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SOME REMARKS ON THE GENERALIZED ORDER AND GENERALIZED TYPE OF ENTIRE MATRIX FUNCTIONS IN COMPLETE REINHARDT DOMAIN

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Abstract. The main aim of this paper is to introduce the definitions of generalized order and generalized type of the entire function of complex matrices and then study some of their properties. By considering the concepts of generalized order and generalized type, we will extend some results of Kishka et al. [5].

1. Introduction

In this paper we represent the field of complex variables by $\mathbb{C}$ and the space of several complex complex variables by $\mathbb{C}^n$. We assume that the readers are familiar with the fundamental results and standard notations of the analytic functions of several complex variables. However, In 1959, Gol'dberg had introduced the definitions of the Gol'dberg order and Gol'dberg type of entire function in several complex variables (cf. [2]). For more details about the study of the order and type of entire functions we refer to ([1,3], [6] to [9]). The main purpose of this present paper is to study of entire function of several complex matrices in complete Rinhardt domains which is also known as poly cylindrical regions. After introducing the definitions of generalized order and generalized type of the entire function of complex matrices in complete Reinhardt domains, we study some of their growth properties which considerably extend the earlier results of [5]. To prove our main results we have followed some of the techniques as used by Kishka et al. [5].

Let $\mathbf{z} = (z_1, z_2, ..., z_n)$ be a point of $\mathbb{C}^n$; the space of several complex variables, a closed complete Reinhardt domain of radii ($\alpha_s r > 0$); $s \in \mathbf{I} = 1, 2, 3, ..., n$ is here denoted by $\overline{\Gamma}_{[\alpha r]}$ and is given by

$$\overline{\Gamma}_{[\alpha r]} = \{\mathbf{z} \in \mathbb{C}^n : |z_s| \leq \alpha_s r; s \in \mathbf{I},$$

where α_s are positive numbers.

The open Reinhardt domain is here denoted by $\Gamma_{[\alpha r]}$ and is given by

$$\Gamma_{[\alpha r]} = \{\mathbf{z} \in \mathbb{C}^n : |z_s| < \alpha_s r; s \in \mathbf{I}.$$

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However, we consider unspecified domain containing the closed complete Reinhardt domain $\overline{\Gamma}_{[\alpha r]}$. This domain will be of radii $\alpha_s r_1; r_1 > r$, then making a contraction to this domain, we will get the domain $\overline{D}([\alpha \mathbf{r}^+]) = \overline{D}([\alpha_1 r^+, \alpha_2 r^+, ..., \alpha_n r^+])$, where r^+ stands for the right-limit of r^+ at r^+ (see [4]).

The order and type of entire functions of several complex variables in Reinhardt domain are given as follows:

Definition 1.1. [2, 4, 7] The order ρ of the entire function $f(\mathbf{z})$ for the closed complete Reinhardt domain $\overline{\Gamma}_{[\alpha r]}$ is defined as follows:

$$\rho = \limsup_{r \to +\infty} \frac{\ln^{[2]} M[\alpha r]}{\ln r},$$

where

$$M[\alpha r] = M[\alpha_1 r, \alpha_2 r, ..., \alpha_n r] = \max_{\overline{\Gamma}_{[\alpha r]}} |f(\mathbf{z})|$$

and $\ln^{[0]} r = r$, $\ln^{[2]} r = \ln(\ln r)$.

Definition 1.2. [2, 4, 7] The type τ of the entire function $f(\mathbf{z})$ for the closed complete Reinhardt domain $\overline{\Gamma}_{[\alpha r]}$ is defined as follows:

$$\tau = \limsup_{r \to +\infty} \frac{\ln M[\alpha r]}{r^{\rho}},$$

where $0 < \rho < +\infty$.

Now we give the following two results relating to the entire function $f(\mathbf{z})$ for the closed complete Reinhardt domain $\overline{\Gamma}_{[\alpha r]}$.

Theorem 1.3. [2,4,7] *The necessary and sufficient condition that the entire function $f(\mathbf{z})$ of several complex variables should be of order ρ in the closed complete Reinhardt domain $\overline{\Gamma}_{[\alpha r]}$ is that*

$$\rho = \limsup_{\langle m \rangle \to +\infty} \frac{\langle \mathbf{m} \rangle \ln \langle \mathbf{m} \rangle}{-\ln \left(|a_{\mathbf{m}}| \prod_{s=1}^{n} \alpha_s^{m_s} \right)},$$

where

$$\langle \mathbf{m} \rangle = m_1 + m_2 + m_3 + ... + m_n \text{ and } \mathbf{m} = (m_1 + m_2 + m_3 + ... + m_n).$$

Theorem 1.4. [2,4,7] *The necessary and sufficient condition that the entire function $f(\mathbf{z})$ of several complex variables should be of type τ in the closed complete Reinhardt domain $\overline{\Gamma}_{[\alpha r]}$ is that*

$$\tau = \frac{1}{e\rho} \limsup_{\langle m \rangle \to +\infty} \langle \mathbf{m} \rangle \left(|a_{\mathbf{m}}| \prod_{s=1}^{n} \alpha_s^{m_s} \right)^{\frac{\rho}{\langle m \rangle}}.$$

1.1. Analytic Functions of Complex Matrices. First of all, it is to be mentioned that, for the simplicity, we consider only two complex matrices, though the results can easily be extended to several complex matrices. Taking this into account, let us consider the space $\mathbb{C}^{N \times N}$ of all matrices $X = [x_{ij}]$ and $Y = [y_{ij}]$, where x_{ij} and y_{ij} are complex numbers;$i, j = 1, 2, 3, ..., N$. Let $F(X, Y)$ be a matrix function such that

$$F = [f_{ij}];\ f_{ij} = f(x_{ij}, y_{ij})\ \forall\ i, j = 1, 2, 3, ..., N.$$

Suppose that the matrix function $F(X,Y)$ of two square complex matrices is given by a power series in the form

$$F(X,Y)=\sum_{m,n}a_{m,n}X^mY^n;\quad m,n\geq 0 \tag{1}$$

where

$$X^m=\sum_{k_1,k_2,...,k_{m-1}}x_{ik_1}x_{k_1k_2}...x_{k_{m-1}j}$$

and

$$Y^n=\sum_{k_1,k_2,...,k_{n-1}}x_{ik_1}x_{k_1k_2}...x_{k_{n-1}j},$$

;in the assumption that $X^0=Y^0=I$, where I is the unit matrix of order N and X^mY^n is equal to a square complex matrix $Z=[z_{ij}]$, where

$$z_{ij}=\sum_{k=1}^{N}\{x^m\}_{ik}\{y^n\}_{kj}.$$

Therefore

$$f_{ij}=\sum_{m,n}a_{m,n}z_{ij};\quad m,n\geq 0. \tag{2}$$

Consequently, we can say that the function $F(X,Y)$ is convergent if the elements f_{ij} given in (2) are convergent series for all $i,j=1,2,...,N$. Now we consider the domain which is a subset of the space determined by the two inequalities

$$|X|\ <\|\alpha_1R\|\quad \text{and}\quad |Y|\ <\|\alpha_2R\|. \tag{3}$$

The symbol $|X|$ denotes the matrix $(|x_{ij}|)$ whose elements are the modulii of the elements x_{ij} of the matrix X, and the symbol $\|\alpha\|$ denotes a matrix each of its elements is equal to the positive number. Hence the above two inequalities implies that

$$|x_{ij}|\ <\alpha_1R\quad \text{and}\quad |y_{ij}|\ <\alpha_2R;\quad i,j=1,2,3,...,N.$$

Hence, there is a number r where $0<r<R$ such that

$$|x_{ij}|\ <\alpha_1r\quad \text{and}\quad |y_{ij}|\ <\alpha_2r;\quad i,j=1,2,3,...,N,$$

where $(x_{ij},y_{ij})\in\overline{\Gamma}_{[\alpha_sR]}$; $\alpha_sR\ (>0)$, α_s are positive numbers, $s=1,2$.

Now, Let $F(z,w)=\sum_{m,n}a_{m,n}z^mw^n$ be the scalar function of two variables z and w associated with the matrix function in (1), that $F(z,w)$ is analytic function in the complete Reinhardt domain $\overline{\Gamma}_{[\alpha_sNR]}$. As

$$F(z,w)=\sum_{m,n}a_{m,n}z^mw^n,\quad M[\alpha_s(NR)]=\max_{\overline{\Gamma}_{[\alpha_sNR]}}|F(z,w)| \tag{4}$$

and

$$|a_{m,n}|=\frac{M}{\alpha_1^m\alpha_2^n(NR)^{m+n}};\quad m,n\geq 0, \tag{5}$$

we get that

$$
\begin{aligned}
|f_{ij}| &= |\sum_{m,n} a_{m,n} z_{ij}| \leq \sum_{m,n} |a_{m,n}| |\sum_{k=1}^{N} \{x^m\}_{ik} \{y^n\}_{kj}| \\
&\leq \sum_{m,n} |a_{m,n}| \sum_{k=1}^{N} N^{m-1} (\alpha_1 r)^m N^{n-1} (\alpha_2 r)^n = \frac{M}{N} \sum_{m,n} \Big(\frac{r}{R}\Big)^{m+n} \\
&= \frac{M}{N} \sum_{\nu=1}^{+\infty} \Big(\frac{r}{R}\Big)^{\nu} = \frac{M}{N\Big(1 - \frac{r}{R}\Big)^2}; \qquad (6) \\
i, j &= 1, 2, 3, ..., N; \quad (x_{ij}, y_{ij}) \in \overline{\Gamma}_{[\alpha_s R]}.
\end{aligned}
$$

Therefore the matrix function $F(X, Y)$ as given in (1) is absolute convergence. Since r can be chose arbitrary near to R, then we state the following theorem.

THEOREM 1.5. (see [5, p. 34]) *If the function $F(z, w)$ as given in (4) is analytic in $\overline{\Gamma}_{[\alpha_s NR]}$, then the function $F(X, Y)$ as given in (1) will be analytic in $\overline{\Gamma}_{[\alpha_s R]}$ and bounded on $\overline{\Gamma}_{[\alpha_s NR]}$, where N is the common order of the matrices X and Y.*

If the matrix function

$$
\begin{aligned}
F(X, Y) &= f_1(X) f_2(Y) = \Big(\sum_{m=0}^{+\infty} a_m^1 X^m\Big)\Big(\sum_{n=0}^{+\infty} a_n^2 Y^n\Big) \\
&= \sum_{m,n=0}^{+\infty} a_{m,n} X^m Y^n; \quad a_{m,n} = a_m^1 a_n^2 \qquad (7)
\end{aligned}
$$

associated with the scalar function

$$
\begin{aligned}
F(z, w) &= f_1(z) f_2(w) = \Big(\sum_{m=0}^{+\infty} a_m^1 z^m\Big)\Big(\sum_{n=0}^{+\infty} a_n^2 w^n\Big) \\
&= \sum_{m,n=0}^{+\infty} a_{m,n} z^m w^n; \quad a_{m,n} = a_m^1 a_n^2, \qquad (8)
\end{aligned}
$$

then we obtain the following theorem:

THEOREM 1.6. (see [5, p. 34]) *If the functions f_1 and f_2 of the single variables z and w are analytic in $|z| < \alpha_1 NR$ and$|w| < \alpha_2 NR$, then the matrix function $F(X, Y)$ of square complex matrices X and Y each of them of order N, as given in (7) will be analytic in $\overline{\Gamma}_{[\alpha_s R]}$.*

Now, if we assume that the scalar functions

$$
F(z, w) = \sum_{m,n=0}^{+\infty} a_{m,n} z^m w^n \text{ and } G(z, w) = \sum_{m,n=0}^{+\infty} b_{m,n} z^m w^n \qquad (9)
$$

are analytic in $\overline{\Gamma}_{[\alpha_s NR]}$, then according to (5), we obtain that

$$
|a_{m,n}| \leq \frac{M_1}{\alpha_1^m \alpha_2^n (NR)^{m+n}}; \qquad m, n \geq 0, \, M_1 \geq 1 \qquad (10)
$$

and

$$|b_{m,n}| \ \leq \frac{M_2}{\alpha_1^m \alpha_2^n (NR)^{m+n}}; \qquad m,n \geq 0,\ M_2 \geq 1. \tag{11}$$

Let $F(X,Y)$ and $G(X,Y)$ be the matrix functions associated with the scalar functions (9) in the form

$$F(X,Y) = \sum_{m,n=0}^{+\infty} a_{m,n} X^m Y^n \ \text{ and } \ G(X,Y) = \sum_{m,n=0}^{+\infty} b_{m,n} X^m Y^n. \tag{12}$$

Then we can write the product matrix function $P(X,Y)$ as follows:

$$P(X,Y) = F(X,Y) \cdot G(X,Y) = \sum_{m,n=0} C_{m,n} X^m Y^n, \tag{13}$$

where

$$C_{m,n} = \sum_{h=0}^{m} \sum_{k=0}^{n} a_{h,k} b_{m-h,n-k}.$$

From (10) and (11), one may deduce that

$$\begin{aligned} C_{m,n} &= \sum_{h=0}^{m} \sum_{k=0}^{n} a_{h,k} b_{m-h,n-k} \\ &\leq \sum_{h=0}^{m} \sum_{k=0}^{n} \frac{M_1 M_2}{\alpha_1^m \alpha_2^n (NR)^{m+n}} = (m+1)(n+1)\frac{M_1 M_2}{\alpha_1^m \alpha_2^n (NR)^{m+n}}. \end{aligned} \tag{14}$$

Thus

$$\begin{aligned} &\max_{\overline{\Gamma}_{[\alpha_s N r]}} \left\| \sum_{m,n=0} C_{m,n} X^m Y^n \right\| \\ &\leq \sum_{m,n=0} |C_{m,n}| \max_{\overline{\Gamma}_{[\alpha_s N r]}} \|X^m Y^n\| \\ &\leq (m+1)(n+1)\frac{M_1 M_2 (r)^{m+n}}{(R)^{m+n}} < +\infty. \end{aligned} \tag{15}$$

Therefore the product matrix function $P(X,Y)$ given in (13) is analytic function in the complete Reinhardt domain $\overline{\Gamma}_{[\alpha_s NR]}$. Since r can be chose arbitrary near to R, then we state the following theorem.

Theorem 1.7. (see [5, p. 35]) *The matrix function $P(X,Y)$ as given in (13) is absolute convergence in $\overline{\Gamma}_{[\alpha_s NR]}$ and analytic in some region if the functions $F(z,w)$ and $G(z,w)$ as given in (9) are analytic in $\overline{\Gamma}_{[\alpha_s NR]}$.*

1.2. On The Order and Type of Entire Matrix Functions. Let

$$F(X,Y) = \sum_{m,n} a_{m,n} X^m Y^n; \quad m,n \geq 0, \tag{16}$$

be an entire function of two square complex matrices X and Y each of them is of order N. Then it follows that

$$M[\alpha_s r] = M[\alpha_1 r, \alpha_2 r] = \max_{ij} \max_{\overline{\Gamma}_{[\alpha_s r]}} |F(X,Y)|. \tag{17}$$

So

$$|a_{m,n}|\alpha_1^m\alpha_2^n \leq \frac{NM[\alpha_s r]}{(rN)^{m+n}}; \qquad m,n \geq 0. \tag{18}$$

Therefore, the radius of regularity of the matrix function $F(X,Y)$ is infinity, i. e.,

$$\limsup_{m+n\to+\infty} \{N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n\}^{\frac{1}{m+n}} = 0. \tag{19}$$

In this connection, we recall the following two definitions.

DEFINITION 1.8. [5] The order Ω of the entire matrix function $F(X,Y)$ is given by

$$\Omega = \limsup_{r\to+\infty} \frac{\ln^{[2]} M[\alpha_s r]}{\ln r}.$$

DEFINITION 1.9. [5] The type Θ of the entire matrix function $F(X,Y)$ with order $\Omega \in (0,+\infty)$ is given by

$$\Theta = \limsup_{r\to+\infty} \frac{\ln M[\alpha_s r]}{r^{\Omega}}.$$

If the entire matrix function $F(X,Y)$ is given by a power series in (16), then we state the following two results due to Kishka et al. [5] concerning the function of two square complex matrices:

THEOREM 1.10. [5] *A necessary and sufficient condition that the entire matrix function* $F(X,Y) = \sum\limits_{m,n} a_{m,n}X^mY^n$ *should be of order* Ω *is that*

$$\Omega = \limsup_{m+n\to+\infty} \frac{(m+n)\ln(m+n)}{-\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)}.$$

THEOREM 1.11. [5] *If the entire matrix function* $F(X,Y) = \sum\limits_{m,n} a_{m,n}X^mY^n$ *is of finite generalized order* Ω*, then the necessary and sufficient condition should be of type* Θ *is that*

$$\Theta = \frac{N^{\Omega}}{e\Omega} \limsup_{m+n\to+\infty} (m+n)\{|a_{m,n}|\alpha_1^m\alpha_2^n\}^{\frac{\Omega}{m+n}}.$$

2. Main Results

First of all let L be a class of continuous non-negative on $(-\infty,+\infty)$ function β such that $\beta(r) = \beta(r_0) \geq 0$ for $r \leq r_0$ and $\beta(r) \uparrow +\infty$ as $r_0 \leq r \to +\infty$. We say that $\beta \in L_1$, if $\beta \in L$ and $\beta((1+o(1))r) = (1+o(1))\beta(r)$ as $r \to +\infty$. Finally, $\beta \in L_{si}$, if $\beta \in L$ and $\beta(cr) = (1+o(1))\beta(r)$ as $r \to +\infty$ for each fixed $c \in (0,+\infty)$, i.e., β is slowly increasing function. Clearly $L_{si} \subset L_1$.

Considering this, Sheremeta [11] in 1967, introduced the concept of generalized order of entire functions in complex context taking two function belonging to L. For details about the generalized order of entire functions, one may see [11]. However, during the past decades, several authors made close investigations on the properties of entire functions related to generalized order in some different direction. For the purpose of further applications, here in this paper we introduce the definitions of the generalized order and the generalized type of the entire matrix function $F(X,Y) = \sum\limits_{m,n} a_{m,n}X^mY^n$ in the following way:

Definition 2.1. The generalized order ϱ of the entire matrix function $F(X,Y)$ is given by

$$\varrho = \limsup_{r\to+\infty} \frac{\beta_1(\ln^{[2]} M[\alpha_s r])}{\beta_2(\ln r)} \ (\beta_1 \in L,\ \beta_2 \in L).$$

Definition 2.2. The generalized type λ of the entire matrix function $F(X,Y)$ with generalized order $\varrho \in (0,+\infty)$ is given by

$$\lambda = \limsup_{r\to+\infty} \frac{\exp(\beta_1(\ln^{[2]} M[\alpha_s r]))}{(\exp(\beta_2(\ln r)))^{\varrho}} \ (\beta_1 \in L,\ \beta_2 \in L).$$

Remark 2.3. If $\beta_1(r) = \beta_2(r) = r$, then Definition 1.8 and Definition 1.9 are special cases of Definition 2.1 and Definition 2.2 respectively.

Now we add three conditions on β_1 and β_2: (i) β_1 and β_2 always denote the functions belonging to L_1, (ii) $\beta_1(r) = o\Big(\beta_2\Big(\frac{\exp r}{r}\Big)\Big)$ as $r \to +\infty$ and (iii) $\beta_1(\ln r) = o(\beta_2(r))$ as $r \to +\infty$. Henceforth, we assume that β_1 and β_2 always satisfy the above three conditions.

Now we present the main results of this paper. In the sequel, we use the following notation due to Sato [10]:

$$\exp^{[0]} r = r, \exp^{[2]} r = \exp(\exp r).$$

Theorem 2.4. *If*

$$\limsup_{r\to+\infty} \frac{\beta_1(\ln^{[2]} M[\alpha_s r])}{\beta_2(\ln r)} \le \gamma, \tag{20}$$

then

$$\limsup_{m+n\to+\infty} \frac{\beta_1(\ln(m+n))}{\beta_2\Big(\frac{-\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)}{(m+n)}\Big)} \le \gamma. \tag{21}$$

Proof. If $\gamma = +\infty$ then there is nothing to prove. If $\gamma_1 > \gamma$, then for a suitable number r_0, we get from (20) that

$$M[\alpha_s r] < \exp^{[2]}(\beta_1^{-1}(\gamma_1\beta_2(\ln r))); \qquad r_0 < r,$$

hence by Cauchy's inequality in (18) gives

$$N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n \le \min_{r_0<r} N\frac{\exp^{[2]}(\beta_1^{-1}(\gamma_1\beta_2(\ln r)))}{(r)^{m+n}}; \qquad r_0 < r. \tag{22}$$

Now we choose the integer μ such that

$$\exp\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(m+n)\Big)\Big) > r_0 \text{.....for } m+n > \mu. \tag{23}$$

So from (22) and (23) we get that

$$\begin{aligned} N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n &\le \min_{r>r_0} N\frac{\exp^{[2]}(\beta_1^{-1}(\gamma_1\beta_2(\ln r)))}{(r)^{m+n}} \\ &= N\frac{\exp^{[2]}(m+n)}{\Big(\exp\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(m+n)\Big)\Big)\Big)^{m+n}}; \\ m+n &> \mu. \end{aligned}$$

Thus we get from above that

$$\begin{aligned}
&\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)\\
\leq\ &\ln N+\ln\Big(\frac{\exp^{[2]}(m+n)}{\Big(\exp\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(m+n)\Big)\Big)\Big)^{m+n}}\Big)
\end{aligned}$$

$$\begin{aligned}
&i.e.,\ \ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)\\
\leq\ &\ln N+\exp(m+n)-\ln\Big(\exp\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(m+n)\Big)\Big)\Big)^{m+n}
\end{aligned}$$

$$\begin{aligned}
&i.e.,\ -\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)\\
\geq\ &-\ln N-\exp(m+n)+(m+n)\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(m+n)\Big)\Big)
\end{aligned}$$

$$\begin{aligned}
&i.e.,\ \frac{(m+n)\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(\ln(m+n))\Big)\Big)}{-\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)}\\
<\ &\frac{(m+n)\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(\ln(m+n))\Big)\Big)}{-\ln N-\exp(m+n)+(m+n)\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(m+n)\Big)\Big)}\\
<\ &\frac{\frac{(m+n)\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(\ln(m+n))\Big)\Big)}{(m+n)\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(m+n)\Big)\Big)}}{-\frac{\ln N}{(m+n)\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(m+n)\Big)\Big)}-\frac{\exp(m+n)}{(m+n)\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(m+n)\Big)\Big)}+1}. \qquad (24)
\end{aligned}$$

Since $\frac{\beta_2\left(\frac{\exp r}{r}\right)}{\beta_1(r)}\to+\infty$ as $r\to+\infty$,

so, $\frac{\frac{(m+n)\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(\ln(m+n))\Big)\Big)}{(m+n)\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(m+n)\Big)\Big)}}{-\frac{\ln N}{(m+n)\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(m+n)\Big)\Big)}-\frac{\exp(m+n)}{(m+n)\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(m+n)\Big)\Big)}+1}\to 0$ as $m+n\to+\infty$. Therefore from (24), we get that

$$\begin{aligned}
&\frac{(m+n)\Big(\beta_2^{-1}\Big(\frac{1}{\gamma_1}\beta_1(\ln(m+n))\Big)\Big)}{-\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)}\to 0 \text{ as } m+n\to+\infty\\
&i.e.,\ \frac{1}{\gamma_1}<\frac{\beta_2\left(\frac{-\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)}{(m+n)}\right)}{\beta_1(\ln(m+n))}\\
&i.e.,\ \limsup_{m+n\to+\infty}\frac{\beta_1(\ln(m+n))}{\beta_2\left(\frac{-\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)}{(m+n)}\right)}\leq\gamma_1.
\end{aligned}$$

Since γ_1 can be chosen arbitrary near to γ, therefore the conclusion of the theorem follows from above. $\square$

THEOREM 2.5. *If*

$$\limsup_{m+n\to+\infty}\frac{\beta_1(\ln(m+n))}{\beta_2\Big(\frac{-\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)}{(m+n)}\Big)}\le\gamma, \tag{25}$$

then

$$\limsup_{r\to+\infty}\frac{\beta_1(\ln^{[2]}M[a_sr])}{\beta_2(\ln r)}\le\gamma. \tag{26}$$

Proof. If $\gamma=+\infty$ then there is nothing to prove. If $\gamma_1>\gamma$,then there is an integer μ such that

$$\frac{\beta_1(\ln(m+n))}{\beta_2\Big(\frac{-\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)}{(m+n)}\Big)}\le\gamma_1;\qquad m+n>\mu,$$

$$i.e.,\ \frac{\beta_1(\ln(m+n))}{\gamma_1}\le\beta_2\Big(\frac{-\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)}{(m+n)}\Big)$$

$$\begin{aligned}&i.e.,\ N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n\\ \le\ &\exp\Big(-(m+n)\beta_2^{-1}\Big(\frac{\beta_1(\ln(m+n))}{\gamma_1}\Big)\Big);\qquad m+n>\mu.\end{aligned} \tag{27}$$

By using (16) and (17), we obtain that

$$M[\alpha_sr]\le\ \max_{ij}\ \max_{\Gamma_{[a_sr]}}\Big|\sum_{m,n}a_{m,n}X^mY^n\Big|\le\frac{1}{N}\sum_{m,n=0}^{+\infty}(Nr)^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n\ . \tag{28}$$

Now for a number $r_0>1$ such that $\exp(\beta_1^{-1}(\gamma_1\beta_2(\ln(2r))))>\mu$ and $r>r_0$, we can fix the integer n_1 such that

$$n_1\le\exp(\beta_1^{-1}(\gamma_1\beta_2(\ln(2r))))<n_1+1;r>r_0.$$

then from (27), (28) and above we get that

$$M[\alpha_sr]\le\frac{1}{N}\left\{\sum_{m,n=0}^{\mu}+\sum_{m,n=\mu+1}^{+\infty}\right\}(Nr)^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n$$

$$=\frac{1}{N}\left\{A+\sum_{m,n=\mu+1}^{+\infty}(r)^{m+n}\exp\Big(-(m+n)\beta_2^{-1}\Big(\frac{\beta_1(\ln(m+n))}{\gamma_1}\Big)\Big)\right\}$$

$$\begin{aligned}=\ &\frac{1}{N}\left\{A+\sum_{m,n=\mu+1}^{n_1}(r)^{m+n}\exp\Big(-(m+n)\beta_2^{-1}\Big(\frac{\beta_1(\ln(m+n))}{\gamma_1}\Big)\Big)+\right.\\ &\left.\sum_{m,n=n_1+1}^{+\infty}(r)^{m+n}\exp\Big(-(m+n)\beta_2^{-1}\Big(\frac{\beta_1(\ln(m+n))}{\gamma_1}\Big)\Big)\right\}.\end{aligned} \tag{29}$$

Now

$$\begin{aligned}
&\sum_{m,n=\mu+1}^{n_1}(r)^{m+n}\exp\Big(-(m+n)\beta_2^{-1}\Big(\frac{\beta_1(\ln(m+n))}{\gamma_1}\Big)\Big)\\
<\;& r^{n_1}\sum_{m,n=\mu}^{n_1}\exp\Big(-(m+n)\beta_2^{-1}\Big(\frac{\beta_1(\ln(\mu+1))}{\gamma_1}\Big)\Big)\\
<\;& r^{n_1}\sum_{m,n=0}^{+\infty}\exp\Big(-(m+n)\beta_2^{-1}\Big(\frac{\beta_1(\ln(\mu+1))}{\gamma_1}\Big)\Big)\\
=\;& Br^{\exp(\beta_1^{-1}(\gamma_1\beta_2(\ln(2r))))}, \qquad (30)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{m,n=n_1+1}^{+\infty}(r)^{m+n}\exp\Big(-(m+n)\beta_2^{-1}\Big(\frac{\beta_1(\ln(m+n))}{\gamma_1}\Big)\Big)\\
<\;& \sum_{m,n=n_1+1}^{+\infty}(r)^{m+n}\exp\Big(-(m+n)\beta_2^{-1}\Big(\frac{\beta_1(\ln(n_1+1))}{\gamma_1}\Big)\Big)\\
<\;& \sum_{m,n=n_1+1}^{+\infty}\Big(\frac{1}{2}\Big)^{m+n}<\sum_{m,n=0}^{\infty}\Big(\frac{1}{2}\Big)^{m+n}=C. \qquad (31)
\end{aligned}$$

Therefore from (29), (30) and (31) we get that

$$M[\alpha_s r]\leq K\exp(\exp(\beta_1^{-1}(\gamma_1\beta_2(\ln(2r))))\ln r), \qquad r>r_0,$$

where$B$, C, and K are constants. Hence from above we get that

$$\ln^{[2]}M[\alpha_s r]\leq \beta_1^{-1}(\gamma_1\beta_2(\ln(2r)))+\ln^{[2]}r+o(1).$$

Since $\frac{\beta_1(\ln r)}{\beta_2(r)}\to 0$ as $r\to+\infty$ and $\beta_2\in L_1$, so it follows from above that

$$\begin{aligned}
\beta_1(\ln^{[2]}M[\alpha_s r]) &\leq (1+o(1))\gamma_1\beta_2(\ln(2r))\\
\text{i.e.,}\quad \frac{\beta_1(\ln^{[2]}M[\alpha_s r])}{(1+o(1))\beta_2(\ln(r))} &\leq (1+o(1))\gamma_1. \qquad (32)
\end{aligned}$$

Making r tend to infinity, we get from (32) that

$$\limsup_{r\to+\infty}\frac{\beta_1(\ln^{[2]}M[\alpha_s r])}{\beta_2(\ln r)}\leq\gamma_1. \qquad (33)$$

Since γ_1 can be chosen arbitrary near to γ, therefore the conclusion of the theorem follows from (33). □

The following theorem is a natural consequence of Theorem 2.4 and Theorem 2.5.

THEOREM 2.6. *A necessary and sufficient condition that the entire matrix function $F(X,Y)=\sum_{m,n}a_{m,n}X^mY^n$ should be of generalized order ϱ is that*

$$\varrho=\limsup_{m+n\to+\infty}\frac{\beta_1(\ln(m+n))}{\beta_2\Big(\frac{-\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)}{(m+n)}\Big)}.$$

The proof is omitted.

THEOREM 2.7. *If*

$$\limsup_{r\to+\infty}\frac{\exp(\beta_1(\ln^{[2]}M[a_s r]))}{(\exp(\beta_2(\ln r)))^{\varrho}}\le\gamma, \tag{34}$$

then

$$\limsup_{m+n\to+\infty}\frac{\exp(\beta_1(\ln(m+n)))}{\Big(\exp\Big(\beta_2\Big(\ln\Big(\frac{1}{\frac{N}{e}(|a_{m,n}|\alpha_1^m\alpha_2^n)^{\frac{1}{m+n}}}\Big)\Big)\Big)\Big)^{\varrho}}\le\gamma. \tag{35}$$

Proof. If $\gamma=+\infty$ then there is nothing to prove. If $\gamma_1>\gamma$, then for a suitable number r_0, we get from (34) that

$$M[\alpha_s r]<\exp^{[2]}(\beta_1^{-1}(\ln(\gamma_1(\exp(\beta_2(\ln r)))^{\varrho})));\qquad r_0<r,$$

hence from (18) we get that

$$N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n\le\min_{r_0<r}N\frac{\exp^{[2]}(\beta_1^{-1}(\ln(\gamma_1(\exp(\beta_2(\ln r)))^{\varrho})))}{(r)^{m+n}};\qquad r_0<r. \tag{36}$$

Now we choose the integer μ such that

$$\exp\Big(\beta_2^{-1}\Big(\ln\Big(\frac{\exp(\beta_1(\ln(m+n)))}{\gamma_1}\Big)^{\frac{1}{\varrho}}\Big)\Big)>r_0.....\text{for } m+n>\mu. \tag{37}$$

So from (36) and (37) we get that

$$\begin{aligned}
N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n &\le \min_{r>r_0}N\frac{\exp^{[2]}(\beta_1^{-1}(\ln(\gamma_1(\exp(\beta_2(\ln r)))^{\varrho})))}{(r)^{m+n}}\\
&= N\frac{\exp(m+n)}{\Big(\exp\Big(\beta_2^{-1}\Big(\ln\Big(\frac{\exp(\beta_1(\ln(m+n)))}{\gamma_1}\Big)^{\frac{1}{\varrho}}\Big)\Big)\Big)^{m+n}};\\
m+n &> \mu,
\end{aligned}$$

$$\begin{aligned}
&i.e.,\ \frac{N}{e}(|a_{m,n}|\alpha_1^m\alpha_2^n)^{\frac{1}{m+n}}\\
&\le \frac{1}{\exp\Big(\beta_2^{-1}\Big(\ln\Big(\frac{\exp(\beta_1(\ln(m+n)))}{\gamma_1}\Big)^{\frac{1}{\varrho}}\Big)\Big)}\\
&i.e.,\ \frac{1}{\frac{N}{e}(|a_{m,n}|\alpha_1^m\alpha_2^n)^{\frac{1}{m+n}}}\\
&\ge \exp\Big(\beta_2^{-1}\Big(\ln\Big(\frac{\exp(\beta_1(\ln(m+n)))}{\gamma_1}\Big)^{\frac{1}{\varrho}}\Big)\Big)\\
&i.e.,\ \Big(\exp\Big(\beta_2\Big(\ln\Big(\frac{1}{\frac{N}{e}(|a_{m,n}|\alpha_1^m\alpha_2^n)^{\frac{1}{m+n}}}\Big)\Big)\Big)\Big)^{\varrho}\\
&\ge \frac{\exp(\beta_1(\exp(m+n)))}{\gamma_1}
\end{aligned}$$

$$
\begin{aligned}
&i.e., \frac{1}{\left(\exp\left(\beta_2\left(\ln\left(\frac{1}{\frac{N}{e}(|a_{m,n}|\alpha_1^m\alpha_2^n)^{\frac{1}{m+n}}}\right)\right)\right)\right)^{\varrho}} \\
&\leq \quad \frac{\gamma_1}{\exp(\beta_1(\exp(m+n)))}
\end{aligned}
$$

$$
i.e., \limsup_{m+n\to+\infty} \frac{\exp(\beta_1(\exp(m+n)))}{\left(\exp\left(\beta_2\left(\ln\left(\frac{1}{\frac{N}{e}(|a_{m,n}|\alpha_1^m\alpha_2^n)^{\frac{1}{m+n}}}\right)\right)\right)\right)^{\varrho}} \leq \gamma_1.
$$

As γ_1 can be taken arbitrary near to γ, hence the required inequality of the theorem is established from above. □

THEOREM 2.8. *If*

$$
\limsup_{m+n\to+\infty} \frac{\exp(\beta_1(\ln(m+n)))}{\left(\exp\left(\beta_2\left(\ln\left(\frac{1}{\frac{N}{e}(|a_{m,n}|\alpha_1^m\alpha_2^n)^{\frac{1}{m+n}}}\right)\right)\right)\right)^{\varrho}} \leq \gamma, \tag{38}
$$

then

$$
\limsup_{r\to+\infty} \frac{\exp(\beta_1(\ln^{[2]} M[a_s r]))}{(\exp(\beta_2(\ln r)))^{\varrho}} \leq \gamma. \tag{39}
$$

Proof. If $\gamma_1 \geq \gamma$,choose an integer $\mu > 1$ such that we can have from (38) that

$$
\begin{aligned}
&\exp\left(\beta_2^{-1}\left(\ln\left(\left(\frac{\exp(\beta_1(\ln(m+n)))}{\gamma}\right)^{\frac{1}{\varrho}}\right)\right)\right) \\
&\leq \quad \frac{1}{\frac{N}{e}(|a_{m,n}|\alpha_1^m\alpha_2^n)^{\frac{1}{m+n}}}; \qquad m+n > \mu,
\end{aligned}
$$

$$
\begin{aligned}
&i.e., \ |a_{m,n}|\alpha_1^m\alpha_2^n \\
&\leq \quad \left(\frac{e}{N\cdot\left(\exp\left(\beta_2^{-1}\left(\ln\left(\left(\frac{\exp(\beta_1(\ln(m+n)))}{\gamma_1}\right)^{\frac{1}{\varrho}}\right)\right)\right)\right)}\right)^{m+n}.
\end{aligned} \tag{40}
$$

Since,

$$
M[\alpha_s r] \leq \frac{1}{N}\sum_{m,n=0}^{+\infty}(Nr)^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n,
$$

so we get in view of (40) that

$$
M[\alpha_s r] \leq \frac{1}{N}\sum_{m,n=0}^{+\infty}(Nr)^{m+n}\left(\frac{e}{N\cdot\left(\exp\left(\beta_2^{-1}\left(\ln\left(\left(\frac{\exp(\beta_1(\ln(m+n)))}{\gamma_1}\right)^{\frac{1}{\varrho}}\right)\right)\right)\right)}\right)^{m+n}.
$$

For a number $r_0 > 1$ such that
$2\exp(\beta_1^{-1}(\ln(\gamma_1(\exp(\beta_2(\ln(re))))^{\varrho}))) > \mu$ and $r > r_0$, we can fix the integer n_1 such that $n_1 \leq 2\exp(\beta_1^{-1}(\ln(\gamma_1(\exp(\beta_2(\ln(re))))^{\varrho}))) < n_1 + 1$; $r > r_0$.

Therefore

$$\begin{aligned} & M[\alpha_s r] \\ \le\ & \frac{1}{N}\left\{\sum_{m,n=0}^{\mu}+\sum_{m,n=\mu+1}^{+\infty}\right\}(Nr)^{m+n} \\ & \cdot\left(\frac{e}{N\cdot\left(\exp\left(\beta_2^{-1}\left(\ln\left(\left(\frac{\exp(\beta_1(\ln(m+n)))}{\gamma_1}\right)^{\frac{1}{\varrho}}\right)\right)\right)\right)}\right)^{m+n} \end{aligned}$$

$$= \frac{1}{N}\left\{A+\sum_{m,n=0}^{n_1}(Nr)^{m+n}\left(\frac{e}{N\cdot\left(\exp\left(\beta_2^{-1}\left(\ln\left(\left(\frac{\exp(\beta_1(\ln(m+n)))}{\gamma_1}\right)^{\frac{1}{\varrho}}\right)\right)\right)\right)}\right)^{m+n}+\right.$$

$$\left.\sum_{m,n=\mu+1}^{+\infty}(Nr)^{m+n}\left(\frac{e}{N\cdot\left(\exp\left(\beta_2^{-1}\left(\ln\left(\left(\frac{\exp(\beta_1(\ln(m+n)))}{\gamma_1}\right)^{\frac{1}{\varrho}}\right)\right)\right)\right)}\right)^{m+n}\right\}$$

$$\le \{A+B\exp^{[2]}(\beta_1^{-1}(\ln(\gamma_1(\exp(\beta_2(\ln r)))^{\varrho})))+C\}$$

$$\le K\exp^{[2]}(\beta_1^{-1}(\ln(\gamma_1(\exp(\beta_2(\ln r)))^{\varrho}))). \tag{41}$$

Making r tend to infinity, we infer from (41) such that

$$\limsup_{r\to+\infty}\frac{\exp(\beta_1(\ln^{[2]}M[\alpha_s r]))}{(\exp(\beta_2(\ln r)))^{\varrho}} \le \gamma_1.$$

As γ_1 can be taken arbitrary near to γ, hence the required inequality of the theorem is established from above. $\square$

Combining Theorem 2.7 and Theorem 2.8 we may state the following theorem.

THEOREM 2.9. *If the entire matrix function $F(X,Y)=\sum_{m,n} a_{m,n}X^mY^n$ is of finite generalized order ϱ, then the necessary and sufficient condition should be of generalized type λ is that*

$$\lambda = \limsup_{m+n\to+\infty}\frac{\exp(\beta_1(\ln(m+n)))}{\left(\exp\left(\beta_2\left(\ln\left(\frac{1}{\frac{N}{e}(|a_{m,n}|\alpha_1^m\alpha_2^n)^{\frac{1}{m+n}}}\right)\right)\right)\right)^{\varrho}}.$$

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