{pmatrix}$$

Note that $M(1)$ is positive semidefinite and has a column relation $X = \mathit{1}$. To build a flat $M(2)$, we need to make sure an extension $M(2)$ is recursively generated; we impose on $M(2)$ to have the column relations $X^2 = \mathit{1}$ and $XY = 1$ and get

$$\begin{pmatrix} M(1) & B(2) \end{pmatrix} = \begin{pmatrix} 5 & 5 & 14 & 5 & 14 & 50 \\ 5 & 5 & 14 & 5 & 14 & 50 \\ 14 & 14 & 50 & 14 & 50 & \beta_{03} \end{pmatrix}$$

Next find W such that $M(1)W = B(2)$ and get, for $k_1, k_2, k_3 \in \mathbb{R}$,

$$W = \begin{pmatrix} 1 - k_1 & -k_2 & (-7\,\beta_{03} - 27\,k_3 + 1250)/27 \\ k_1 & k_2 & k_3 \\ 0 & 1 & (5\beta_{03} - 700)/54 \end{pmatrix}.$$

We then evaluate $C(2) = W^* M(1) W$ and obtain

$$\begin{pmatrix} 5 & 5 & 14 & 5 & 14 & 50 \\ 5 & 5 & 14 & 5 & 14 & 50 \\ 14 & 14 & 50 & 14 & 50 & \beta_{03} \\ 5 & 5 & 14 & 5 & 14 & 50 \\ 14 & 14 & 50 & 14 & 50 & \beta_{03} \\ 50 & 50 & \beta_{03} & 50 & \beta_{03} & 5(\beta_{03}^2 - 280\beta_{03} + 25000)/54 \end{pmatrix}$$

The column relations in $M(2)$ are $X = \mathit{1}, X^2 = \mathit{1}, XY = Y$, and

$$Y^2 = \frac{-7\beta_{03} - 27k + 1250}{27}\mathit{1} + kX + \frac{(5\beta_{03} - 700)}{54}Y$$

for some $k \in \mathbb{R}$; that is, $M(1)$ admits a flat extension and it follows from the flat extension theorem that it has a 2-atomic representing measure. If we take a specific $\beta_{03} = 194$, then the algebraic variety becomes $\mathcal{V} = \{(1, 1), (1, 4)\}$. To find the densities, we need to solve the Vandermonde equation:

$$\begin{pmatrix} 1 & 1 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} \beta_{00} \\ \beta_{01} \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 14 \end{pmatrix}.$$

We finally have $\rho_1 = 2$, $\rho_2 = 3$ and a representing measure for $\beta^{(4)}$ is $\mu = 2\,\delta_{(1,1)} + 3\,\delta_{(1,4)}$.

Lastly, we recall a result which is very useful to locate the support of a representing measure for a truncated moment sequence.

PROPOSITION 2.3. *[2, Proposition 3.1] Suppose μ is a representing measure for β. For $p \in \mathcal{P}_n$,*

$$\text{supp } \mu \subseteq \mathcal{Z}(p) \iff p(X, Y) = \mathbf{0}.$$

3. Main results

We are ready to see how we find the Cayley-Bachrach point as a tool of truncated moment problems.

THEOREM 3.1. *Let $P_1 = (x_1, y_1), \dots, P_8 = (x_8, y_8)$ be eight distinct points in the plane, no three on a line and no six on a conic. If $M(3)$ is the sextic moment matrix generated by the 8-atomic measure $\mu = \sum_{n=1}^{8} \delta_{(x_i,y_i)}$, then the Cayley-Bacharach point is contained in $\mathcal{V}(M(3))$.*

Proof. Suppose $P_1 = (x_1, y_1), \dots, P_8 = (x_8, y_8)$ are eight distinct points in the plane such that no three of them lie on a line and no six of them lie on a conic. Let $\mu = \sum_{k=1}^{8} \delta_{(x_k,y_k)}$ and suppose $M(3)$ is the sextic moment matrix generated by μ. Observe from Proposition 2.3 that $M(3)$ cannot have a linear or conic column relation, since no three of $P_1, \dots, P_8$ lie on a line and no six of them lie on a conic. Thus, we know that $M(3)$ has cubic column relations whose zero set contains all the 8 points $P_1, \dots, P_8$. We now claim that $M(3)$ has at most 3 different cubic column relations. For, the variety condition says rank $M(3) \leq$ card supp $\mu = 8$ and there is no linear or conic column relation in $M(3)$; thus, rank $M(3)$ is at least 6. If rank $M(3) = 6$, then $M(3)$ is flat and has a unique 6-atomic representing measure, which is not possible. The polynomials obtained from the column relations of $M(3)$ share the 8 points $P_1, \dots, P_8$, and hence $\mathcal{V}(M(3))$ contains the Cayley-Bacharach point by Theorem 1.2. □

Due to Theorem 3.1, we can easily find the Cayley-Bacharach point for the given 8 points. Indeed, we only need to use basic knowledge about linear algebra. The next example illustrates how the ninth point can be discovered.

EXAMPLE 3.2. We revisit Example 1.4. Construct $M(3)$ with the 8-atomic representing measure $\mu = \sum_{k=1}^{8} \delta_{(x_k,y_k)}$, where $x_k = 1, \dots, 8$ and $y_k = x_k^3$; that is, $M(3)$ has the moments $\beta_{ij} = \sum_{k=1}^{8} x_k^i y_k^j$ of the form. After row reduction of $M(3)$, we get

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1451520 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3904704 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 2304100 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -4142880 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -740880 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -15120 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 140847 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 750 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

We write the 2 column relations from the 7th and 10th columns as follows:

$$\begin{aligned}
&X^3 = Y, \\
&Y^3 = -1451520 \cdot \mathit{1} + 3904704x + 2304100y - 4142880x^2 \\
&\qquad - 740880xy - 15120y^2 + 140847x^2y + 750xy^2.
\end{aligned} \tag{3.1}$$

To compute the algebraic variety $\mathcal{V}(M(3))$, we find the intersection of the 2 polynomials corresponding to (3.1) and see that $\mathcal{V}(M(3))$ contains $(-36, (-36)^3)$ besides the given 8 points. This confirms our approach is relevant to Theorem 1.3.

Concluding Remark. Moment theory is widely applied in algebraic geometry, optimization theory, image reconstruction, and so on. The main result of this paper shows that a certain geometric phenomenon can be interpreted through the results of the truncated moment problem. In most cases where the moment matrix $M(3)$ has a column relation, the necessary and sufficient conditions for having a representing measure for $M(3)$ have been discovered. However, there should be a case of rank $M(3) = 7$ and card $\mathcal{V}(M(3)) = 8$ (or 9) according to the moment theory, but no concrete example has been found yet. Because of the rigidity appeared in the Cayley-Bacharach theorem, such an example is probably not expected to exist. This topic seems to be interesting for future research.

References

[1] M. Chasles, *Traité des sections coniques*, Gauthier-Villars, Paris, 1885.
[2] R. Curto and L. Fialkow, *Solution of the truncated complex moment problem with flat data*, *Memoirs Amer. Math. Soc.* no. 568, Amer. Math. Soc., Providence, 1996.
[3] D. Eisenbud, M. Green, Mark, and J. Harris, *Cayley-Bacharach theorems and conjectures*, *Bull. Amer. Math. Soc.* **33** (3) (1996), 295–324.
[4] Q. Ren, J. Richter-Gebert, and B. Sturmfels,*Cayley–Bacharach Formulas,* The American Mathematical Monthly, **122** (9) (2015), 845–854
[5] J.L. Smul'jan, *An operator Hellinger integral* (Russian), *Mat. Sb.* **91** (1959), 381–430.
[6] Wolfram Research, Inc., *Mathematica*, Version 12.3.1, Champaign, IL, 2021.

Seonguk Yoo
Department of Mathematics Education and RINS,
Gyeongsang National University, Jinju 52828, Republic of Korea
E-mail: seyoo@gnu.ac.kr