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A GENERALIZED APPROACH OF FRACTIONAL FOURIER TRANSFORM TO STABILITY OF FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. This research article deals with the Mittag-Leffler-Hyers-Ulam stability of linear and impulsive fractional order differential equation which involves the Caputo derivative. The application of the generalized fractional Fourier transform method and fixed point theorem, evaluates the existence, uniqueness and stability of solution that are acquired for the proposed non-linear problems on Lizorkin space. Finally, examples are introduced to validate the outcomes of main result.

1. Introduction

The fractional integral transform has been implemented as an unavoidable position in the field of mathematics to solve fractional order differential equation (FODE) and applied for numerous branch of science and technology. In the literature, there are many transform to solve differential equations such as fractional Fourier [1], fractional wavelet [2], fractional Laplace [3], fractional Mellin [4] and fractional Hankel [5] etc.

An effective and convenient method for solving fractional order differential equation (FODE) is needed. Method for integer order differential equation is not applicable to the case of arbitrary order. However, we discover the presence of fractional Fourier transform (FrFT) solves FODE. The Fourier representation of a function is the most important mathematical formulation for modeling and analysis of physical phenomena. The FrFT is a kind of integral transform and it was utilized by Wiener in the paper [6] as yearly 1929. It converts differential equation into simple algebraic equation. After solving the algebraic equation, we can find the solution of the original equation by inverse FrFT.

Hyers-Ulam stability, which expects a significant job inside research of take a look at of stability of various equations. It means that a function satisfying the differential equation approximately is close to an exact solution of differential equation. In recent years, only few works have been reported on the development of stability of differential equation by transforms approach. For instance, the stability result for linear differential equation of various order with constant co-coefficients using transform

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technique are proposed in [7, 8]. Unyong et al. [9] discussed the Hyers-Ulam stability of FODE with Riemann-Liouville derivative in which the results are obtained with help of FrFT.

At present, some results associated with Mittag-Leffler-Hyers-Ulam(M-L-H-U) stability of FODE have been reported [10]- [12]. Recently, the authors [13] used the method of Laplace transform to prove the stability of linear Caputo-Fabrizio FODE with Mittag-Leffler kernel and they also presented a existence and uniqueness of solution of nonlinear FODE by fixed point method. To the exceptional of our understanding, there are no results on M-L-H-U stability of FODE with Caputo derivative by using generalized FrFT method.

The impulsive phenomena and their models are examined and investigated in various practical issues. The theory of impulsive mathematical models based on FODE has significant application in many problems in applied sciences and engineering. The Hyers-Ulam stability of impulsive FODE are studied in different method and can be referred to [14]- [16].

In this paper, we will discuss about M-L-H-U stability of

$$ {}_a^C D_t^\alpha h(t) - \lambda h(t) = g(t), \quad 0 < \alpha \leq 1, \quad t \in [-T, T], \tag{1} $$

and

$$ {}_a^C D_t^\alpha h(t) = \lambda h(t) + \sum_{i=1}^{m} C_i \chi(t-t_i) u_i(t-t_i) h(t_i) + g(t), \quad 0 < \alpha \leq 1, \quad t \in [-T, T], \tag{2} $$

and the existence and uniqueness solution of

$$ {}_a^C D_t^\alpha h(t) - \lambda h(t) = g(t, h(t)), \quad 0 < \alpha \leq 1, \quad t \in [-T, T], \tag{3} $$

subject to the condition $h^k(a) = a_k$, $k = 0, 1, 2 ... n-1$.
The paper is outlined as follows: In Section 2, basic definitions, properties and lemmas are related to Mittag-Leffler function, fractional derivative and fractional integrals have been summarized. In Section 3, M-L-H-U stability of linear FODE are provided. In Section 4, we investigate M-L-H-U stability of impulse FODE. In Section 5, the existence and uniqueness results for nonlinear FODE by fixed point theorems. Examples and conclusions in Sections 5 and 6 respectively.

2. Preliminaries

In this section, we give some definitions, properties and preliminary concept to our work. We note that $\mathbb{F}$ represents a real $\mathbb{R}$ or a complex $\mathbb{C}$ field.

Definition 2.1. [17] If $0 < \alpha < 1$, then the Riemann-Liouville fractional derivative for the function $h(t)$ is expressed as

$$ {}_a D_t^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-u)^{-\alpha} h(u) du, \quad Re(\alpha) > 0. $$

Definition 2.2. [18] The non-integer-order derivative in the Caputo fractional sense of a function h over the interval $[a, t]$ is defined as

$$ {}_a^C D_t^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-u)^{-\alpha} h'(u) du, \tag{4} $$

where $0 < \alpha < 1$ and $Re(\alpha) > 0$.

Lemma 2.3. [18] *Let $h(t)$ to be $(n-1)$-times continuously differentiable and $h^{(n)}(t)$ to be integrable. Then,*

$$ {}_a^C D_t^\alpha h(t) = {}_a D_t^\alpha h(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} h^{(k)}(a), \tag{5} $$

where $h^{(k)}(t)$ stands for k-th order derivative of $h(t)$.

Definition 2.4. (Lizorkin space) Let $V(\mathbb{R})$ be the set of functions

$$ V(\mathbb{R}) = \{ v \in S(\mathbb{R}) : v^n(0) = 0, n = 0, 1, 2, ...\}. $$

The Lizorkin space $\Phi(\mathbb{R})$ is defined as

$$ \Phi(\mathbb{R}) = \{ \varphi \in S(\mathbb{R}) : \widehat{\varphi} \in V(\mathbb{R})\}, $$

where $S(\mathbb{R})$ is the set of rapidly decreasing test functions on $\mathbb{R}$ and $\widehat{\varphi}$ denotes the FrFT of the function φ.

Definition 2.5. The function $(h_1 * h_2)(t) = \int_{\mathbb{R}} h_1(t-\tau)h_2(\tau)d\tau$ is the convolution of the function h_1, h_2 defined on $\mathbb{R}$.

Definition 2.6. (Generalized fractional Fourier Transform) If a function $h(t) : \mathbb{R} \to \mathbb{F}$ is absolutely integrable and piecewise α^{th} continuously differentiable, then the generalized FrFT of order α, $\alpha \le \eta$ is defined by

$$ \widehat{\mathcal{H}}_\alpha(\omega) = (F_\alpha h)(\omega) = \int_{-\infty}^{\infty} h(t) e^{i\omega^{\frac{\eta}{\alpha}} t} dt, \quad 0 < \alpha \le 1, \quad \omega > 0. $$

Its inverse formula is given by

$$ h(t) = \frac{\eta}{2\pi\alpha} \int_{-\infty}^{\infty} e^{i\omega^{\frac{\eta}{\alpha}} t} \omega^{\frac{\eta-\alpha}{\alpha}} \widehat{\mathcal{H}}_\alpha(\omega) d\omega, \quad \omega > 0. $$

The following are some requisite properties of generalized FrFT related to our work.

1. If $(F_\alpha h_1)(\omega) = (F_\alpha h_2)(\omega)$, then $h_1(t) = h_2(t)$; (one-to-one)
2. $F_\alpha (h_1 * h_2)(\omega) = F_\alpha (h_1)(\omega) F_\alpha (h_2)(\omega)$. (convolution)

Definition 2.7. [19] The Mittag-Leffler function with two parameters is defined as

$$ E_{\rho,\mu}(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(m\rho+\mu)}, \quad t \in \mathbb{C}, \tag{6} $$

where ρ and μ are positive constants.

Lemma 2.8. *The generalized FrFT of the Mittag-Leffler function in two parameters is*

$$ F_\alpha(E_{\rho,\mu}(t)) = \int_{-\infty}^{\infty} E_{\rho,\mu}(t) e^{i\omega^{\frac{\eta}{\alpha}} t} dt = \sum_{m=0}^{\infty} \frac{e^{i\pi(m-1)/2}}{\Gamma(m\rho+\mu)} \omega^{\frac{-\eta(m+1)}{\alpha}} m!, \quad \alpha \le \eta. \tag{7} $$

Proof. We first express generalized Mittag-Leffler function in (7) in a series form and then interchange the order of integration and summation since the series occurring in (6) is absolutely convergent. By substituting $i\omega^{\frac{\eta}{\alpha}} t = -z$ and we get

$$ F_\alpha (E_{\rho,\mu}(t)) = \sum_{m=0}^{\infty} \frac{e^{i\pi(m-1)/2} \omega^{\frac{-\eta(m+1)}{\alpha}})}{\Gamma(m\rho+\mu)} \int_0^\infty e^{-z} z^m dz. \tag{8} $$

Now, using the following results in (8)

$$\int_0^\infty e^{-z} z^\lambda dz = \Gamma(\lambda+1) = \lambda!, \quad Re(\lambda) > 0.$$

We get the desired result (7). □

LEMMA 2.9. *For any $t, \rho, \mu \in \mathbb{C}$ with $min\{Re(\rho), Re(\mu)\} > 0$,*

$$F_\alpha(t^{\mu-1} E_{\rho,\mu}(\lambda t^\rho)) = \frac{(i\omega^{\frac{\eta}{\alpha}})^{-\mu+\rho}}{((i\omega^{\frac{\eta}{\alpha}})^\rho - \lambda)}.$$

Proof. From the equation (6), it follows that

$$\begin{aligned} F_\alpha(t^{\mu-1} E_{\rho,\mu}(\lambda t^\rho)) &= F_\alpha\left(\sum_{k=0}^\infty \frac{t^{\mu-1}(\lambda t^\rho)^k}{\Gamma(k\rho+\mu)}\right) \\ &= \sum_{k=0}^\infty \frac{\lambda^k}{\Gamma(k\rho+\mu)} F_\alpha(t^{k\rho+\mu-1}) \\ &= \sum_{k=0}^\infty \frac{\lambda^k}{\Gamma(k\rho+\mu)} \frac{\Gamma(k\rho+\mu)}{(i\omega^{\frac{\eta}{\alpha}})^{k\rho+\mu}} \\ &= (i\omega^{\frac{\eta}{\alpha}})^{-\mu}\left(1 - \lambda(i\omega^{\frac{\eta}{\alpha}})^{-\rho}\right)^{-1} \\ &= \frac{(i\omega^{\frac{\eta}{\alpha}})^{-\mu+\rho}}{((i\omega^{\frac{\eta}{\alpha}})^\rho - \lambda)}. \end{aligned}$$

This completes the proof. □

REMARK 2.10. If $\lambda = 1$ and $\mu = \rho$, we have

$$F_\alpha(t^{\rho-1} E_{\rho,\rho}(t^\rho)) = \frac{1}{((i\omega^{\frac{\eta}{\alpha}})^\rho - 1)}.$$

LEMMA 2.11. *The generalized FrFT of Caputo fractional derivative is*

$$F_\alpha\left({}_a^C D_t^\alpha h\right)(t) = \left(i\omega^{\frac{\eta}{\alpha}}\right)^\alpha \widehat{H}_\alpha(\omega),$$

where $\widehat{H}_\alpha(\omega)$ denotes the generalized FrFT of $h(t)$.

Proof. Using (4) and concept of convolution, we get

$$\begin{aligned} F_\alpha({}_a^C D_t^\alpha h)(t) &= F_\alpha\left(\frac{d}{dt}\left(\frac{t^{-\alpha}}{\Gamma(1-\alpha)} * h(t)\right)\right) \\ &= \frac{-\alpha}{\Gamma(1-\alpha)} F_\alpha(t^{-\alpha-1}) F_\alpha(h(t)) \\ &= \frac{-\alpha\Gamma(-\alpha)}{\Gamma(1-\alpha)}(i\omega^{\frac{\eta}{\alpha}})^{\alpha+1-1}\widehat{F}_\alpha(\omega) \\ &= (i\omega^{\frac{\eta}{\alpha}})^\alpha \widehat{F}_\alpha(\omega). \end{aligned}$$

The proof is completed. □

For $\epsilon > 0$, we consider the inequality:

$$\left|{}_a^C D_t^\alpha h(t) - \lambda h(t) - g(t)\right| \le \epsilon E_\alpha(t^\alpha), \quad t \in [-T, T]. \tag{9}$$

Definition 2.12. The FODE (1) is said to be M-L-H-U stability if there exists $K \in \mathbb{R}^+$ and such that for every function $h : [-T,T] \to \mathbb{F}$ of the satisfying inequality (9), there exists a solution $h_\alpha : [-T,T] \to \mathbb{F}$ of FODE (1) with

$$|h(t) - h_\alpha(t)| \leq K\epsilon E_\alpha(t^\alpha), \quad t \in [-T,T].$$

Further, the FODE (1) is generalized M-L-H-U stable if we can find a function Ψ : $\mathbb{R} \to \mathbb{R}$ with $\Psi(0) = 0$ such that $|h(t) - h_\alpha(t)| \leq \Psi(\epsilon)E_\alpha(t^\alpha)$, $t \in [-T,T]$.

Definition 2.13. The FODE (1) is stable in the generalized M-L-H-U-Rassias sense subject to $\psi : \mathbb{R} \to \mathbb{R}$ if there exist $K_\Psi \in \mathbb{R}^+$, in a way that given any $\epsilon > 0$ and any solution satisfying the inequality

$$\left|{}_a^C D_t^\alpha h(t) - \lambda h(t) - g(t)\right| \leq \epsilon \Psi(t) E_\alpha(t^\alpha), \quad t \in [-T,T],$$

there exist a unique solution $h_\alpha : [-T,T] \to \mathbb{F}$ of FODE (1), in which

$$|h(t) - h_\alpha(t)| \leq K_\Psi \epsilon \Psi(t) E_\alpha(t^\alpha), \quad t \in [-T,T].$$

Definition 2.14. The FODE (1) is said to be generalized M-L-H-U-Rassias stable with respect to non zero positive real number K_Ψ, such that for any solution h_α : $[-T,T] \to \mathbb{F}$ of the inequality

$$\left|{}_a^C D_t^\alpha h(t) - \lambda h(t) - g(t)\right| \leq \Psi(t) E_\alpha(t^\alpha), \quad t \in [-T,T],$$

there exists a unique solution $h_\alpha : [-T,T] \to \mathbb{F}$ of FODE (1), such that

$$|h(t) - h_\alpha(t)| \leq K_\Psi \Psi(t) E_\alpha(t^\alpha), \quad t \in [-T,T].$$

Theorem 2.15. *(**Banach Contaction Principle**) Let ψ be a Banach space, $\Delta : \psi \to \psi$ a contraction mapping. Then Δ has a unique fixed point in ψ.*

Theorem 2.16. *(**Schaefer's fixed point theorem**) Let $\Delta : \psi \to \psi$ be a completely continuous mapping. If*

$$\mathcal{B} = \{h \in \psi | h = \zeta \Delta h, 0 < \zeta < 1\},$$

is a bounded set, then there is at least one fixed point ψ for Δ.

3. M-L-H-U stability of linear FODE

In this section, we prove the M-L-H-U stability of linear or non-linear FODE (1) on Lizorkin space by using the FrFT method.

Theorem 3.1. *Suppose a given real continuous function $g(t)$ in $\Phi(\mathbb{R})$ and $0 < \alpha \leq 1$. If a function h(t) satisfies the inequality*

$$\left|{}_a^C D_t^\alpha h(t) - \lambda h(t) - g(t)\right| \leq \epsilon E_\alpha(t^\alpha), \quad t \in [-T,T], \tag{10}$$

$\lambda \in \mathbb{R}$ and for some $\epsilon > 0$, then there exists a solution $h_\alpha(t) : [-T,T] \to \mathbb{F}$ of FODE (1) such that

$$|h(t) - h_\alpha(t)| \leq \epsilon t^{\alpha-2} E_{2\alpha,\alpha+1}(\lambda t^{2\alpha}), \quad t \in [-T,T],$$

where $E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})$ is the two-parameter Mittag-Leffler function.

Proof. A function $y : [-T, T] \to \mathbb{F}$ is defined in which

$$\begin{aligned} y(t) &= {}^C_a D^\alpha_t h(t) - \lambda h(t) - g(t) \\ &= {}_a D^\alpha_t h(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} h^k(a) - \lambda h(t) - g(t). \end{aligned} \tag{11}$$

Suppose that $h(t)$ is a continuously differentiable function satisfying inequality (10). We have,

$$|y(t)| \le \epsilon E_\alpha(t^\alpha).$$

Taking generalized FrFT operator F_α of (11), we get

$$F_\alpha(y(t)) = (i\omega^{\frac{\eta}{\alpha}})^\alpha F_\alpha(h(t)) - e^{i\omega^{\frac{\eta}{\alpha}} a} \sum_{k=0}^{n-1} a_k \frac{\Gamma(k-\alpha+1)}{\Gamma(k-\alpha+1)(i\omega^{\frac{\eta}{\alpha}})^{k-\alpha+1}} - \lambda F_\alpha(h(t)) - \widehat{G}_\alpha(\omega)$$

(12)

$$F_\alpha(h(t)) = \frac{F_\alpha(y(t))}{((i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda)} + \frac{e^{i\omega^{\frac{\eta}{\alpha}} a}}{((i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda)} \sum_{k=0}^{n-1} \frac{a_k}{(i\omega^{\frac{\eta}{\alpha}})^{k-\alpha+1}} + \frac{\widehat{G}_\alpha(\omega)}{((i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda)}.$$

At this point, set

$$h_\alpha(t) = e^{i\omega^{\frac{\eta}{\alpha}} a} \sum_{k=0}^{n-1} a_k t^k E_{\alpha,k+1}(\lambda t^\alpha) + \int_{-t}^{t} (t-x)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-x)^\alpha) g(x) dx. \tag{13}$$

Using the convolution property of generalized FrFT, we obtain

$$\begin{aligned} h_\alpha(t) &= e^{i\omega^{\frac{\eta}{\alpha}} a} \sum_{k=0}^{n-1} a_k F_\alpha^{-1} \left(\frac{(i\omega^{\frac{\eta}{\alpha}})^{-k-1+\alpha}}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \right) + (t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha)) * g(t) \\ &= \frac{e^{i\omega^{\frac{\eta}{\alpha}}} a}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \sum_{k=0}^{n-1} a_k F_\alpha^{-1} \left(\left(i\omega^{\frac{\eta}{\alpha}} \right)^{-k-1+\alpha} \right) + F_\alpha^{-1} \left(\frac{1}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \right) * F_\alpha^{-1}(\widehat{G}_\alpha(\omega)) \end{aligned}$$

(14)

$$F_\alpha(h_\alpha(t)) = \frac{e^{i\omega^{\frac{\eta}{\alpha}}} a}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \sum_{k=0}^{n-1} a_k (i\omega^{\frac{\eta}{\alpha}})^{-k-1+\alpha} + \left(\frac{\widehat{G}_\alpha(\omega)}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \right).$$

Now,

$$\begin{aligned} F_\alpha({}^C_a D^\alpha_t h_\alpha(t) - \lambda h_\alpha(t)) &= (i\omega^{\frac{\eta}{\alpha}})^\alpha F_\alpha(h_\alpha(t)) - \sum_{k=0}^{n-1} \frac{F_\alpha((t-a)^{k-\alpha})}{\Gamma(k-\alpha+1)} a_k - \lambda F_\alpha(h_\alpha(t)) \\ &= \left((i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda \right) F_\alpha(h_\alpha(t)) - \sum_{k=0}^{n-1} a_k e^{i\omega^{\frac{\eta}{\alpha}} a} \frac{\Gamma(k-\alpha+1)(i\omega^{\frac{\eta}{\alpha}})^{\alpha-k-1}}{\Gamma(k-\alpha+1)} \\ &= \left((i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda \right) \left(\frac{e^{i\omega^{\frac{\eta}{\alpha}}} a}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \sum_{k=0}^{n-1} a_k (i\omega^{\frac{\eta}{\alpha}})^{-k-1+\alpha} + \left(\frac{\widehat{G}_\alpha(\omega)}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \right) \right) \\ &\quad - \sum_{k=0}^{n-1} a_k e^{i\omega^{\frac{\eta}{\alpha}} a} (i\omega^{\frac{\eta}{\alpha}})^{\alpha-k-1} \\ &= \widehat{G}_\alpha(\omega). \end{aligned}$$

By considering one-to-one property of generalized FrFT, we obtain

$$ {}_a^C D_t^\alpha h_\alpha(t) - \lambda h_\alpha(t) = g(t), $$

which gives that $h_\alpha(t)$ is a solution of equation (1). From the relations (12) and (14), we get

$$ F_\alpha(h(t)) - F_\alpha(h_\alpha(t)) = \frac{F_\alpha(y(t))}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda}. $$

By the convolution property of generalized FrFT and by Lemma 2.2 , we obtain

$$ h(t) - h_\alpha(t) = t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha) * y(t). $$

Taking modulus on both sides of above equation, we obtain

$$ \begin{aligned} |h(t) - h_\alpha(t)| &= \left| \int_{-t}^{t} (t-x)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-x))^\alpha y(x) dx \right| \\ &\leq \epsilon \int_{-t}^{t} (t-x)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-x))^\alpha E_\alpha(x^\alpha) dx \\ &= \epsilon \sum_{m=0}^{\infty} \int_{-t}^{t} \frac{\lambda^m (t-x)^{\alpha m+\alpha-1}}{\Gamma(\alpha m+\alpha)} \frac{x^{\alpha m}}{\Gamma(\alpha m+1)} dx \\ &\leq \epsilon \sum_{m=0}^{\infty} t^{\alpha m+\alpha-2} \lambda^m \int_0^1 \frac{(1-u)^{\alpha m+\alpha-1}(tu)^{\alpha m}}{\Gamma(\alpha m+\alpha)\Gamma(\alpha m+1)} du \\ &= \epsilon \sum_{m=0}^{\infty} \frac{\lambda^m t^{2\alpha m+\alpha-2}}{\Gamma(\alpha m+\alpha)\Gamma(\alpha m+1)} \int_0^1 (1-u)^{\alpha m+\alpha-1} u^{\alpha m} du \\ &= \epsilon \sum_{m=0}^{\infty} \frac{\lambda^m t^{2\alpha m+\alpha-2}}{\Gamma(\alpha m+\alpha)\Gamma(\alpha m+1)} \frac{\Gamma(\alpha m+1)\Gamma(\alpha m+\alpha)}{\Gamma(2\alpha m+\alpha+1)} \\ &\leq \epsilon t^{\alpha-2} E_{2\alpha,\alpha+1}(\lambda t^{2\alpha}), \quad t \in [-T, T]. \end{aligned} $$

From Definition 3.1, the FODE (1) is M-L-H-U stable. The proof is complete. □

Remark 3.2. Putting $\Psi(\epsilon) = K\epsilon$, we obtain FODE (1) is generalized M-L-H-U stable.

Similarly, we can prove that the FODE (1) is generalized M-L-H-U Rassias stable with the help of generalized FrFT.

Corollary 3.3. *For any function $h \in \Phi(\mathbb{R})$ fulfilling the inequality*

$$ \left| {}_a^C D_t^\alpha h(t) - \lambda h(t) - g(t) \right| \leq \Psi(t) E_\alpha(t^\alpha), \quad \forall t \in \mathbb{R}, $$

$0 < \alpha \leq 1$ and a continuous function $g(t) \in \Phi(\mathbb{R})$, there exists a solution $h_\alpha \in \Phi(\mathbb{R})$ of FODE (1) wherein

$$ |h(t) - h_\alpha(t)| \leq K_\Psi \Psi(t) E_\alpha(t^\alpha), \quad \forall t \in \mathbb{R}. $$

4. M-L-H-U stability of impulsive FODE

In this section, we will discuss about M-L-H-U stability of impulsive FODE of the form:

$$ {}_a^C D_t^{\alpha} h(t) = \lambda h(t) + \sum_{i=1}^{m} C_i \chi(t-t_i) u_i(t-t_i) h(t_i) + g(t), \quad h^k(a) = a_k, \quad k = 0,1,2,...n-1 $$

on $[-T,T]$, where $-T < t_1 < t_2..... < t_m = T$. We assume that λ, C_i are constant for $i = 1,2,3...m$ and $g(t)$ is continuous function on $[-T,T]$. Also, $\chi(t-t_i)$ is the Heaviside unit step function which is given by

$$ \chi(t-t_i) = \begin{cases} 1 & ; t > t_i \\ 0 & ; t \leq t_i. \end{cases} $$

In addition, $u_i(t-t_i)$ are continuous function on (t_i, T), for $i = 1,2,3...m$ such that $lim_{t \to t_i} u_i(t-t_i)$ exists.

THEOREM 4.1. *Impulsive FODE 2 is M-L-H-U stable.*

Proof. Let $\epsilon > 0$ and for each solution $h(t)$ satisfying

$$ \left| {}_a^C D_t^{\alpha} h(t) - \lambda h(t) - \sum_{i=1}^{m} C_i \chi(t-t_i) u_i(t-t_i) h(t_i) - g(t) \right| \leq \epsilon E_{\alpha}(t^{\alpha}), \tag{15} $$

for all $t \in [-T,T]$. Define a function $y(t) : [-T,T] \to \mathbb{F}$ by $y(t) =_a^C D_t^{\alpha} h(t) - \lambda h(t) - \sum_{i=1}^{m} C_i \chi(t-t_i) u_i(t-t_i) h(t_i) - g(t)$ for all $t > 0$. In view of (15), we get $|y(t)| \leq \epsilon E_{\alpha}(t^{\alpha})$. Using (5), we have

$$ y(t) =_a D_t^{\alpha} h(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} h^k(a) - \lambda h(t) - \sum_{i=1}^{m} C_i \chi(t-t_i) u_i(t-t_i) h(t_i) - g(t). \tag{16} $$

Taking fractional Fourier transform operator F_{α} to y(t), we obtain

$$ \begin{aligned} F_{\alpha}(y(t)) =& F_{\alpha}({}_a D_t^{\alpha} h(t)) - \sum_{k=0}^{n-1} \frac{a_k}{\Gamma(k-\alpha+1)} F_{\alpha}((t-a)^{k-\alpha}) - \lambda F_{\alpha}(h(t)) \\ & - \sum_{i=1}^{m} C_i e^{i\omega^{\frac{\eta}{\alpha}} t_i} h(t_i) F_{\alpha}(u_i(t)) - \widehat{G}_{\alpha}(\omega) \\ =& (i\omega^{\frac{\eta}{\alpha}})^{\alpha} \widehat{H}_{\alpha}(\omega) - e^{i\omega^{\frac{\eta}{\alpha}} a} \sum_{k=0}^{n-1} a_k \frac{\Gamma(k-\alpha+1)}{\Gamma(k-\alpha+1)(i\omega^{\frac{\eta}{\alpha}})^{k-\alpha+1}} - \lambda \widehat{H}_{\alpha}(\omega) \\ & - \sum_{i=1}^{m} C_i e^{i\omega^{\frac{\eta}{\alpha}} t_i} h(t_i) F_{\alpha}(u_i(t)) - \widehat{G}_{\alpha}(\omega) \end{aligned} $$

$$ ((i\omega^{\frac{\eta}{\alpha}})^{\alpha} - \lambda) F_{\alpha}(h(t)) = F_{\alpha}(y(t)) + e^{i\omega^{\frac{\eta}{\alpha}} a} \sum_{k=0}^{n-1} \frac{a_k}{(i\omega^{\frac{\eta}{\alpha}})^{k-\alpha+1}} + \sum_{i=1}^{m} C_i e^{i\omega^{\frac{\eta}{\alpha}} t_i} h(t_i) F_{\alpha}(u_i(t)) + \widehat{G}_{\alpha}(\omega) $$

$$ \begin{aligned} F_{\alpha}(h(t)) =& \frac{F_{\alpha}(y(t))}{((i\omega^{\frac{\eta}{\alpha}})^{\alpha} - \lambda)} + \frac{e^{i\omega^{\frac{\eta}{\alpha}} a}}{((i\omega^{\frac{\eta}{\alpha}})^{\alpha} - \lambda)} \sum_{k=0}^{n-1} \frac{a_k}{(i\omega^{\frac{\eta}{\alpha}})^{k-\alpha+1}} \\ & + \sum_{i=1}^{m} C_i e^{i\omega^{\frac{\eta}{\alpha}} t_i} \frac{h(t_i) F_{\alpha}(u_i(t))}{((i\omega^{\frac{\eta}{\alpha}})^{\alpha} - \lambda)} + \frac{\widehat{G}_{\alpha}(\omega)}{((i\omega^{\frac{\eta}{\alpha}})^{\alpha} - \lambda)}. \end{aligned} \tag{17} $$

Set

(18) $$h_\alpha(t) = e^{i\omega^{\frac{\eta}{\alpha}}} \sum_{k=0}^{n-1} a_k t^k E_{\alpha,k+1}(\lambda t^\alpha) + \sum_{i=1}^{m} C_i \chi_i(t-t_i) h(t_i) \widehat{U}_i(t-t_i) + \int_{-t}^{t} (t-x)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-x)^\alpha) g(x) dx,$$

where $\widehat{U}_i(t) =$ Fourier inverse of $\frac{F_\alpha(u_i(t))}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda}$. By the definition of convolution,

$$\begin{aligned} h_\alpha(t) &= e^{i\omega^{\frac{\eta}{\alpha}}} \sum_{k=0}^{n-1} a_k F_\alpha^{-1} \left(\frac{(i\omega^{\frac{\eta}{\alpha}})^{-k-1+\alpha}}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \right) + \sum_{i=1}^{m} C_i \chi_i(t-t_i) F_\alpha^{-1} \left(\frac{F_\alpha(u_i(t-t_i))}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \right) h(t_i) \\ &+ (t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha)) * g(t) \\ &= \frac{e^{i\omega^{\frac{\eta}{\alpha}}}}{(i\omega^{\eta\alpha})^\alpha - \lambda} \sum_{k=0}^{n-1} a_k F_\alpha^{-1} \left((i\omega^{\frac{\eta}{\alpha}})^{-k-1+\alpha} \right) + \sum_{i=1}^{m} C_i \chi_i(t-t_i) F_\alpha^{-1} \left(\frac{F_\alpha(u_i(t-t_i))}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \right) h(t_i) \\ &+ F_\alpha^{-1} \left(\frac{1}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \right) * F_\alpha^{-1}(\widehat{G}_\alpha(\omega)), \end{aligned}$$

(19)
$$F_\alpha(h_\alpha(t)) = \frac{e^{i\omega^{\frac{\eta}{\alpha}}}}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \sum_{k=0}^{n-1} a_k (i\omega^{\frac{\eta}{\alpha}})^{-k-1+\alpha} + \sum_{i=1}^{m} C_i \chi_i(t-t_i) \left(\frac{F_\alpha(u_i(t-t_i))}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \right) h(t_i) + \left(\frac{\widehat{G}_\alpha(\omega)}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \right).$$

Now,

$${}_a^C D_t^\alpha h_\alpha(t) - \lambda h_\alpha(t) = {}_a D_t^\alpha h_\alpha(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} h_\alpha^k(a) - \lambda h_\alpha(t),$$

$$\begin{aligned} F_\alpha({}_a^C D_t^\alpha h_\alpha(t) - \lambda h_\alpha(t)) &= (i\omega^{\frac{\eta}{\alpha}})^\alpha F_\alpha(h_\alpha(t)) - \sum_{k=0}^{n-1} \frac{F_\alpha((t-a)^{k-\alpha})}{\Gamma(k-\alpha+1)} a_k - \lambda F_\alpha(h_\alpha(t)) \\ &= \left((i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda \right) F_\alpha(h_\alpha(t)) - \sum_{k=0}^{n-1} a_k e^{i\omega^{\frac{\eta}{\alpha}} a} \frac{\Gamma(k-\alpha+1)(i\omega^{\frac{\eta}{\alpha}})^{\alpha-k-1}}{\Gamma(k-\alpha+1)} \\ &= \left((i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda \right) \left(\frac{e^{i\omega^{\frac{\eta}{\alpha}}}}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \sum_{k=0}^{n-1} a_k (i\omega^{\frac{\eta}{\alpha}})^{-k-1+\alpha} \right. \\ &\left. + \sum_{i=1}^{m} C_i \chi_i(t-t_i) \left(\frac{F_\alpha(u_i(t-t_i))}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \right) h_\alpha(t_i) + \left(\frac{\widehat{G}_\alpha(\omega)}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda} \right) \right) \\ &- \sum_{k=0}^{n-1} a_k e^{i\omega^{\frac{\eta}{\alpha}} a} (i\omega^{\frac{\eta}{\alpha}})^{\alpha-k-1}, \\ &= \sum_{i=1}^{m} C_i \chi_i(t-t_i) F_\alpha(u_i(t-t_i)) h_\alpha(t_i) + F_\alpha(g(t)), \end{aligned}$$

which shows that $h_\alpha(t)$ is solution of equation (2) since F_α is one-to-one. Now, the relations (17) and (19) necessitate that

$$F_\alpha(h(t)) - F_\alpha(h_\alpha(t)) = \frac{F_\alpha(y(t))}{(i\omega^{\frac{\eta}{\alpha}})^\alpha - \lambda}.$$

By the convolution property FrFT and Lemma 2.3, we obtain

$$F_\alpha(h(t)-h_\alpha(t)) = F_\alpha\left(t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha) * y(t)\right),$$

$$h(t)-h_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha) * y(t). \tag{20}$$

Taking modulus on both sides of equation (20) and $|y(t)| \le \epsilon E_\alpha(t^\alpha)$, we get

$$|h(t)-h_\alpha(t)| \le \epsilon t^{\alpha-2}E_{2\alpha,\alpha+1}(\lambda t^{2\alpha}), \quad t \in [-T,T]. \tag{21}$$

Hence impulse FODE (2) has Mittag-Leffler-Hyers-Ulam stability. □

5. M-L-H-U stability of non-linear FODE

Now, we give the proof of the existence and uniqueness theorem for non-linear FODE (3).
We provide the following assumptions:

(B1) The function $g : [-T,T]$ x $\mathbb{R} \to \mathbb{F}$ is continuous, there exists constant $\mathbb{L} > 0$ such that $|g(t,h_1) - g(t,h_2)| \le \mathbb{L}\,|h_1 - h_2|, \quad \forall h_1, h_2 \in \mathbb{R}$.
(B2) There exists a constant $\mathbb{L}_h > 0$ such that $|g(t,h)| \le \mathbb{L}_h(1+|h|), \quad \forall h \in \mathbb{R}$.

Theorem 5.1. *Suppose that* $(B1)$ *holds. Then the FODE (3) have a unique solution, provided that* $\frac{\mathbb{L}(2T)^\alpha}{\Gamma(\alpha+1)} < 1$.

Proof. The space $\psi = C(\mathbb{R},\mathbb{F})$ is define with the norm $\|h\| = sup\{|h(t)|\,; t \in \mathbb{R}\}$. Define the operator $\Lambda : \psi \to \psi$ as

(22)
$$(\Lambda h)(t) = e^{i\omega\frac{\eta}{\alpha}a}\sum_{k=0}^{n-1} a_k t^k E_{\alpha,k+1}(\lambda t^\alpha) + \int_{-t}^{t}(t-x)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-x)^\alpha)g(x,h(x))dx,$$

for any $t \in [-T,T]$. Λ is well defined because of $(B1)$. Let $h_1, h_2 \in \psi$, then

$$\begin{aligned}\|(\Lambda h_1)(t) - (\Lambda h_2)(t)\| &\le \int_{-t}^{t}(t-x)^{\alpha-1}\left|E_{\alpha,\alpha}(\lambda(t-x)^\alpha)\right|\left|g(x,h_1(x)) - g(x,h_2(x))\right|dx\\ &\le \frac{L}{\Gamma\alpha}\int_{-t}^{t}(t-x)^{\alpha-1}\left|h_1(x)-h_2(x)\right|dx\\ &\le \mathbb{L}\frac{(2T)^\alpha}{\Gamma(\alpha+1)}\|h_1-h_2\|.\end{aligned}$$

Condition $\frac{\mathbb{L}(2T)^\alpha}{\Gamma(\alpha+1)} < 1$ shows that Λ is a contraction mapping. By Banach contraction principle, Λ has a unique fixed point. □

Theorem 5.2. *Suppose that assumptions* $(B1)$ *and* $(B2)$ *hold. Then there exists at least one solution of nonlinear FODE (3).*

Proof. Define Λ as in (22). We will complete the proof into four steps.
Step 1: Λ is continuous.

Let h_n be a sequence such that $h_n \to h$, as $n \to \infty$ in ψ. For all $t \in [-T, T]$, one has

$$\begin{aligned}\|(\Lambda h_n)(t) - (\Lambda h)(t)\| &\leq \int_{-t}^{t} (t-x)^{\alpha-1} \left|E_{\alpha,\alpha}(\lambda(t-x)^\alpha)\right| \left|g(x, h_n(x)) - g(x, h(x))\right| dx \\ &\leq \frac{(2T)^\alpha}{\Gamma(\alpha+1)} \|g(., h_n) - g(., h)\|.\end{aligned}$$

Since g is continuous, the operator Λ is also continuous. Then, $\|(\Lambda h_n)(t) - (\Lambda h)(t)\| \to 0$ as $n \to 0$.

Step 2: Λ maps bounded set in ψ.

For all $l > 0$, there exists a $\mathcal{N} > 0$, in which for any $h \in \mathbb{B}_l = \{h \in \psi : \|h\| \leq l\}$, we have $\|\Lambda h\| \leq \mathcal{N}$. Form $(B2)$, $\|\Lambda h\| \leq \frac{2T\mathbb{L}_h(1+|h|)}{\Gamma(\alpha+1)} = K$. Hence $\Lambda(\mathbb{B}_l)$ is bounded.

Step 3: Λ maps bounded set into equicontinuous set in ψ.

Let $t_1, t_2 \in [-T, T]$ with $a \leq t_1 < t_2 \leq T$, $h \in \mathbb{B}_l$. Then, by using $(B2)$, we have

$$\begin{aligned}\|\Lambda h(t_2) - \Lambda h(t_1)\| \leq{}& \frac{1}{\Gamma\alpha} \sum_{k=0}^{n-1} a_k(t_2^k - t_1^k) + \left| \int_a^{t_2} (t_2-x)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_2-x)^\alpha) g(x, h(x)) dx \right. \\ & \left. - \int_a^{t_1} (t_1-x)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_1-x)^\alpha) g(x, h(x)) dx \right| \\ \leq{}& \frac{1}{\Gamma\alpha} \sum_{k=0}^{n-1} a_k(t_2^k - t_1^k) + \int_a^{t_1} \Big((t_1-x)^{\alpha-1} \left|E_{\alpha,\alpha}(\lambda(t_1-x)^\alpha)\right| \\ & - (t_2-x)^{\alpha-1} \left|E_{\alpha,\alpha}(\lambda(t_2-x)^\alpha)\right| \Big) \left|g(x, h(x))\right| dx \\ & + \int_{t_1}^{t_2} (t_2-x)^{\alpha-1} \left|E_{\alpha,\alpha}(\lambda(t_2-x)^\alpha)\right| \left|g(x, h(x))\right| dx \\ \leq{}& \frac{1}{\Gamma\alpha} \sum_{k=0}^{n-1} a_k(t_2^k - t_1^k) + \mathbb{L}_h(1+|h|) \frac{(t_2-t_1)^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

Then, $\|\Lambda h(t_2) - \Lambda h(t_1)\| \to 0$ as $t_2 \to t_1$. From Steps $(1) - (3)$, Λ is completely continuous by Arzela-Ascoli theorem.

Step 4: We show that the set $E(\Lambda) = \{h \in \psi : h = \zeta \Lambda h, \zeta \in (0, 1)\}$ is bounded.

Let $h \in E(\Lambda)$. Then $h = \zeta \Lambda h$ for some $\zeta \in (0, 1)$. For each $t \in [-T, T]$, we have

$$\begin{aligned}|h(t)| \leq{}& \frac{1}{\Gamma\alpha} \int_{-t}^{t} (t-x)^{\alpha-1} \left|g(x, h(x))\right| dx \\ \leq{}& \frac{\mathbb{L}_h}{\Gamma\alpha} \int_{-t}^{t} (t-x)^{\alpha-1} dx + \frac{\mathbb{L}_h}{\Gamma\alpha} \int_{-t}^{t} (t-x)^{\alpha-1} \left|h(x)\right| dx \\ \leq{}& \frac{\mathbb{L}_h (2T)^\alpha}{\Gamma(\alpha+1)} + \frac{\mathbb{L}_h}{\Gamma\alpha} \int_{-t}^{t} (t-x)^{\alpha-1} \left|h(x)\right| dx \\ \leq{}& \mathbb{R}_1 + \mathbb{R}_2 \int_{-t}^{t} (t-x)^{\alpha-1} \left|h(x)\right| dx,\end{aligned}$$

where $\mathbb{R}_1 = \frac{\mathbb{L}_h(2T)^\alpha}{\Gamma(\alpha+1)}$ and $\mathbb{R}_2 = \frac{\mathbb{L}_h}{\Gamma\alpha}$. Grownwall's inequality gives that $|h(t)| \leq \mathbb{R}_1 exp\left(\frac{\mathbb{R}_2(2T)^\alpha}{\alpha}\right)$. Then the set $E(\Lambda)$ is bounded. By Schaefer's fixed point theorem, Λ has a fixed point which is a solution of FODE (3). $\square$

THEOREM 5.3. *Assume that hypothesis $(B1)$ holds. If a continuously differential function $h : \mathbb{R} \to \mathbb{F}$ satisfies*

$$\left|{}_a^C D_t^\alpha h(t) - \lambda h(t) - g(t, h(t))\right| \leq \epsilon E_\alpha(t^\alpha), \tag{23}$$

for all $t \in [-T, T]$, then non-linear FODE (3) is M-L-H-U stable.

Proof. Let $h \in C(\mathbb{R}, \mathbb{F})$ be a solution of (23). From Theorem 3.2, the FODE (3) has a unique solution

$$h_\alpha(t) = e^{i\omega \frac{\eta}{\alpha} a} \sum_{k=0}^{n-1} a_k t^k E_{\alpha,k+1}(\lambda t^\alpha) + \int_{-t}^{t} (t-x)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-x)^\alpha) g(x, h_\alpha(x)) dx.$$

From equation (23), we have $-\epsilon E_\alpha(t^\alpha) \leq_a^C D_t^\alpha h(t) - \lambda h(t) - g(t, h(t)) \leq \epsilon E_\alpha(t^\alpha)$, for all $t \in [-T, T]$. If we integrate each term of above inequality, we obtain

$$\left| h(t) - e^{i\omega \frac{\eta}{\alpha} a} \sum_{k=0}^{n-1} a_k t^k E_{\alpha,k+1}(\lambda t^\alpha) - \int_{-t}^{t} (t-x)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-x)^\alpha) g(x, h(x)) dx \right|$$

$$\leq \epsilon \int_{-t}^{t} (t-x)^{\alpha-1} \left| E_{\alpha,\alpha}(\lambda (t-x)^\alpha) \right| E_\alpha(x^\alpha) dx \quad \leq \quad \epsilon t^{\alpha-2} E_{2\alpha,\alpha+1}(\lambda t^{2\alpha}).$$

Thus,

$$\begin{aligned}
&|h(t) - h_\alpha(t)| \\
&= \left| h(t) - e^{i\omega \frac{\eta}{\alpha} a} \sum_{k=0}^{n-1} a_k t^k E_{\alpha,k+1}(\lambda t^\alpha) - \int_{-t}^{t} (t-x)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-x)^\alpha) g(x, h_\alpha(x)) dx \right| \\
&\leq \left| h(t) - e^{i\omega \frac{\eta}{\alpha} a} \sum_{k=0}^{n-1} a_k t^k E_{\alpha,k+1}(\lambda t^\alpha) - \int_{-t}^{t} (t-x)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-x)^\alpha) g(x, h(x)) dx \right| \\
&+ \int_{-t}^{t} (t-x)^{\alpha-1} \left| E_{\alpha,\alpha}(\lambda (t-x)^\alpha) \right| \left| g(x, h(x)) - g(x, h_\alpha(x)) \right| dx \\
&\leq \epsilon t^{\alpha-2} E_{2\alpha,\alpha+1}(\lambda t^{2\alpha}) + \frac{\mathbb{L}}{\Gamma\alpha} \int_{-t}^{t} (t-x)^{\alpha-1} \left| h(x) - h_\alpha(x) \right| dx.
\end{aligned}$$

From Grownwall's inequality,

$$|h(t) - h_o(t)| \leq K^* \epsilon E_{2\alpha,\alpha+1}(\lambda t^{2\alpha}),$$

where $K^* = t^{\alpha-2} exp\left(\frac{\mathbb{L}(2T)^\alpha}{\Gamma(\alpha+1)}\right)$. □

6. Examples

EXAMPLE 6.1. Consider the composite fractional oscillation equation [20]

$${}_0^C\mathcal{D}_t^\alpha h(t) + h(t) - t^2 - \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} = 0, \quad 0 \leq t \leq 1, \tag{24}$$

with the initial conditions $h^k(0) = 0,\ k = 0, 1.$
If we set $\alpha = \frac{1}{2}$ and $\lambda = 1$, we obtain $g(t) = t^2 + \frac{8}{3}\frac{1}{\sqrt{\pi}} t^{\frac{3}{2}}$.
Note $h(t) = t^2$ satisfies

$$\left|{}_0^C\mathcal{D}_t^\alpha h(t) + h(t) - g(t)\right| = \left|{}_0^C\mathcal{D}_t^\alpha h(t) + h(t) - \frac{8}{3}\frac{1}{\sqrt{\pi}} t^{\frac{3}{2}} - t^2\right| \leq \frac{1}{2} = \epsilon.$$

From (13), we get solution of equation (24), that is

$$h_\alpha(t) = \int_0^1 (t-x)^{\frac{-1}{2}} E_{0.5,0.5}\Big((t-x)^{\frac{1}{2}}\Big)\Big(t^2 + \frac{8}{3}\frac{1}{\sqrt{\pi}}t^{\frac{3}{2}}\Big)dx.$$

By Theorem 3.1, it is M-L-H-U stable whereby

$$|h(t) - h_\alpha(t)| \leq \frac{1}{2}t^{-1.5}E_{1,1.5}(t), \quad 0 \leq t \leq 1.$$

Hence, our outcomes can be applied to equation (24).

Example 6.2. Consider the problem

$$(25) \qquad {}_0^C\mathcal{D}_t^{\frac{3}{4}} h(t) + h(t) = 1.77t^{\frac{5}{4}} + t^2 + \frac{1}{2}, \quad 0 \leq t \leq 1,$$

with the initial conditions $h(0) = 0, h'(0) = 0$.
with $\alpha = \frac{3}{4}$, $a = 0$, $\lambda = 1$, $g(t) = 1.77t^{\frac{5}{4}} + t^2 + \frac{1}{2}$. Note $h(t) = t^2$ satisfies

$$\Big|{}_0^C\mathcal{D}_t^{\frac{3}{4}} h(t) + h(t) - g(t)\Big| = \Big|{}_0^C\mathcal{D}_t^{\frac{3}{4}} t^2 + t^2 - 1.77t^{\frac{5}{4}} - t^2 - \frac{1}{2}\Big| \leq \frac{1}{2}.$$

From (13), we get solution of equation (25), that is

$$h_\alpha(t) = \int_0^1 (t-x)^{\frac{-1}{4}} E_{0.75,0.75}\Big((t-x)^{\frac{3}{4}}\Big)\Big(1.77x^{\frac{5}{4}} + x^2 + \frac{1}{2}\Big)dx.$$

By Theorem 3.1,

$$|h(t) - h_\alpha(t)| \leq \frac{1}{2}t^{\frac{-1}{4}}E_{\frac{3}{2},\frac{7}{4}}\left(t^{\frac{3}{2}}\right), \quad 0 \leq t \leq 1.$$

Hence, the solution of equation (25) is M-L-H-U stable.

Example 6.3. Consider

$$(26) \quad {}_0^C\mathcal{D}_t^{\frac{1}{2}} h(t) = h(t) + \sum_{i=1}^m i\chi(t-t_i)h(t_i) + t^2, \quad h^k(0) = 0, \quad k = 0,1,2...n-1.$$

We have taken $\alpha = \frac{1}{2}$, $\lambda = 1$, $u(t-t_i) = 1$, $C_i = i$ and $g(t) = t^2$. For $\epsilon = \frac{1}{2}$, the function $h(t) = t^2$ satisfies

$$\left|{}_0^C\mathcal{D}_t^{\frac{1}{2}} h(t) - h(t) - \sum_{i=1}^m i\chi(t-t_i)h(t_i) - t^2\right| < \frac{1}{2}.$$

From (18), there exists a solution of equation (26)

$$h_\alpha(t) = \sum_{i=1}^m i\chi_i(t-t_i)h(t_i) + \int_{-t}^t (t-x)^{\frac{-1}{2}} E_{\frac{1}{2},\frac{1}{2}}((t-x)^{\frac{1}{2}})x^2 dx.$$

By Theorem 4.1, we gets

$$|h(t) - h_\alpha(t)| \leq \frac{1}{2}t^{-1.5}E_{1,1.5}(t).$$

Therefore, we can ascertain that equation (26) is M-L-H-U stable.

7. Conclusion

In this paper, the researcher had made an attempt to analyze the M-L-H-U stability of FODE with Caputo derivative using generalized FrFT. Also we have showed that the existence and uniqueness solution of non-linear FODE by Banach contraction principle and fixed point theorems. The article went further in the process and highlighted an immodest role of Mittag-Leffler function and generalized FrFT plays in solving FODE.

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