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REPDIGITS AS DIFFERENCE OF TWO PELL OR PELL-LUCAS NUMBERS

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Abstract. In this paper, we determine all repdigits, which are difference of two Pell and Pell-Lucas numbers. It is shown that the largest repdigit which is difference of two Pell numbers is $99 = 169 - 70 = P_7 - P_6$ and the largest repdigit which is difference of two Pell-Lucas numbers is $444 = 478 - 34 = Q_7 - Q_4$.

1. Introduction

Let $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ be the sequences of Pell and Pell-Lucas numbers defined by $P_0 = 0, P_1 = 1,\ P_{n+2} = 2P_{n+1} + P_n$, and $Q_0 = 2, Q_1 = 2, Q_{n+2} = 2Q_{n+1} + Q_n$ for $n \geq 0$, respectively. Binet formulas for these numbers are

$$P_n = \frac{\lambda^n - \delta^n}{2\sqrt{2}} \text{ and } Q_n \ = \lambda^n + \delta^n,$$

where $\lambda = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$, which are the roots of the characteristic equation $x^2 - 2x - 1 = 0$. It can be seen that $2 < \lambda < 3$, $-1 < \delta < 0$, and $\lambda\delta = -1$. The relation between n–th Pell number P_n and λ is given by

$$\lambda^{n-2} \leq P_n \leq \lambda^{n-1} \tag{1}$$

for $n \geq 1$. Also, the relation between n–th Pell–Lucas number Q_n and λ is given by

$$\lambda^{n-1} \leq Q_n < 2\lambda^n \tag{2}$$

for $n \geq 1$. The inequalities (1) and (2) can be proved by induction on n.

A non-negative integer N is called a base b-repdigit if all of its base b-digits are equal. Particularly, we say to simplify notation, for $b = 10$ that N is a repdigit. Recently, several authors have dealt with the problem of finding the repdigits in the second-order linear recurrence sequences. In [7], the author has found all Fibonacci and Lucas numbers which are repdigits. The largest repdigits in the Fibonacci and Lucas sequences are $F_{10} = 55$ and $L_5 = 11$. In [6], the authors have found all Pell and Pell-Lucas numbers which are repdigits. The largest repdigits in the Pell and Pell-Lucas sequences are $P_3 = 5$ and $Q_2 = 6$. In [11], the authors solved the problem of finding the repdigits as product of any two numbers in the sequences of

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Pell numbers or Pell-Lucas numbers. In [12], the authors determined base-b repdigits that are difference of two Fibonacci numbers. In this paper, we solve the Diophantine equations

$$P_n - P_m = \frac{d \cdot (10^k - 1)}{9} \tag{3}$$

$$Q_n - Q_m = \frac{d \cdot (10^k - 1)}{9} \tag{4}$$

where $1 \leq d \leq 9, k \geq 1$, and $1 \leq m < n$. Note that, the case $m = 0$ in the equation (3) has been also resolved in [6]. Furthermore, Q_0 and Q_1 values are the same. Thus, we will assumed that $m \geq 1$.

Recently, many of the above mentioned equations are solved by Baker's theory of lower bounds for a nonzero linear form in logarithms of algebraic numbers. Now we give some well known results, which are useful in proving our main theorems.

2. Auxiliary results

Let η be an algebraic number of degree d with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + ... + a_d = a_0 \prod_{i=1}^{d} \left(x - \eta^{(i)}\right) \in \mathbb{Z}[x],$$

where the a_i's are relatively prime integers with $a_0 > 0$ and the $\eta^{(i)}$'s are conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\left\{|\eta^{(i)}|, 1\right\}\right)\right) \tag{5}$$

is called the logarithmic height of η. In particular, if $\eta = a/b$ is a rational number with $\gcd(a, b) = 1$ and $b > 0$, then $h(\eta) = \log\left(\max\left\{|a|, b\right\}\right)$.

We give some properties of the logarithmic height whose proofs can be found in [3].

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{6}$$

$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \tag{7}$$

$$h(\eta^m) = |m| h(\eta). \tag{8}$$

Now we give a theorem which is deduced from Corollary 2.3 of Matveev [8] and provides a large upper bound for the subscript n in the equations (3) and (4) (also see Theorem 9.4 in [4]).

Theorem 1. *Assume that $\gamma_1, \gamma_2, ..., \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, ..., b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 A_2 \cdots A_t\right),$$

where

$$B \geq \max\{|b_1|, ..., |b_t|\},$$

and $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ *for all* $i = 1, ..., t$.

The following lemma was given in [2]. This lemma is an immediate variation of the lemma of Dujella and Pethő in [5]. The result (Lemma 5 (a)) given in [5] is a variation of a lemma of Baker and Davenport [1]. This lemma will be used to reduce the upper bound for the subscript n in the equations (3) and (4). For any real number x, we let $||x|| = \min\{|x-n| : n \in \mathbb{Z}\}$ be the distance from x to the nearest integer.

Lemma 2. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational number γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\gamma q||$. If $\epsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

The following lemma can be found in [13].

Lemma 3. *Let $a, x \in \mathbb{R}$. If $0 < a < 1$ and $|x| < a$, then*

$$|\log(1+x)| < \frac{-\log(1-a)}{a} \cdot |x|$$

and

$$|x| < \frac{a}{1-e^{-a}} \cdot |e^x - 1|.$$

The following lemmas can be deduced from [9] and [10].

Lemma 4. *All nonnegative integer solutions $(n, m, d, k, P_n + P_m)$ of the equation,*

$$P_n + P_m = \frac{d \cdot (10^k - 1)}{9}$$

with the $d \in \{1, 2, ..., 9\}$ have

$$(n, m, d, k, P_n + P_m) \in \left\{ \begin{array}{l} (1,0,1,1,1), (1,1,2,1,2), (2,0,2,1,2), \\ (2,1,3,1,3), (2,2,4,1,4), (3,0,5,1,5), \\ (3,1,6,1,6), (3,2,7,1,7), (6,5,9,2,99) \end{array} \right\}.$$

Lemma 5. *All positive integer solutions $(n, m, d, k, Q_n + Q_m)$ of the equation,*

$$Q_n + Q_m = \frac{d \cdot (10^k - 1)}{9}$$

with the $d \in \{1, 2, ..., 9\}$ have

$$(n, m, d, k, Q_n + Q_m) \in \{(1,1,4,1,4), (2,1,8,1,8), (5,2,8,2,88)\}.$$

3. Main Theorems

THEOREM 6. *Let* $1 \leq m < n$, $k \geq 1$,*and* $1 \leq d \leq 9$. *If the equations* (3) *has a solution* $(n, m, d, k, P_n - P_m)$, *then*

$$(n, m, d, k, P_n - P_m) \in \left\{ \begin{array}{c} (2,1,1,1,1), (3,1,4,1,4), (3,2,3,1,3), \\ (4,1,1,2,11), (4,3,7,1,7), (7,6,9,2,99) \end{array} \right\}.$$

Proof. Assume that $P_n - P_m$ is a repdigit. Then the equation (3) holds for $1 \leq m < n$ with $k \geq 1$. Let us suppose that $1 \leq m < n \leq 99$. Then by using Mathematica program, we obtain the only solutions displayed in the statement of Theorem 6. Let $n - m = 1$. Then we get

$$P_{m+1} - P_m = P_m + P_{m-1}.$$

Thus by Lemma 4, we get the solutions

$$(n, m, d, k, P_n - P_m) = (2,1,1,1,1), (3,2,3,1,3), (4,3,7,1,7), (7,6,9,2,99),$$

which is displayed in the statement of Theorem 6. From now on, assume that $n \geq 100, m \geq 1$ and $n - m \geq 2$. Then, by using (1), we get

$$\lambda^{2k-2} < 10^{k-1} < \frac{d \cdot (10^k - 1)}{9} = P_n - P_m \leq \lambda^{n-1} - 1 < \lambda^{n-1}.$$

This shows that $2k < n + 1$. That is, $k < n + 1$. On the other hand, rearranging the equation (3) as

$$\frac{\lambda^n}{\sqrt{8}} - \frac{d \cdot 10^k}{9} = P_m + \frac{\delta^n}{\sqrt{8}} - \frac{d}{9} \tag{9}$$

and taking absolute values of both sides of (9), we get

$$\left| \frac{\lambda^n}{\sqrt{8}} - \frac{d \cdot 10^k}{9} \right| \leq P_m + \frac{|\delta|^n}{\sqrt{8}} + \frac{d}{9} < \lambda^{m-1} + 1.1. \tag{10}$$

Dividing both sides of (10) by $\frac{\lambda^n}{\sqrt{8}}$ yields

$$\begin{aligned} \left| 1 - \frac{\lambda^{-n} \cdot 10^k \cdot \sqrt{8} \cdot d}{9} \right| &\leq \sqrt{8} \cdot \lambda^{m-n-1} + 1.1 \cdot \sqrt{8} \cdot \lambda^{-n} \\ &< \sqrt{8} \cdot \lambda^{m-n} \cdot (\lambda^{-1} + 1.1 \cdot \lambda^{-m}) \\ &< 2.5 \cdot \lambda^{m-n}, \end{aligned} \tag{11}$$

where we have used the facts that $m \geq 1$. Now, let us apply Theorem 1 with $(\gamma_1, b_1) := (\lambda, -n), (\gamma_2, b_2) := (10, k), (\gamma_3, b_3) := \left(\frac{\sqrt{8} \cdot d}{9}, 1 \right)$. The number field containing positive real numbers γ_1, γ_2, and γ_3 is $\mathbb{K} := \mathbb{Q}(\sqrt{2})$, which has degree 2. That is, $D = 2$. Now, we show that

$$\Lambda_1 := 1 - \frac{\lambda^{-n} \cdot 10^k \cdot \sqrt{8} \cdot d}{9}$$

is nonzero. Contrast to this, we assume that $\Lambda_1 = 0$. Then we get $\lambda^n = \sqrt{8} \cdot d \cdot 10^k / 9$. Conjugating in $\mathbb{Q}(\sqrt{2})$ gives us $\delta^n = -\sqrt{8} \cdot d \cdot 10^k / 9$ and so $Q_n = \lambda^n + \delta^n = 0$, which is impossible. Moreover, since

$$h(\gamma_1) = h(\lambda) = \frac{\log \lambda}{2}, h(\gamma_2) = h(10) = \log 10$$

and

$$h(\gamma_3) \leq h(\sqrt{8}) + h(d) + h(9) \leq \frac{\log 8}{2} + \log 9 + \log 9 < 5.44$$

by (7) we can take $A_1 := 0.9$, $A_2 := 4.61$ and $A_3 := 10.88$. Also, since $k < n+1$, we can take $B := n+1$. Thus, taking into account the inequality (11) and using Theorem 1, we obtain

$$2.5 \cdot \lambda^{m-n} > |\Lambda_1| > \exp\left(C \cdot (1 + \log(n+1))\,(0.9)\,(4.61)\,(10.88)\right),$$

where $C = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2)$. This implies that

$$(n-m)\log\lambda - \log 2.5 < 4.38 \cdot 10^{13} \cdot (1 + \log(n+1)). \tag{12}$$

Now, let rearrange the equation (3) as

$$\frac{\lambda^n}{\sqrt{8}} - \frac{\lambda^m}{\sqrt{8}} - \frac{d \cdot 10^k}{9} = \frac{\delta^n}{\sqrt{8}} - \frac{\delta^m}{\sqrt{8}} - \frac{d}{9}. \tag{13}$$

Taking absolute values of both sides of (13), we get

$$\left|\frac{\lambda^n \cdot (1 - \lambda^{m-n})}{\sqrt{8}} - \frac{d \cdot 10^k}{9}\right| \leq \frac{|\delta|^n + |\delta|^m}{\sqrt{8}} + \frac{d}{9} < 1.2. \tag{14}$$

We divide both sides of (14) by $\frac{\lambda^n \cdot (1-\lambda^{m-n})}{\sqrt{8}}$ to obtain

$$\begin{aligned}\left|1 - \frac{\lambda^{-n} \cdot (1 - \lambda^{m-n})^{-1} \cdot \sqrt{8} \cdot d \cdot 10^k}{9}\right| &\leq 3.4 \cdot \lambda^{-n} \cdot (1 - \lambda^{m-n})^{-1} \\ &< (4.2) \cdot \lambda^{-n}.\end{aligned} \tag{15}$$

Put $(\gamma_1, b_1) := (\lambda, -n)$, $(\gamma_2, b_2) := (10, k)$, and $(\gamma_3, b_3) := \left((1 - \lambda^{m-n})^{-1} \cdot \sqrt{8} \cdot d/9, 1\right)$. The numbers γ_1, γ_2, and γ_3 are positive real numbers and elements of the field $\mathbb{K} = \mathbb{Q}(\sqrt{2})$ and so $D = 2$. Let

$$\Lambda_2 := 1 - \frac{\lambda^{-n} \cdot (1 - \lambda^{m-n})^{-1} \cdot \sqrt{8} \cdot d \cdot 10^k}{9}.$$

Then Λ_2 is nonzero. For, if $\Lambda_2 = 0$, then $\lambda^n = (1-\lambda^{m-n})^{-1} \cdot \sqrt{8} \cdot d \cdot 10^k/9$. Conjugating in $\mathbb{Q}(\sqrt{2})$ gives us $\delta^n = -(1 - \delta^{m-n})^{-1} \cdot \sqrt{8} \cdot d \cdot 10^k/9$. By a simple computation, it seen that $Q_n = Q_m$, which is impossible since $n > m$. Since

$$h(\gamma_1) = h(\lambda) = \frac{\log\lambda}{2}, h(\gamma_2) = h(10) = \log 10$$

and

$$\begin{aligned} h(\gamma_3) &\leq h(\sqrt{8}) + h(d) + h(9) + h((1 - \lambda^{m-n})^{-1}) \\ &\leq \frac{\log 8}{2} + \log 9 + \log 9 + (n-m)\frac{\log\lambda}{2} + \log 2 \\ &< 6.13 + (n-m)\frac{\log\lambda}{2} \end{aligned}$$

by (6),(7), and (8), we can take $A_1 := 0.9$, $A_2 := 4.61$ and $A_3 := 12.26 + (n-m)\log\lambda$. The same argument as above shows that we can take $B := n+1$. Thus, taking into account the inequality (15) and using Theorem 1, we obtain

$$4.2 \cdot \lambda^{-n} > |\Lambda_2| > \exp\left(C \cdot (1 + \log(n+1))\,(0.9)\,(4.61)\,(12.26 + (n-m)\log\lambda)\right),$$

where $C = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2)$. This implies that

$$n \log \lambda - \log 4.2 < 4.03 \cdot 10^{12} \cdot (1 + \log(n+1)) \cdot (12.26 + (n-m) \log \lambda) . \tag{16}$$

Combining the inequalities (12) and (16), we get

$$n \log \lambda - \log 4.2 < 4.03 \cdot 10^{12} (1 + \log(n+1)) \big(12.26 + (\log 2.5 + 4.38 \cdot 10^{13} (1 + \log(n+1))\big) \tag{17}$$

Hence, a computer search with Mathematica gives us that $n < 9.84 \cdot 10^{29}$. Now, let us try to reduce the upper bound on n by applying Lemma 2. Let

$$z_1 := k \log 10 - n \log \lambda + \log(\sqrt{8} d/9)$$

and $\Lambda_1 := 1 - e^{z_1}$. From (11), we have

$$|\Lambda_1| = |1 - e^{z_1}| < 2.5 \cdot \lambda^{m-n} < 0.45$$

for $n - m \geq 2$. Choosing $a := 0.45$, we get the inequality

$$|z_1| < -\frac{\log 0.55}{0.45} \cdot \frac{2.5}{\lambda^{n-m}} < (3.33) \cdot \lambda^{-(n-m)}$$

by Lemma 3. Thus, it follows that

$$0 < \left| k \log 10 - n \log \lambda + \log(\sqrt{8} d/9) \right| < (3.33) \cdot \lambda^{-(n-m)}.$$

Dividing this inequality by $\log \lambda$, we get

$$0 < |k\gamma - n + \mu| < (3.78) \cdot \lambda^{-(n-m)}, \tag{18}$$

where

$$\gamma := \frac{\log 10}{\log \lambda} \notin \mathbb{Q} \text{ and } \mu := \frac{\log(\sqrt{8} d/9)}{\log \lambda}.$$

Put $M := 9.84 \cdot 10^{29}$, which is an upper bound on k since $k < n + 1$ and $n < 9.84 \cdot 10^{29}$. We found that q_{69}, the denominator of the 69 th convergent of γ exceeds $6M$. Considering the fact that $1 \leq d \leq 9$, a quick computation with Mathematica gives us the inequality

$$0.001 < \epsilon := ||\mu q_{69}|| - M||\gamma q_{69}|| < 0.43.$$

Let $A := 3.78$, $B := \lambda$, and $w := n - m$. Thus, Lemma 2 says to us that the inequality (18) has a solutions for

$$n - m < \frac{\log(A q_{69}/\epsilon)}{\log B} < 91.52,$$

which implies that $n - m \leq 91$. Consequently, substituting this upper bound for $n - m$ into (16), we obtain $n < 1.63 \cdot 10^{16}$. Now, let

$$z_2 := k \log 10 - n \log \lambda + \log \left(\frac{(1 - \lambda^{m-n})^{-1} \cdot \sqrt{8} \cdot d}{9} \right).$$

and $\Lambda_2 := 1 - e^{z_2}$. It is clear that

$$|\Lambda_2| = |1 - e^{z_2}| < (4.2) \cdot \lambda^{-n} < 0.01$$

by (15), where we have used the fact that $n \geq 100$. Thus, taking $a := 0.01$ in Lemma 3 and making necessary calculations, we get

$$|z_2| < \frac{\log(100/99)}{0.01} \cdot \frac{4.2}{\lambda^n} < 4.23 \cdot \lambda^{-n}.$$

That is,

$$0 < \left| k \log 10 - n \log \lambda + \log \left(\frac{(1-\lambda^{m-n})^{-1} \cdot \sqrt{8} \cdot d}{9} \right) \right| < 4.23 \cdot \lambda^{-n}.$$

Dividing both sides of the above inequality by $\log \lambda$, we obtain

$$0 < |k\gamma - n + \mu| < 4.8 \cdot \lambda^{-n}, \tag{19}$$

where

$$\gamma := \frac{\log 10}{\log \lambda} \text{ and } \mu := \frac{\log \left(\frac{(1-\lambda^{m-n})^{-1} \cdot \sqrt{8} \cdot d}{9} \right)}{\log \lambda}.$$

Since $k < n+1$, we can take $M := 1.63 \cdot 10^{16}$, which is an upper bound on k. We found that q_{46}, the denominator of the 46 th convergent of γ exceeds $6M$. For $2 \leq n-m \leq 91$ and $1 \leq d \leq 9$, a quick computation with Mathematica gives us the inequality

$$0.0002 < \epsilon := ||\mu q_{46}|| - M||\gamma q_{46}|| < 0.499.$$

Let $A := 4.8,\ B := \lambda$, and $w := n$ in Lemma 2. Thus, with the help of Mathematica, we can say that if the inequality (19) has a solution, then

$$n < \frac{\log(A q_{46}/\epsilon)}{\log B} < 64.1,$$

which yields $n \leq 64$. This contradicts our assumption that $n \geq 100$. Thus, the proof is completed. □

Now, we can give the following result.

Corollary 7. *The largest repdigit, which is difference of two Pell numbers is* $99 = 169 - 70 = P_7 - P_6$.

Theorem 8. *Let* $1 \leq m < n$, $k \geq 1$, *and* $1 \leq d \leq 9$. *If* $Q_n - Q_m$ *is a repdigit, then*

$$(n, m, d, k, Q_n - Q_m) \in \{(2,1,4,1,4), (3,2,8,1,8), (7,4,4,3,444)\}.$$

Proof. Assume that $Q_n - Q_m$ is a repdigit. Then the equation (4) holds for $1 \leq m < n$ with $k \geq 1$. Let us suppose that $1 \leq m < n \leq 99$. Then by using Mathematica program, we obtain only the solutions displayed in the statement of Theorem 8. Let $n - m = 1$. Then we get

$$Q_{m+1} - Q_m = Q_m + Q_{m-1}.$$

Thus by Lemma 5, we get the solution $(m, m-1, d, k, Q_{m+1} - Q_m) = (2,1,8,1,8)$, which gives the solution $(n, m, d, k, Q_n - Q_m) = (3,2,8,1,8)$. From now on, assume that $n \geq 100, m \geq 1$ and $n - m \geq 2$. Since Q_n is even for all n, $Q_n - Q_m$ is even. Therefore, we get $d = 2,4,6,8$. Then, by using (2), we get

$$\lambda^{2k-4} < 10^{k-2} < \frac{8}{9} \cdot 10^{k-1} < \frac{d \cdot (10^k - 1)}{9} = Q_n - Q_m < \lambda^{n+1}.$$

This shows that $2k < n + 5$. That is, $k < n + 5$. On the other hand, rearranging the equation (4) as

$$\lambda^n - \frac{d \cdot 10^k}{9} = Q_m - \delta^n - \frac{d}{9} \tag{20}$$

and taking absolute values of both sides of (20), we get

$$\left|\lambda^n - \frac{d\cdot 10^k}{9}\right| \le Q_m + |\delta|^n + \frac{d}{9} < 2\lambda^m + 1. \tag{21}$$

Dividing both sides of (21) by λ^n yields

$$\begin{aligned}\left|1 - \frac{\lambda^{-n}\cdot d\cdot 10^k}{9}\right| &\le 2\lambda^{m-n} + \lambda^{-n}\\ &< \lambda^{m-n}(2+\lambda^{-m})\\ &< 2.5\cdot\lambda^{m-n}\end{aligned} \tag{22}$$

where we have used the fact that $m \ge 1$. Now, let us apply Theorem 1 with $(\gamma_1, b_1) := (\lambda, -n), (\gamma_2, b_2) := (10, k), (\gamma_3, b_3) := (d/9, 1)$. Observe that the numbers γ_1, γ_2, and γ_3 are positive real numbers and belong to the field $\mathbb{K} = \mathbb{Q}(\sqrt{2})$. It is obvious that the degree of the field $\mathbb{K}$ is 2. So $D = 2$. Now, we show that

$$\Lambda_1 := 1 - \frac{\lambda^{-n}\cdot d\cdot 10^k}{9}$$

is nonzero. Contrast to this, we assume that $\Lambda_1 = 0$. Then $\lambda^n = d\cdot 10^k/9$, which is impossible since λ^n is irrational. Moreover, since

$$h(\gamma_1) = h(\lambda) = \frac{\log\lambda}{2}, h(\gamma_2) = h(10) = \log 10$$

and

$$h(\gamma_3) \le h(d) + h(9) \le \log 8 + \log 9 < 4.3$$

by (7), we can take $A_1 := 0.9$, $A_2 := 4.61$ and $A_3 := 8.6$. Also, since $k < n+5$, we can take $B := n+5$. Thus, taking into account the inequality (22) and using Theorem 1, we obtain

$$(2.5)\cdot\lambda^{m-n} > |\Lambda_1| > \exp\left(C\cdot(1+\log(n+5))\,(0.9)\,(4.61)\,(8.6)\right),$$

where $C = -1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2\cdot(1+\log 2)$. This implies that

$$(n-m)\log\lambda - \log 2.5 < 3.47\cdot 10^{13}\cdot(1+\log(n+5)). \tag{23}$$

Now, let rearrange the equation (4) as

$$\lambda^n - \lambda^m - \frac{d\cdot 10^k}{9} = -\delta^n + \delta^m - \frac{d}{9}. \tag{24}$$

Taking absolute values of both sides of (24), we get

$$\left|\lambda^n\cdot(1-\lambda^{m-n}) - \frac{d\cdot 10^k}{9}\right| \le |\delta|^n + |\delta|^m + \frac{d}{9} < 1.4. \tag{25}$$

Dividing both sides of (25) by $\lambda^n\cdot(1-\lambda^{m-n})$, we obtain

$$\begin{aligned}\left|1 - \frac{\lambda^{-n}\cdot(1-\lambda^{m-n})^{-1}\cdot d\cdot 10^k}{9}\right| &< (1.4)\cdot\lambda^{-n}\cdot(1-\lambda^{m-n})^{-1}\\ &< (1.7)\cdot\lambda^{-n}.\end{aligned} \tag{26}$$

Put $(\gamma_1, b_1) := (\lambda, -n), (\gamma_2, b_2) := (10, k)$, and $(\gamma_3, b_3) := ((1-\lambda^{m-n})^{-1} \cdot d/9, -1)$. The number field containing γ_1, γ_2, and γ_3 is $\mathbb{K} = \mathbb{Q}(\sqrt{2})$, which has degree $D = 2$. Let

$$\Lambda_2 := 1 - \frac{\lambda^{-n} \cdot (1-\lambda^{m-n})^{-1} \cdot d \cdot 10^k}{9}.$$

Then Λ_2 is nonzero. For, if $\Lambda_2 = 0$, then $\lambda^n = (1-\lambda^{m-n})^{-1} \cdot d \cdot 10^k/9$. Conjugating in $\mathbb{Q}(\sqrt{2})$ gives us $\delta^n = -(1-\delta^{m-n})^{-1} \cdot d \cdot 10^k/9$. By a simple computation, it seen that $Q_n = Q_m$, which is impossible since $n > m$. Since

$$h(\gamma_1) = h(\lambda) = \frac{\log\lambda}{2}, h(\gamma_2) = h(10) = \log 10,$$

and

$$\begin{aligned} h(\gamma_3) &\le h(d) + h(9) + h((1-\lambda^{m-n})^{-1}) \\ &\le \log 8 + \log 9 + (n-m)\frac{\log\lambda}{2} + \log 2 \\ &< 4.97 + (n-m)\frac{\log\lambda}{2} \end{aligned}$$

by (6),(7), and (8), we can take $A_1 := 0.9$, $A_2 := 4.61$ and $A_3 := 9.94 + (n-m)\log\lambda$. The same argument as above shows that we can take $B := n+5$. Thus, taking into account the inequality (26) and using Theorem 1, we obtain

$$(1.7) \cdot \lambda^{-n} > |\Lambda_2| > \exp\left(C \cdot (1+\log(n+5))\,(0.9)\,(4.61)\,(9.94 + (n-m)\log\lambda)\right),$$

where $C = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1+\log 2)$. This implies that

$$n\log\lambda - \log(1.7) < 4.03 \cdot 10^{12}(1+\log(n+5))\,(9.94 + (n-m)\log\lambda)\,. \tag{27}$$

Combining the inequalities (23) and (27), we get

$$n\log\lambda - \log(1.7) < 4.03 \cdot 10^{12}(1+\log(n+5))(9.94 + \log(2.5) + 3.47 \cdot 10^{13} \cdot (1+\log(n+5))). \tag{28}$$

Hence, a computer search with Mathematica gives us that $n < 7.74 \cdot 10^{29}$. Now, let us try to reduce the upper bound on n by applying Lemma 2. Now, let

$$z_1 := k\log 10 - n\log\lambda + \log(d/9)$$

and $\Lambda_1 := 1 - e^{z_1}$. From (22), we have

$$|\Lambda_1| = |1 - e^{z_1}| < \frac{2.5}{\lambda^{n-m}} < 0.45$$

for $n - m \ge 2$. Choosing $a := 0.45$, we get the inequality

$$|z_1| < -\frac{\log(0.55)}{0.45} \cdot \frac{2.5}{\lambda^{n-m}} < (3.33) \cdot \lambda^{-(n-m)}$$

by Lemma 3. Thus, it follows that

$$0 < |k\log 10 - n\log\lambda + \log(d/9)| < (3.33) \cdot \lambda^{-(n-m)}. \tag{29}$$

Dividing this inequality by $\log\lambda$, we get

$$0 < \left| k\left(\frac{\log 10}{\log\lambda}\right) - n + \left(\frac{\log(d/9)}{\log\lambda}\right) \right| < (3.78) \cdot \lambda^{-(n-m)}. \tag{30}$$

Take $\gamma := \dfrac{\log 10}{\log \lambda} \notin \mathbb{Q}$ and $M := 7.74 \cdot 10^{29}$. We found that q_{69}, the denominator of the 69 th convergent of γ exceeds $6M$. Now let

$$\mu := \frac{\log(d/9)}{\log \lambda}.$$

Considering the fact that $d = 2, 4, 6, 8$ a quick computation with Mathematica gives us that the inequality

$$0.07 < \epsilon := ||\mu q_{69}|| - M||\gamma q_{69}|| < 0.36.$$

Let $A = 3.78,\ B = \lambda$, and $w = n - m$ in Lemma 2. Thus, if the inequality (30) has a solution, then

$$n - m < \frac{\log(Aq_{69}/\epsilon)}{\log B} < 87.46,$$

which implies that $n - m \leq 87$. Substituting this upper bound for $n - m$ into (27), we obtain $n < 1.52 \cdot 10^{16}$. Now, let

$$z_2 := k \log 10 - n \log \lambda + \log\left(\frac{(1 - \lambda^{m-n})^{-1} \cdot d}{9}\right). \tag{31}$$

and $\Lambda_2 := 1 - e^{z_2}$. It is clear that

$$|\Lambda_2| = |1 - e^{z_2}| < (1.7) \cdot \lambda^{-n} < 0.01$$

by (26), where we have used the fact that $n \geq 100$. Thus, taking $a := 0.01$ in Lemma 3 and making necessary calculations, we get

$$|z_2| < \frac{\log(100/99)}{0.01} \cdot \frac{1.7}{\lambda^n} < (1.71) \cdot \lambda^{-n}.$$

That is,

$$0 < \left| k \log 10 - n \log \lambda + \log\left(\frac{(1 - \lambda^{m-n})^{-1} \cdot d}{9}\right)\right| < (1.71) \cdot \lambda^{-n}. \tag{32}$$

Dividing both sides of the above inequality by $\log \lambda$, we obtain

$$0 < |k\gamma - n + \mu| < A \cdot B^{-w}, \tag{33}$$

where

$$\gamma := \frac{\log 10}{\log \lambda}, \mu := \frac{\log\left(\frac{(1-\lambda^{m-n})^{-1} \cdot d}{9}\right)}{\log \lambda},\ \ A := 1.95,\ \ B := \lambda,$$

and $w := n$. Since $k < n + 5$, we can take $M := 1.52 \cdot 10^{16}$. We found that q_{44}, the denominator of the 44 th convergent of γ exceeds $6M$. Applying Lemma 2 to the inequality (33) for $2 \leq n - m \leq 87$, a quick computation with Mathematica gives us that

$$0.002 < \epsilon := ||\mu q_{44}|| - M||\gamma q_{44}|| < 0.496$$

and thus, we can say that if the inequality (33) has a solution, then

$$n < \frac{\log(Aq_{44}/\epsilon)}{\log B} < 55.92.$$

This yields $n \leq 55$, which contraicts our assumption that $n \geq 100$. □

Now, we can give the following result.

Corollary 9. *The largest repdigit which is difference of two Pell-Lucas numbers is* $444 = 478 - 34 = Q_7 - Q_4$.

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