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CONTROLLED K-FRAMES IN HILBERT C*-MODULES

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Abstract. Controlled frames have been the subject of interest because of their ability to improve the numerical efficiency of iterative algorithms for inverting the frame operator. In this paper, we introduce the notion of controlled K-frame or controlled operator frame in Hilbert C^*-modules. We establish the equivalent condition for controlled K-frame. We investigate some operator theoretic characterizations of controlled K-frames and controlled Bessel sequences. Moreover, we establish the relationship between the K-frames and controlled K-frames. We also investigate the invariance of a controlled K-frame under a suitable map T. At the end, we prove a perturbation result for controlled K-frame.

1. Introduction

Frames a more flexible substitutes of bases in Hilbert spaces were first proposed by Duffin and Schaeffer [15] in 1952 while studying nonharmonic Fourier series. Daubechies, Grossmann and Meyer [6] reintroduced and developed the theory of frames in 1986. Due to their rich structure the subject drew the attention of many mathematician, physicists and engineers because of its applicability in signal processing [14], image processing [4], coding and communications [16], sampling [20, 21], numerical analysis, filter theory [5]. Nowadays it is used in compressive sensing, data analysis, and other areas. In general, frames can be viewed as a redundant representation of basis. Due to its redundancy, it becomes more applicable not only from a theoretical point of view but also in various kinds of applications.
Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of real or complex numbers. They were introduced and investigated initially by Kaplansky [7]. Frank and Larson [10] defined the concept of standard frames in finitely or countably generated Hilbert C^*-modules over a unital C^*-algebra. For more details of frames in Hilbert C^*-modules one may refer to Doctoral Dissertation [17], Han et al. [2] and Han et al. [3]. In 2012, L. Gavruta [9] introduced the notion of K-frames in Hilbert space to study the atomic systems with respect to a bounded linear operator K. Controlled frames in Hilbert spaces have been introduced by P. Balazs [13] to improve the numerical efficiency of

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iterative algorithms for inverting the frame operator. Rahimi [11] defined the concept of controlled K-frames in Hilbert spaces and showed that controlled K-frames are equivalent to K-frames due to which the controlled operator C can be used as preconditions in applications. In [1], Najati et al. introduced the concepts of an atomic system for operators and K-frames in Hilbert C^*-modules. Controlled frames in Hilbert C^*-modules were introduced by Rashidi and Rahimi [12], and the authors showed that they share many useful properties with their corresponding notions in a Hilbert space. Motivated by the above literature, we introduce the notion of a controlled K-frame in Hilbert C^*-modules.

2. Preliminaries

In this section we give some basic definitions related to Hilbert C^*-modules, frames, K-frames, controlled frames in Hilbert C^*-modules. Hilbert C^*-modules are generalization of Hilbert spaces by allowing the inner product to take values in C^*-algebra rather than $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. Let $\mathcal{A}$ be a C^*-algebra. An *inner product $\mathcal{A}$-module* is a complex vector space $\mathcal{H}$ such that
(i) $\mathcal{H}$ is a right $\mathcal{A}$-module i.e there is a bilinear map

$$\mathcal{H} \times \mathcal{A} \to \mathcal{A} \colon (x, a) \to x \cdot a$$

satisfying $(x \cdot a) \cdot b = x \cdot (ab)$ and $(\lambda x) \cdot a = x \cdot (\lambda a)$, and $x \cdot 1 = x$ where $\mathcal{A}$ has a unit 1.
(ii) There is a map $\mathcal{H} \times \mathcal{H} \to \mathcal{A} \colon (x, y) \to \langle x, y\rangle$ satisfying
1. $\langle x, x\rangle \geq 0$
2. $\langle x, y\rangle^* = \langle y, x\rangle$
3. $\langle ax, y\rangle = a\langle x, y\rangle$
4. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
5. $\langle x, x\rangle = 0$ if and only if $x = 0$ (for every $x, y, z \in \mathcal{H}$, $a \in \mathcal{A}$).

Definition 2.2. A *Hilbert C^*-module* over $\mathcal{A}$ is an inner product $\mathcal{A}$-module with the property that $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|_{\mathcal{A}}^{\frac{1}{2}}$, where $\|.\|_{\mathcal{A}}$ denotes the norm on $\mathcal{A}$.

Let $\mathcal{A}$ be a C^*-algebra and consider

$$l^2(\mathcal{A}) = \{\{a_j\}_{j\in\mathbb{J}} \subseteq \mathcal{A} : \sum_{j\in\mathbb{J}} a_j a_j^* \text{ converges in norm in } \mathcal{A}\}$$

It is easy to see that $l^2(\mathcal{A})$ is a Hilbert C^*-module with pointwise operations and the inner product defined as

$$\langle\{a_j\}, \{b_j\}\rangle = \sum_{j\in\mathbb{J}} a_j b_j^*, \ \{a_j\}, \{b_j\} \in l^2(\mathcal{A})$$

and

$$\|\{a_j\}\| = \sqrt{\|\sum_{j\in\mathbb{J}} a_j a_j^*\|}.$$

Definition 2.3. [17] Let $\mathcal{A}$ be a unital C^*-algebra and $j \in \mathbb{J}$ be a finite or countable index set. A sequence $\{\psi_j\}_{j\in\mathbb{J}}$ of elements in a Hilbert $\mathcal{A}$ -module $\mathcal{H}$ is said to be a frame if there exist two constants $C, D > 0$ such that

$$C\langle f, f\rangle \leq \sum_{j\in\mathbb{J}} \langle f, \psi_j\rangle\langle \psi_j, f\rangle \leq D\langle f, f\rangle, \forall f \in \mathcal{H}. \tag{2.1}$$

The frame $\{\psi_j\}_{j\in\mathbb{J}}$ is said to be a tight frame if $C = D$, and is said to be Parseval or a normalized tight frame if $C = D = 1$.
Suppose that $\{\psi_j\}_{j\in\mathbb{J}}$ is a frame of a finitely or countably generated Hilbert C^*-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. The operator $T \colon \mathcal{H} \to l^2(\mathcal{A})$ defined by

$$Tf = \{\langle f, \psi_j\rangle\}_{j\in\mathbb{J}}$$

is called the *analysis operator*.
The adjoint operator $T^* \colon l^2(\mathcal{A}) \to \mathcal{H}$ is given by

$$T^*\{c_j\}_{j\in\mathbb{J}} = \sum_{j\in\mathbb{J}} c_j\psi_j$$

T^* is called *pre-frame operator or the synthesis operator*.
By composing T and T^*, we obtain the *frame operator* $S \colon \mathcal{H} \to \mathcal{H}$

$$Sf = T^*Tf = \sum_{j\in\mathbb{J}} \langle f, \psi_j\rangle\psi_j. \tag{2.2}$$

Definition 2.4. [1] A sequence $\{\psi_j\}_{j\in\mathbb{J}}$ of elements in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ is said to be a K-frame ($K \in L(\mathcal{H})$) if there exist constants $C, D > 0$ such that

$$C\langle K^*f, K^*f\rangle \leq \sum_{j\in\mathbb{J}} \langle f, \psi_j\rangle\langle \psi_j, f\rangle \leq D\langle f, f\rangle, \forall f \in \mathcal{H}. \tag{2.3}$$

Definition 2.5. [12] Let $\mathcal{H}$ be a Hilbert C^*-module and $C \in GL(\mathcal{H})$. A frame controlled by the operator C or C -controlled frame in Hilbert C^*-module $\mathcal{H}$ is a family of vectors $\{\psi_j\}_{j\in\mathbb{J}}$, such that there exist two constants $A, B > 0$ satisfying

$$A\langle f, f\rangle \leq \sum_{j\in\mathbb{J}} \langle f, \psi_j\rangle\langle C\psi_j, f\rangle \leq B\langle f, f\rangle, \forall f \in \mathcal{H}.$$

Likewise, $\{\psi_j\}_{j\in\mathbb{J}}$ is called a C-controlled Bessel sequence with bound B, if there exists $B > 0$ such that

$$\sum_{j\in\mathbb{J}} \langle f, \psi_j\rangle\langle C\psi_j, f\rangle \leq B\langle f, f\rangle, \forall f \in \mathcal{H},$$

where the sum in the above inequalities converges in norm.
If $A = B$, we call $\{\psi_j\}_{j\in\mathbb{J}}$ as C-controlled tight frame, and if $A = B = 1$ it is called a C-controlled Parseval frame.

3. Controlled K-frames

For the rest of the paper we assume that $\mathcal{H}$ is a Hilbert C^*-module over unital C^*-algebra $\mathcal{A}$ with $\mathcal{A}$-valued inner product $\langle ., .\rangle$ and norm $\|.\|$. $L(\mathcal{H})$ denotes the set of all adjointable operators on Hilbert C^*-module $\mathcal{H}$, and $GL^+(\mathcal{H})$ indicates the set of all bounded linear positive invertible operators on $\mathcal{H}$ with bounded inverse. We define below the controlled operator frame or C-controlled K-frame on a Hilbert C^*-module $\mathcal{H}$.

Definition 3.1. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module over a unital C^*-algebra, $C \in GL^+(\mathcal{H})$ and $K \in L(\mathcal{H})$. A sequence $\{\psi_j\}_{j\in\mathbb{J}}$ in $\mathcal{H}$ is said to be a C-controlled K-frame if there exist two constants $0 < A \leq B < \infty$ such that

$$A\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle \leq \sum_{j\in\mathbb{J}}\langle f,\psi_j\rangle\langle C\psi_j, f\rangle \leq B\langle f,f\rangle, \forall f \in \mathcal{H}. \tag{3.4}$$

If $C = I$, the C-controlled K-frame $\{\psi_j\}_{j\in\mathbb{J}}$ is simply K-frame in $\mathcal{H}$ which was discussed in [1]. The sequence $\{\psi_j\}_{j\in\mathbb{J}}$ is called a C-controlled Bessel sequence with bound B, if there exists $B > 0$ such that

$$\sum_{j\in\mathbb{J}}\langle f,\psi_j\rangle\langle C\psi_j, f\rangle \leq B\langle f,f\rangle, \forall f \in \mathcal{H}, \tag{3.5}$$

where the sum in the above inequalities converges in norm.
We now give an example of C-controlled K-frame in Hilbert C^*-module.

Example 3.1. Let $\mathcal{H} = C_0$ be the set of all sequences converging to zero and $\{e_j\}_{j=1}^{\infty}$ be the standard orthonormal basis for $\mathcal{H}$.
For any $u = \{u_j\}_{j=1}^{\infty} \in \mathcal{H}$ and $v = \{v_j\}_{j=1}^{\infty} \in \mathcal{H}$

$$\langle u, v\rangle = uv^* = \{u_j v_j^*\}_{j=1}^{\infty}$$

We define $\{\psi_j\}_{j\in\mathbb{J}}$ as follows:

$$\{\psi_j\}_{j\in\mathbb{J}} = \{0, 0, e_3, e_4, e_5, ...\}$$

Let K be the orthogonal projection from $\mathcal{H}$ onto $\overline{span}\{e_j\}_{j=3}^{\infty}$ and $C \in GL^+(\mathcal{H})$ be such that

$$C(e_i) = \begin{cases} e_1 + e_2, & i = 1 \\ e_i, & \text{otherwise} \end{cases}$$

Let $f = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, ...\} \in \mathcal{H}$. Then $\langle f, f\rangle = \{\alpha_1\alpha_1^*, \alpha_2\alpha_2^*, \alpha_3\alpha_3^*, \alpha_4\alpha_4^*, ...\}$
Now, for the upper bound, we have

$$\begin{aligned}
\sum_{j\in\mathbb{J}}\langle f,\psi_j\rangle\langle C\psi_j, f\rangle &= \langle f, e_3\rangle\langle C(e_3), f\rangle + \langle f, e_4\rangle\langle C(e_4), f\rangle + \langle f, e_5\rangle\langle C(e_5), f\rangle + ... \\
&= \langle f, e_3\rangle\langle e_3, f\rangle + \langle f, e_4\rangle\langle e_4, f\rangle + \langle f, e_5\rangle\langle e_5, f\rangle + ... \\
&\leq \sum_{j\in\mathbb{J}}\langle f, e_j\rangle\langle e_j, f\rangle \\
&= \langle f, f\rangle
\end{aligned}$$

On the other hand, f can be written as $f = \sum_{j=1}^{\infty} \alpha_j e_j$. Thus, we have

$$\begin{aligned}
\langle C^{\frac{1}{2}} K^* f, C^{\frac{1}{2}} K^* f \rangle &= \langle C K^* f, K^* f \rangle \\
&= \big\langle C K^* (\sum_{j=1}^{\infty} \alpha_j e_j), K^* (\sum_{j=1}^{\infty} \alpha_j e_j) \big\rangle \\
&= \big\langle C (\sum_{j=3}^{\infty} \alpha_j e_j), \sum_{j=3}^{\infty} \alpha_j e_j \big\rangle \\
&= \big\langle \sum_{j=3}^{\infty} \alpha_j e_j, \sum_{j=3}^{\infty} \alpha_j e_j \big\rangle \\
&= \sum_{j=3}^{\infty} \langle f, e_j \rangle \langle e_j, f \rangle \\
&\leq \sum_{j \in \mathbb{J}} \langle f, \psi_j \rangle \langle C \psi_j, f \rangle
\end{aligned}$$

Hence $\{\psi_j\}_{j \in \mathbb{J}}$ is a C-controlled K-frame with lower and upper frame bound 1.

Let $\{\psi_j\}_{j \in \mathbb{J}}$ be a C-controlled Bessel sequence for Hilbert module $\mathcal{H}$ over $\mathcal{A}$. The operator $T \colon \mathcal{H} \to l^2(\mathcal{A})$ defined by

$$Tf = \{\langle f, \psi_j \rangle\}_{j \in \mathbb{J}}, f \in \mathcal{H} \tag{3.6}$$

is called the *analysis operator*. The adjoint operator $T^* \colon l^2(\mathcal{A}) \to \mathcal{H}$ given by

$$T^*(\{c_j\})_{j \in \mathbb{J}} = \sum_{j \in \mathbb{J}} c_j C \psi_j \tag{3.7}$$

is called *pre-frame operator or the synthesis operator*. By composing T and T^*, we obtain the *C-controlled frame operator* $S_C \colon \mathcal{H} \to \mathcal{H}$ as

$$S_C f = T^* T f = \sum_{j \in \mathbb{J}} \langle f, \psi_j \rangle C \psi_j. \tag{3.8}$$

We quote the following results from the literature that will be used in our work.

Lemma 3.1. [8] *Let $\mathcal{A}$ be a C^*-algebra. Let U and V be two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(U, V)$. Then the following statements are equivalent:*

1. *T is surjective.*
2. *T^* is bounded below with respect to norm i.e there exists $m > 0$ such that $\|T^* f\| \geq m \|f\|$ for all $f \in U$.*
3. *T^* is bounded below with respect to inner product i.e there exists $m > 0$ such that $\langle T^* f, T^* f \rangle \geq m \langle f, f \rangle$ for all $f \in U$.*

Lemma 3.2. [18] *Let U and V be Hilbert $\mathcal{A}$-modules over a C^*-algebra $\mathcal{A}$ and let $T : U \to V$ be a linear map. Then the following conditions are equivalent:*

1. *The operator T is bounded and $\mathcal{A}$-linear.*
2. *There exists $k \geq 0$ such that $\langle Tx, Tx \rangle \leq k \langle x, x \rangle$ holds for all $x \in U$.*

Theorem 3.1. [19] *Let E, F and G be Hilbert $\mathcal{A}$-modules over a C^*-algebra $\mathcal{A}$. Let $T \in L(E,F)$ and $T' \in L(G,F)$ with $\overline{R(T^*)}$ be orthogonally complemented. Then the following statements are equivalent:*

1. *$T'T'^* \leq \lambda TT^*$ for some $\lambda > 0$;*
2. *There exists $\mu > 0$ such that $\|T'^*z\| \leq \mu\|T^*z\|$ for all $z \in F$;*
3. *There exists $D \in L(G,E)$ such that $T' = TD$, that is the equation $TX = T'$ has a solution;*
4. *$R(T') \subseteq R(T)$.*

For the rest of the paper we indicate that S_C stands for the controlled frame operator as we have defined in (3.8), and S stands for the classical frame operator in Hilbert C^*-module $\mathcal{H}$ as defined in (2.2).

Lemma 3.3. *Let $C \in GL^+(\mathcal{H})$, $KC = CK$ and $R(C^{\frac{1}{2}}) \subseteq R(K^*C^{\frac{1}{2}})$ with $\overline{R((C^{\frac{1}{2}})^*)}$ is orthogonally complemented. Then $\|C^{\frac{1}{2}}f\|^2 \leq \lambda'\|K^*C^{\frac{1}{2}}f\|^2$ for some $\lambda' > 0$.*

Proof. Suppose $R(C^{\frac{1}{2}}) \subseteq R(K^*C^{\frac{1}{2}})$ with $\overline{R((C^{\frac{1}{2}})^*)}$ orthogonally complemented. Then by using Theorem 3.1, there exist some $\lambda' > 0$ such that

$$(C^{\frac{1}{2}})(C^{\frac{1}{2}})^* \leq \lambda'(K^*C^{\frac{1}{2}})(K^*C^{\frac{1}{2}})^*.$$

This implies that $\langle (C^{\frac{1}{2}})(C^{\frac{1}{2}})^*f, f\rangle \leq \lambda'\langle (K^*C^{\frac{1}{2}})(K^*C^{\frac{1}{2}})^*f, f\rangle$.
Now by taking norm on both sides, we get

$$\|C^{\frac{1}{2}}f\|^2 \leq \lambda'\|K^*C^{\frac{1}{2}}f\|^2.$$

□

In the following theorem, we establish an equivalence condition for C-controlled K-frame in a Hilbert C^*-module $\mathcal{H}$.

Theorem 3.2. *Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$ -module over a unital C^*-algebra $\mathcal{A}$, $\{\psi_j\}_{j\in\mathbb{J}} \subset \mathcal{H}$ be a sequence, $C \in GL^+(\mathcal{H})$, $K \in L(\mathcal{H})$, $KC = CK$ and $R(C^{\frac{1}{2}}) \subseteq R(K^*C^{\frac{1}{2}})$ with $\overline{R((C^{\frac{1}{2}})^*)}$ be orthogonally complemented. Then $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame in Hilbert C^*-module if and only if there exist constants $0 < A \leq B < \infty$ such that*

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \|\sum_{j\in\mathbb{J}}\langle f,\psi_j\rangle\langle C\psi_j, f\rangle\| \leq B\|f\|^2,\ \forall f \in \mathcal{H}. \tag{3.9}$$

Proof. ($\Longrightarrow$) Obvious.
Now we assume that there exist constants $0 < A \leq B < \infty$ such that

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \|\sum_{j\in\mathbb{J}}\langle f,\psi_j\rangle\langle C\psi_j, f\rangle\| \leq B\|f\|^2,\ \forall f \in \mathcal{H}.$$

We prove that $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame for Hilbert C^*-module $\mathcal{H}$. As S and C are both positive operator, they are self adjoint. Thus we have

$$\begin{aligned}
A\|C^{\frac{1}{2}}K^*f\|^2 &\leq \|\sum_{j\in\mathbb{J}}\langle f,\psi_j\rangle\langle C\psi_j, f\rangle\| \\
&= \|\langle S_Cf, f\rangle\| = \|\langle CSf, f\rangle\| = \|\big\langle (CS)^{\frac{1}{2}}f, (CS)^{\frac{1}{2}}f\big\rangle\|,\ \text{as } S_C = CS \\
&= \|(CS)^{\frac{1}{2}}f\|^2. \qquad (3.10)
\end{aligned}$$

Since $R(C^{\frac{1}{2}}) \subseteq R(K^*C^{\frac{1}{2}})$ with $\overline{R((C^{\frac{1}{2}})^*)}$ is orthogonally complemented, then using Lemma 3.3, there exist some $\lambda^{'} > 0$ such that

$$\|C^{\frac{1}{2}}f\|^2 \leq \lambda^{'}\|K^*C^{\frac{1}{2}}f\|^2.$$

Multiplying both side by A, we get

$$\begin{aligned} A\|C^{\frac{1}{2}}f\|^2 &\leq A\lambda^{'}\|K^*C^{\frac{1}{2}}f\|^2 \\ &\leq \lambda^{'}\|(CS)^{\frac{1}{2}}f\|^2, \end{aligned}$$

which implies

$$\frac{A}{\lambda^{'}}\|C^{\frac{1}{2}}f\|^2 \leq \|S^{\frac{1}{2}}C^{\frac{1}{2}}f\|^2$$

$$\Rightarrow \sqrt{\frac{A}{\lambda^{'}}}\|C^{\frac{1}{2}}f\| \leq \|S^{\frac{1}{2}}C^{\frac{1}{2}}f\|. \tag{3.11}$$

Now by using Lemma 3.1, we have

$$\langle S^{\frac{1}{2}}C^{\frac{1}{2}}f, S^{\frac{1}{2}}C^{\frac{1}{2}}f\rangle \geq \sqrt{\frac{A}{\lambda^{'}}}\langle C^{\frac{1}{2}}f, C^{\frac{1}{2}}f\rangle$$

$$\Rightarrow \langle C^{\frac{1}{2}}f, C^{\frac{1}{2}}f\rangle \leq \sqrt{\frac{\lambda^{'}}{A}}\langle S_Cf, f\rangle.$$

Also

$$\begin{aligned} \langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle &\leq \|K^*\|^2\langle C^{\frac{1}{2}}f, C^{\frac{1}{2}}f\rangle \\ &\leq \|K^*\|^2\sqrt{\tfrac{\lambda^{'}}{A}}\langle S_Cf, f\rangle. \end{aligned}$$

This implies that

$$\frac{1}{\|K^*\|^2}\sqrt{\frac{A}{\lambda^{'}}}\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle \leq \langle S_Cf, f\rangle. \tag{3.12}$$

Since S_C is positive, self adjoint and bounded $\mathcal{A}$-linear map, we can write

$$\langle S_C^{\frac{1}{2}}f, S_C^{\frac{1}{2}}f\rangle = \langle S_Cf, f\rangle = \sum_{j\in\mathbb{J}}\langle f, \psi_j\rangle\langle C\psi_j, f\rangle,$$

and hence by using Lemma 3.2, there exists some $B' > 0$ such that

$$\langle S_C^{\frac{1}{2}}f, S_C^{\frac{1}{2}}f\rangle \leq B^{'}\langle f, f\rangle$$

$$\Longrightarrow \langle S_Cf, f\rangle \leq B^{'}\langle f, f\rangle, \forall f \in \mathcal{H}. \tag{3.13}$$

Therefore from (3.12) and (3.13), we conclude that $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame in Hilbert C^*-module $\mathcal{H}$ with frame bounds $\frac{1}{\|K^*\|^2}\sqrt{\frac{A}{\lambda^{'}}}$ and $B^{'}$. □

Lemma 3.4. *Let $C \in GL^+(\mathcal{H})$, $CS_C = S_CC$ and $R(S_C^{\frac{1}{2}}) \subseteq R((CS_C)^{\frac{1}{2}})$ with $\overline{R((S_C^{\frac{1}{2}})^*)}$ is orthogonally complemented. Then $\|S_C^{\frac{1}{2}}f\|^2 \leq \lambda\|(CS_C)^{\frac{1}{2}}f\|^2$ for some $\lambda > 0$.*

Proof. By the assumption that $R(S_C^{\frac{1}{2}}) \subseteq R((CS_C)^{\frac{1}{2}})$ with $\overline{R((S_C^{\frac{1}{2}})^*)}$ orthogonally complemented. Then by using Theorem 3.1, there exists some $\lambda > 0$ such that

$$(S_C^{\frac{1}{2}})(S_C^{\frac{1}{2}})^* \leq \lambda((CS_C)^{\frac{1}{2}})((CS_C)^{\frac{1}{2}})^*.$$

This implies that

$$\begin{aligned} &\big\langle (S_C^{\frac{1}{2}})(S_C^{\frac{1}{2}})^* f, f\big\rangle \leq \lambda \big\langle ((CS_C)^{\frac{1}{2}})((CS_C)^{\frac{1}{2}})^* f, f\big\rangle \\ \Rightarrow\quad &\|S_C^{\frac{1}{2}} f\|^2 \leq \lambda \|(CS_C)^{\frac{1}{2}} f\|^2,\ \forall f \in \mathcal{H}. \end{aligned}$$

□

In the following theorem, we prove a characterization of C-controlled Bessel sequence.

THEOREM 3.3. *Let $\{\psi_j\}_{j\in\mathbb{J}}$ be a sequence of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. Suppose that C commutes with the controlled frame operator S_C and $R(S_C^{\frac{1}{2}}) \subseteq R((CS_C)^{\frac{1}{2}})$ with $\overline{R((S_C^{\frac{1}{2}})^*)}$ is orthogonally complemented. Then $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled Bessel sequence if and only if the operator $U\colon l^2(\mathcal{A}) \to \mathcal{H}$ defined by*

$$U\{a_j\}_{j\in\mathbb{J}} = \sum_{j\in\mathbb{J}} a_j C\psi_j$$

is a well defined bounded operator from $l^2(\mathcal{A})$ into $\mathcal{H}$ with $\|U\| \leq \sqrt{B}\|C^{\frac{1}{2}}\|$.

Proof. Suppose that $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled Bessel sequence with bound B. Therefore, we have

$$\|\sum_{j\in\mathbb{J}} \langle f, \psi_j\rangle\langle C\psi_j, f\rangle\| = \|\langle S_C f, f\rangle\| \leq B\|f\|^2, \forall f \in \mathcal{H}.$$

We first show that U is a well-defined operator. For arbitrary $n > m$, we have

$$\begin{aligned} \|\sum_{j=1}^{n} a_j C\psi_j - \sum_{j=1}^{m} a_j C\psi_j\|^2 &= \|\sum_{j=m+1}^{n} a_j C\psi_j\|^2 \\ &= \sup_{\|f\|=1} \big\|\big\langle \sum_{j=m+1}^{n} a_j C\psi_j, f\big\rangle\big\|^2 \\ &= \sup_{\|f\|=1} \big\| \sum_{j=m+1}^{n} a_j \langle C\psi_j, f\rangle\big\|^2 \\ &\leq \sup_{\|f\|=1} \big\| \sum_{j=m+1}^{n} \langle f, C\psi_j\rangle\langle C\psi_j, f\rangle\big\|\big\| \sum_{j=m+1}^{n} a_j a_j^*\big\| \\ &= \sup_{\|f\|=1} \big\|\big\langle \sum_{j=m+1}^{n} \langle f, C\psi_j\rangle C\psi_j, f\big\rangle\big\|\big\| \sum_{j=m+1}^{n} a_j a_j^*\big\| \\ &\leq \sup_{\|f\|=1} \big\|\langle CS_C f, f\rangle\big\|\big\| \sum_{j=m+1}^{n} a_j a_j^*\big\| \end{aligned}$$

$$
\begin{aligned}
&= \sup_{\|f\|=1} \Big\| \langle (CS_C)^{\frac{1}{2}} f, (CS_C)^{\frac{1}{2}} f \rangle \Big\| \Big\| \sum_{j=m+1}^{n} a_j a_j^* \Big\| \\
&\leq \sup_{\|f\|=1} \|(CS_C)^{\frac{1}{2}} f\|^2 \|a_j\|^2 \\
&\leq \sup_{\|f\|=1} \|C^{\frac{1}{2}}\|^2 \|S_C^{\frac{1}{2}} f\|^2 \|a_j\|^2 \\
&\leq \sup_{\|f\|=1} B\|f\|^2 \|C^{\frac{1}{2}}\|^2 \|a_j\|^2 = B\|C^{\frac{1}{2}}\|^2 \|a_j\|^2.
\end{aligned}
$$

This shows that $\sum_{j\in\mathbb{J}} a_j C\psi_j$ is a Cauchy sequence which is convergent in $\mathcal{H}$. Thus $U\{a_j\}_{j\in\mathbb{J}}$ is a well defined operator from $l^2(\mathcal{A})$ into $\mathcal{H}$.
For boundedness of U, we consider

$$
\begin{aligned}
\|U\{a_j\}_{j\in\mathbb{J}}\|^2 &= \sup_{\|f\|=1} \|\langle U\{a_j\}, f\rangle\|^2 \\
&= \sup_{\|f\|=1} \Big\| \sum_{j\in\mathbb{J}} a_j \langle C\psi_j, f\rangle \Big\|^2 \\
&\leq \sup_{\|f\|=1} \Big\| \sum_{j\in\mathbb{J}} \langle f, C\psi_j\rangle \langle C\psi_j, f\rangle \Big\| \Big\| \sum_{j\in\mathbb{J}} a_j a_j^* \Big\| \\
&= \sup_{\|f\|=1} \Big\| \Big\langle \sum_{j\in\mathbb{J}} \langle f, C\psi_j\rangle C\psi_j, f \Big\rangle \Big\| \Big\| \sum_{j\in\mathbb{J}} a_j a_j^* \Big\| \\
&= \sup_{\|f\|=1} \Big\| \langle CS_C f, f\rangle \Big\| \Big\| \sum_{j\in\mathbb{J}} a_j a_j^* \Big\| \\
&= \sup_{\|f\|=1} \Big\| \Big\langle (CS_C)^{\frac{1}{2}} f, (CS_C)^{\frac{1}{2}} f \Big\rangle \Big\| \Big\| \sum_{j\in\mathbb{J}} a_j a_j^* \Big\| \\
&= \sup_{\|f\|=1} \Big\| (CS_C)^{\frac{1}{2}} f \Big\|^2 \|a_j\|^2 \\
&\leq \sup_{\|f\|=1} \|C^{\frac{1}{2}}\|^2 \|S_C^{\frac{1}{2}} f\|^2 \|a_j\|^2 \\
&\leq B\|C^{\frac{1}{2}}\|^2 \|a_j\|^2.
\end{aligned}
$$

This implies that $\|U\| \leq \sqrt{B}\|C^{\frac{1}{2}}\|$.

Now assume that U is well defined operator from $l^2(\mathcal{A})$ into $\mathcal{H}$ and $\|U\| \leq \sqrt{B}\|C^{\frac{1}{2}}\|$.
We now prove that $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled Bessel sequence.
For arbitrary $f \in \mathcal{H}$ and $\{a_j\} \in l^2(\mathcal{A})$, we have

$$
\begin{aligned}
\Big\langle f, U\{a_j\}_{j\in\mathbb{J}} \Big\rangle &= \Big\langle f, \sum_{j\in\mathbb{J}} a_j C\psi_j \Big\rangle \\
&= \Big\langle \sum_{j\in\mathbb{J}} a_j^* C f, \psi_j \Big\rangle \\
&= \sum_{j\in\mathbb{J}} \langle Cf, \psi_j\rangle a_j^*.
\end{aligned}
$$

Therefore we get

$$\langle f, U\{a_j\}_{j\in\mathbb{J}}\rangle = \langle\{\langle Cf,\psi_j\rangle\},\{a_j\}\rangle.$$

This implies that U has an adjoint, and $U^*f = \{\langle Cf,\psi_j\rangle\}$. Also, $\|U\| = \|U^*\|$. So we have

$$\begin{aligned}\|U^*f\|^2 = \|\langle U^*f,U^*f\rangle\| = \|\langle UU^*f,f\rangle\| = \|\langle CS_Cf,f\rangle\| &= \|(CS_C)^{\frac{1}{2}}f\|^2\\ &\le B\|C^{\frac{1}{2}}\|^2\|f\|^2. \qquad (3.14)\end{aligned}$$

By using Lemma 3.4, we have $\|S_C^{\frac{1}{2}}f\|^2 \le \lambda\|(CS_C)^{\frac{1}{2}}f\|^2$ for some $\lambda > 0$. Using (3.14) we get

$$\|S_C^{\frac{1}{2}}f\|^2 \le \lambda\|(CS_C)^{\frac{1}{2}}f\|^2 \le \lambda B\|C^{\frac{1}{2}}\|^2\|f\|^2.$$

Therefore $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled Bessel sequence with Bessel bound $\lambda B\|C^{\frac{1}{2}}\|^2$. □

Proposition 3.1. *Let $\{\psi_j\}_{j\in\mathbb{J}}$ be a C-controlled K-frame in $\mathcal{H}$. Then $ACKK^*I \le S_c \le BI$.*

Proof. Suppose $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame with bounds A and B. Then

$$\begin{aligned}&A\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle \le \sum_{j\in\mathbb{J}}\langle f,\psi_j\rangle\langle C\psi_j,f\rangle \le B\langle f,f\rangle, \forall f\in\mathcal{H}.\\ \Rightarrow\ &A\langle CKK^*f,f\rangle \le \langle S_Cf,f\rangle \le B\langle f,f\rangle.\\ \Rightarrow\ &ACKK^*I \le S_C \le BI.\end{aligned}$$

□

Proposition 3.2. *Let $\{\psi_j\}_{j\in\mathbb{J}}$ be a C-controlled Bessel sequence in $\mathcal{H}$ and $C \in GL^+(\mathcal{H})$. Then $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame for $\mathcal{H}$, if and only if there exists $A > 0$ such that $CS \ge ACKK^*$.*

Proof. The sequence $\{\psi_j\}_{j\in\mathbb{J}}$ is a controlled K-frame for $\mathcal{H}$ with frame bounds A, B and frame operator S_C, if and only if

$$\begin{aligned}&A\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle \le \sum_{j\in\mathbb{J}}\langle f,\psi_j\rangle\langle C\psi_j,f\rangle \le B\langle f,f\rangle, \forall f\in\mathcal{H}.\\ &\Leftrightarrow A\langle CKK^*f,f\rangle \le \langle S_Cf,f\rangle \le B\langle f,f\rangle.\\ &\Leftrightarrow A\langle CKK^*f,f\rangle \le \langle CSf,f\rangle \le B\langle f,f\rangle.\\ &\Leftrightarrow ACKK^*I \le CS.\end{aligned}$$

□

In the following two propositions we establish the inter-relationship between K-frame and C-controlled K-frame.

Proposition 3.3. *Let $C \in GL^+(\mathcal{H})$, $K \in L(\mathcal{H})$, $KC = CK$, $R(C^{\frac{1}{2}}) \subseteq R(K^*C^{\frac{1}{2}})$ with $\overline{R((C^{\frac{1}{2}})^*)}$ is orthogonally complemented, and $\{\psi_j\}_{j\in\mathbb{J}}$ be a C-controlled K-frame for $\mathcal{H}$ with lower and upper frame bounds A and B, respectively. Then $\{\psi_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$ with lower and upper frame bounds $A\|C^{\frac{1}{2}}\|^{-2}$ and $B\|C^{\frac{-1}{2}}\|^2$, respectively.*

Proof. Suppose $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame for $\mathcal{H}$ with bound A and B. Then by Theorem 3.2, we have

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \|\sum_{j\in\mathbb{J}}\langle f,\psi_j\rangle\langle C\psi_j,f\rangle\| \leq B\|f\|^2, \forall f\in\mathcal{H}.$$

Now,

$$\begin{aligned}A\|K^*f\|^2 &= A\|C^{\frac{-1}{2}}C^{\frac{1}{2}}K^*f\|^2\\ &\leq A\|C^{\frac{1}{2}}\|^2\|C^{\frac{-1}{2}}K^*f\|^2\\ &\leq \|C^{\frac{1}{2}}\|^2\|\sum_{j\in\mathbb{J}}\langle f,\psi_j\rangle\langle\psi_j,f\rangle\|.\end{aligned}$$

This implies that

$$A\|C^{\frac{1}{2}}\|^{-2}\|K^*f\|^2 \leq \|\sum_{j\in\mathbb{J}}\langle f,\psi_j\rangle\langle\psi_j,f\rangle\|$$

On the other hand for every $f\in\mathcal{H}$,

$$\begin{aligned}\|\sum_{j\in\mathbb{J}}\langle f,\psi_j\rangle\langle\psi_j,f\rangle\| \quad &= \|\langle Sf,f\rangle\|\\ &= \|\langle C^{-1}CSf,f\rangle\|\\ &= \|\langle (C^{-1}CS)^{\frac{1}{2}}f,(C^{-1}CS)^{\frac{1}{2}}f\rangle\|\\ &= \|(C^{-1}CS)^{\frac{1}{2}}f\|^2\\ &\leq \|C^{\frac{-1}{2}}\|^2\|(CS)^{\frac{1}{2}}f\|^2\\ &= \|C^{\frac{-1}{2}}\|^2\|\langle (CS)^{\frac{1}{2}}f,(CS)^{\frac{1}{2}}f\rangle\|\\ &= \|C^{\frac{-1}{2}}\|^2\|\langle CSf,f\rangle\|\\ &\leq \|C^{\frac{-1}{2}}\|^2B\|f\|^2.\end{aligned}$$

Therefore, $\{\psi_j\}_{j\in\mathbb{J}}$ is a K-frame with lower and upper frame bounds $A\|C^{\frac{1}{2}}\|^{-2}$ and $B\|C^{\frac{-1}{2}}\|^2$, respectively. $\square$

PROPOSITION 3.4. *Let $C\in GL^+(\mathcal{H})$, $K\in L(\mathcal{H})$, $KC=CK$, $R(C^{\frac{1}{2}})\subseteq R(K^*C^{\frac{1}{2}})$ with $\overline{R((C^{\frac{1}{2}})^*)}$ is orthogonally complemented. Let $\{\psi_j\}_{j\in\mathbb{J}}$ be a K-frame for $\mathcal{H}$ with lower and upper frame bounds A and B, respectively. Then $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame for $\mathcal{H}$ with lower and upper frame bounds A and $\|C\|\|S\|$, respectively.*

Proof. Suppose $\{\psi_j\}_{j\in\mathbb{J}}$ is a K-frame with frame bounds A and B. Then by equivalence condition [9] of K-frame, we have

$$A\|K^*f\|^2 \leq \|\sum_{j\in\mathbb{J}}\langle f,\psi_j\rangle\langle\psi_j,f\rangle\| \leq B\|f\|^2, \forall f\in\mathcal{H}.$$

For any $f \in \mathcal{H}$,

$$\begin{aligned} A\|C^{\frac{1}{2}}K^*f\|^2 &= A\|K^*C^{\frac{1}{2}}f\|^2 \\ &\leq \|\sum_{j\in\mathbb{J}}\langle C^{\frac{1}{2}}f,\psi_j\rangle\langle\psi_j,C^{\frac{1}{2}}f\rangle\| \\ &= \|\sum_{j\in\mathbb{J}}\langle C^{\frac{1}{2}}f,\psi_j\rangle\psi_j,C^{\frac{1}{2}}f\rangle\| \\ &= \|\langle C^{\frac{1}{2}}Sf,C^{\frac{1}{2}}f\rangle\| \\ &= \|\langle CSf,f\rangle\|. \end{aligned} \tag{3.15}$$

On the other hand for every $f \in \mathcal{H}$,

$$\begin{aligned} \|\langle CSf,f\rangle\| &= \|\langle Sf,C^*f\rangle\| \\ &= \|\langle Sf,Cf\rangle\| \\ &\leq \|Sf\|\|Cf\| \\ &\leq \|C\|\|S\|\|f\|^2. \end{aligned} \tag{3.16}$$

Therefore from (3.15),(3.16) and Theorem 3.2, we conclude that $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame with lower and upper frame bounds A and $\|C\|\|S\|$, respectively. □

Theorem 3.4. *Let $C \in GL^+(\mathcal{H})$, $\{\psi_j\}_{j\in\mathbb{J}}$ be a C-controlled K-frame for $\mathcal{H}$ with bounds A and B. Let $M, K \in L(\mathcal{H})$ with $R(M) \subset R(K)$, $\overline{R(K^*)}$ orthogonally complemented, and C commutes with M and K both. Then $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled M-frame for $\mathcal{H}$.*

Proof. Suppose $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame for $\mathcal{H}$ with bounds A and B. Then

$$A\langle C^{\frac{1}{2}}K^*f,C^{\frac{1}{2}}K^*f\rangle \leq \sum_{j\in\mathbb{J}}\langle f,\psi_j\rangle\langle C\psi_j,f\rangle \leq B\langle f,f\rangle,\ \forall f\in\mathcal{H}. \tag{3.17}$$

Since $R(M) \subset R(K)$, from Theorem 3.1, there exists some $\lambda^{'} > 0$ such that $MM^* \leq \lambda^{'}KK^*$. So we have

$$\langle MM^*C^{\frac{1}{2}}f,C^{\frac{1}{2}}f\rangle \leq \lambda^{'}\langle KK^*C^{\frac{1}{2}}f,C^{\frac{1}{2}}f\rangle.$$

Multiplying the above inequality by A, we get

$$\tfrac{A}{\lambda^{'}}\langle MM^*C^{\frac{1}{2}}f,C^{\frac{1}{2}}f\rangle \leq A\langle KK^*C^{\frac{1}{2}}f,C^{\frac{1}{2}}f\rangle.$$

From (3.17), we have

$$\frac{A}{\lambda^{'}}\langle MM^*C^{\frac{1}{2}}f,C^{\frac{1}{2}}f\rangle \leq \sum_{j\in\mathbb{J}}\langle f,\psi_j\rangle\langle C\psi_j,f\rangle \leq B\langle f,f\rangle, \text{ for all } f\in\mathcal{H}.$$

Therefore, $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled M-frame with lower and upper frame bounds $\dfrac{A}{\lambda^{'}}$ and B, respectively. □

In the following result, we investigate the invariance of a C-controlled Bessel sequence under an adjointable operator.

PROPOSITION 3.5. *Let $\{\psi_j\}_{j\in\mathbb{J}}$ be a C-controlled Bessel sequence with bound D. Let $T \in L(\mathcal{H})$ and $CT = TC$. Then $\{T\psi_j\}_{j\in\mathbb{J}}$ is also C-controlled Bessel sequence with bound $D\|T^*\|^2$.*

Proof. Suppose $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled Bessel sequence with bound D. Then we have

$$\sum_{j\in\mathbb{J}}\langle f, \psi_j\rangle\langle C\psi_j, f\rangle \leq D\langle f, f\rangle, \forall f \in \mathcal{H}.$$

For every $f \in \mathcal{H}$,

$$\begin{aligned}\sum_{j\in\mathbb{J}}\langle f, T\psi_j\rangle\langle CT\psi_j, f\rangle &= \sum_{j\in\mathbb{J}}\langle T^*f, \psi_j\rangle\langle TC\psi_j, f\rangle \\ &= \sum_{j\in\mathbb{J}}\langle T^*f, \psi_j\rangle\langle C\psi_j, T^*f\rangle \\ &\leq D\langle T^*f, T^*f\rangle \\ &\leq D\|T^*\|^2\langle f, f\rangle.\end{aligned}$$

Thus $\{T\psi_j\}_{j\in\mathbb{J}}$ is also C-controlled Bessel sequence with bound $D\|T^*\|^2$. □

Now, we investigate the invariance of a C-controlled K-frame under an adjointable operator.

THEOREM 3.5. *Let $C \in GL^+(\mathcal{H})$, $K \in L(\mathcal{H})$ and $\{\psi_j\}_{j\in\mathbb{J}}$ be a C-controlled K-frame for $\mathcal{H}$ with lower and upper bounds A and B, respectively. If $T \in L(\mathcal{H})$ with closed range such that $\overline{R(TK)}$ is orthogonally complemented and C, K, T commute with each other. Then $\{T\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame for $R(T)$.*

Proof. Suppose $\{\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame for $\mathcal{H}$ with bound A and B. Then

$$A\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle \leq \sum_{j\in\mathbb{J}}\langle f, \psi_j\rangle\langle C\psi_j, f\rangle \leq B\langle f, f\rangle, \forall f \in \mathcal{H}.$$

We know that if T has closed range then T has Moore-Penrose inverse $T^\dagger$ such that $TT^\dagger T = T$ and $T^\dagger TT^\dagger = T^\dagger$. So $TT^\dagger|_{R(T)} = I_{R(T)}$ and $(TT^\dagger)^* = I^* = I = TT^\dagger$.
We have

$$\begin{aligned}\langle K^*C^{\frac{1}{2}}f, K^*C^{\frac{1}{2}}f\rangle &= \big\langle (TT^\dagger)^*K^*C^{\frac{1}{2}}f, (TT^\dagger)^*K^*C^{\frac{1}{2}}f\big\rangle \\ &= \big\langle T^{\dagger *}T^*K^*C^{\frac{1}{2}}f, T^{\dagger *}T^*K^*C^{\frac{1}{2}}f\big\rangle \\ &\leq \|(T^\dagger)^*\|^2\big\langle T^*K^*C^{\frac{1}{2}}f, T^*K^*C^{\frac{1}{2}}f\big\rangle.\end{aligned}$$

This implies that

$$\|(T^\dagger)^*\|^{-2}\big\langle K^*C^{\frac{1}{2}}f, K^*C^{\frac{1}{2}}f\big\rangle \leq \big\langle T^*K^*C^{\frac{1}{2}}f, T^*K^*C^{\frac{1}{2}}f\big\rangle. \tag{3.18}$$

Since $R(T^*K^*) \subset R(K^*T^*)$, by using Theorem 3.1, there exists some $\lambda' > 0$ such that

$$\big\langle T^*K^*C^{\frac{1}{2}}f, T^*K^*C^{\frac{1}{2}}f\big\rangle \leq \lambda'\big\langle K^*T^*C^{\frac{1}{2}}f, K^*T^*C^{\frac{1}{2}}f\big\rangle. \tag{3.19}$$

Therefore, using (3.18) and (3.19) we get

$$\begin{aligned}\sum_{j\in\mathbb{J}}\langle f,T\psi_j\rangle\langle CT\psi_j,f\rangle &= \sum_{j\in\mathbb{J}}\langle T^*f,\psi_j\rangle\langle TC\psi_j,f\rangle\\ &= \sum_{j\in\mathbb{J}}\langle T^*f,\psi_j\rangle\langle C\psi_j,T^*f\rangle\\ &\geq A\big\langle C^{\frac{1}{2}}K^*T^*f,C^{\frac{1}{2}}K^*T^*f\big\rangle\\ &\geq A\lambda'\langle T^*C^{\frac{1}{2}}K^*f,T^*C^{\frac{1}{2}}K^*f\rangle\\ &\geq A\lambda'\|(T^\dagger)^*\|^{-2}\langle C^{\frac{1}{2}}K^*f,C^{\frac{1}{2}}K^*f\rangle.\end{aligned}$$

This gives the lower frame inequality for $\{T\psi_j\}_{j\in\mathbb{J}}$. On the other hand by Proposition 3.5, $\{T\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled Bessel sequence. So $\{T\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame for $R(T)$. □

THEOREM 3.6. *Let $C\in GL^+(\mathcal{H})$, $K\in L(\mathcal{H})$ and $\{\psi_j\}_{j\in\mathbb{J}}$ be a C-controlled K-frame for $\mathcal{H}$ with lower and upper bound A, B respectively. If $T\in L(\mathcal{H})$ is a isometry such that $R(T^*K^*)\subset R(K^*T^*)$ with $\overline{R(TK)}$ is orthogonally complemented and C,K,T commute with each other. Then $\{T\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame for $\mathcal{H}$.*

Proof. By Theorem 3.1, there exist some $\lambda>0$ such that $\|T^*K^*C^{\frac{1}{2}}f\|^2\leq\lambda\|K^*T^*C^{\frac{1}{2}}f\|^2$. Suppose A is a lower bound for the C-controlled K-frame $\{\psi_j\}_{j\in\mathbb{J}}$. Since T is an isometry, then

$$\begin{aligned}\frac{A}{\lambda}\|C^{\frac{1}{2}}K^*f\|^2 &= \frac{A}{\lambda}\|T^*C^{\frac{1}{2}}K^*f\|^2\\ &\leq A\|K^*T^*C^{\frac{1}{2}}f\|^2\\ &= A\|C^{\frac{1}{2}}K^*T^*f\|^2\\ &\leq \sum_{j\in\mathbb{J}}\langle T^*f,\psi_j\rangle\langle C\psi_j,T^*f\rangle\\ &= \sum_{j\in\mathbb{J}}\langle f,T\psi_j\rangle\langle TC\psi_j,f\rangle\\ &= \sum_{j\in\mathbb{J}}\langle f,T\psi_j\rangle\langle CT\psi_j,f\rangle \qquad (3.20)\end{aligned}$$

Therefore from Proposition 3.5 and inequality (3.20), we conclude that $\{T\psi_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame for $\mathcal{H}$ with bounds $\dfrac{A}{\lambda}$ and $B\|T^*\|^2$. □

Now we prove a perturbation result for C-controlled K-frame.

THEOREM 3.7. *Let $F=\{f_j\}_{j\in\mathbb{J}}$ be a C-controlled K-frame for $\mathcal{H}$, with controlled frame operator S_C. Suppose $K\in L(\mathcal{H})$, $KC=CK$, $R(C^{\frac{1}{2}})\subseteq R(K^*C^{\frac{1}{2}})$ with $\overline{R((C^{\frac{1}{2}})^*)}$ is orthogonally complemented . If $G=\{g_j\}_{j\in\mathbb{J}}$ is a non zero sequence in $\mathcal{H}$, and $E=T_F-T_G$ be a compact operator, where $T_G(\{c_j\}_{j\in\mathbb{J}})=\sum_{j\in\mathbb{J}}c_jg_j$ for $\{c_j\}_{j\in\mathbb{J}}\in l^2(\mathcal{A})$, then $G=\{g_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame for $\mathcal{H}$.*

Proof. Let $\{f_j\}_{j\in\mathbb{J}}$ be a C-controlled K-frame with bounds A and B, then because of Theorem 3.2, we have

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \|\sum_{j\in\mathbb{J}}\langle f,f_j\rangle\langle Cf_j,f\rangle\| \leq B\|f\|^2, \forall f\in\mathcal{H}.$$

This implies $\|T_F\|^2 \leq B\|C^{\frac{-1}{2}}\|^2$.
Let $V = T_F - E$ be an operator from $l^2(\mathcal{A})$ into $\mathcal{H}$. Since T_F and E are bounded, then the operator V is bounded. Therefore $\|V\| = \|V^*\|$.
For any $f\in\mathcal{H}$,

$$\begin{aligned}
V^*f &= T_F^*f - E^*f\\
&= \{\langle f,f_j\rangle\}_{j\in\mathbb{J}} - \{\langle f,f_j-g_j\rangle\}_{j\in\mathbb{J}}\\
&= \{\langle f,f_j\rangle\}_{j\in\mathbb{J}} - \{\langle f_j-g_j,f\rangle^*\}_{j\in\mathbb{J}}\\
&= \{\langle f,f_j\rangle\}_{j\in\mathbb{J}} - \{\langle f_j,f\rangle^* - \langle g_j,f\rangle^*\}_{j\in\mathbb{J}}\\
&= \{\langle f,f_j\rangle\}_{j\in\mathbb{J}} - \{\langle f,f_j\rangle - \langle f,g_j\rangle\}_{j\in\mathbb{J}}\\
&= \{\langle f,g_j\rangle\}_{j\in\mathbb{J}}.
\end{aligned}$$

We have

$$V(\{c_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} c_jg_j, \text{ and } S_G = VV^*. \tag{3.21}$$

Now using (3.21), we have

$$\begin{aligned}
\|\langle f,CS_Gf\rangle\| = \|\langle f,CVV^*f\rangle\| &= \|\langle C^{\frac{1}{2}}Vf, C^{\frac{1}{2}}Vf\rangle\|\\
&= \|C^{\frac{1}{2}}Vf\|^2\\
&\leq \|C^{\frac{1}{2}}\|^2\|Vf\|^2\\
&= \|C^{\frac{1}{2}}\|^2\|(T_F-E)f\|^2\\
&\leq \|C^{\frac{1}{2}}\|^2\|T_F-E\|^2\|f\|^2\\
&\leq (\|T_F\|^2 + 2\|T_F\|\|E\| + \|E\|^2)\|C^{\frac{1}{2}}\|^2\|f\|^2\\
&\leq \big(B\|C^{\frac{-1}{2}}\|^2 + 2\sqrt{B}\|C^{\frac{-1}{2}}\|\|E\| + \|E\|^2\big)\|C^{\frac{1}{2}}\|^2\|f\|^2\\
&= B\Big(\|C^{\frac{-1}{2}}\| + \frac{\|E\|}{\sqrt{B}}\Big)^2\|C^{\frac{1}{2}}\|^2\|f\|^2. \qquad (3.22)
\end{aligned}$$

This inequality shows that $\{g_j\}_{j\in\mathbb{J}}$ is a controlled Bessel sequence with bound
$B\Big(\|C^{\frac{-1}{2}}\| + \frac{\|E\|}{\sqrt{B}}\Big)^2\|C^{\frac{1}{2}}\|^2$.
Again we have

$$\begin{aligned}
VV^* &= (T_F-E)(T_F-E)^*\\
&= (T_F-E)(T_F^*-E^*)\\
&= T_FT_F^* - T_FE^* - ET_F^* + EE^*\\
&= S_F - T_FE^* - ET_F^* + EE^*
\end{aligned}$$

Since E, T_F and S_F are compact operators, then $S_F - T_FE^* - ET_F^* + EE^*$ is a compact operator. Therefore $S_F - T_FE^* - ET_F^* + EE^* + I$ is a bounded operator with closed range. Thus, $VV^* = S_F - T_FE^* - ET_F^* + EE^*$ is a bounded operator with closed range. Clearly, V and its adjoint operator $V^*f = \{\langle f,g_j\rangle\}_{j\in\mathbb{J}}$ is injective. This implies VV^*

is injective as composition of two injective operator is injective. Hence $VV^*(= S_G)$ is bounded below. So there exists some constant $A > 0$ such that

$$A\|C^{\frac{1}{2}}f\| \leq \|S_G C^{\frac{1}{2}}f\|. \tag{3.23}$$

Now

$$\begin{aligned} \|C^{\frac{1}{2}}K^*f\|^2 &= \|K^*C^{\frac{1}{2}}f\|^2 \\ &\leq \|K^*\|^2\|C^{\frac{1}{2}}f\|^2 \\ &\leq \frac{1}{A^2}\|K^*\|^2\|S_G C^{\frac{1}{2}}f\|^2. \end{aligned}$$

This implies that

$$\frac{A^2}{\|K^*\|^2}\|C^{\frac{1}{2}}K^*f\|^2 \leq \|S_G C^{\frac{1}{2}}f\|^2. \tag{3.24}$$

Therefore from (3.22) and (3.24), we conclude that $G = \{g_j\}_{j\in\mathbb{J}}$ is a C-controlled K-frame for $\mathcal{H}$ with frame bounds $\frac{A^2}{\|K^*\|^2}$ and $B\Big(\|C^{\frac{-1}{2}}\| + \frac{\|E\|}{\sqrt{B}}\Big)^2\|C^{\frac{1}{2}}\|^2$. □

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