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R-NOTION OF CONJUGACY IN PARTIAL TRANSFORMATION SEMIGROUP

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Abstract. In this paper, we present a new definition of conjugacy that can be applied to an arbitrary semigroup and it does not reduce to the universal relation in semigroups with a zero. We compare the new notion of conjugacy with existing notions, characterize the conjugacy in subsemigroups of partial transformations through digraphs and restrictive partial homomorphisms.

1. Introduction

Let G be a group. For $x, y \in G$, we say x is conjugate to y if there exists $p \in G$ such that $y = p^{-1}xp$ which is equivalent to $xp = py$. Using the latter formulation one may try to extend the notion of conjugacy to semigroups in the following way: define a relation $\sim_l$ on a semigroup S by

$$x \sim_l y \Leftrightarrow \exists\ p \in S^1 \text{ such that } xp = py$$

where S^1 is S with an identity adjoined. If $x \sim_l y$, we say x is left conjugate to y [1,9,10]. The relation $\sim_l$ is always reflexive and transitive in any semigroup but not symmetric in general. The relation $\sim_l$ gets reduced to a universal relation in a semigroup with zero. However the relation $\sim_l$ is an equivalence relation on a free semigroup. Lallement [4] has defined the conjugate elements of a free semigroup S as those related by $\sim_l$ and showed that $\sim_l$ is equal to the following equivalence on the free semigroup S:

$$x \sim_p y \Leftrightarrow \exists\ u, v \in S^1 \text{ such that } x = uv \text{ and } y = vu$$

The relation $\sim_p$ is always reflexive and symmetric but not transitive in general. The relation $\sim_l$ has been restricted to $\sim_o$ in [1], and $\sim_p$ has been extended to $\sim_p^*$ (the transitive closure of $\sim_p$) in [2, 3], in such a way that the modified relations are equivalences on an arbitrary semigroup S.

Otto in [1] introduced the $\sim_o$ notion of conjugacy in semigroup S defined as:

$$x \sim_o y \Leftrightarrow \exists\ p, q \in S^1 \text{ such that } xp = py \text{ and } yq = qx,$$

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The relation $\sim_o$ is not useful for semigroups S with zero since for every such S, we have $\sim_o = S \times S$. This deficiency has been remedied in [6], where the following relation has been defined on an arbitrary semigroup S,

$$x \sim_c y \Leftrightarrow \exists\ p \in \mathbb{P}^1(x), q \in \mathbb{P}^1(y) \text{ such that } xp = py \text{ and } yq = qx,$$

where for $x \neq 0$, $\mathbb{P}(x) = \{p \in S^1 : (mx)p \neq 0 \text{ for all } mx \in S^1(x)\backslash\{0\}\}$ and $\mathbb{P}(0) = \{1\}$. The relation $\sim_c$ is an equivalence relation on any semigroup S and it does not get reduce to $S \times S$ if S has a zero, and it is equal to $\sim_o$ if S does not have a zero.

Further, J. Konieczny in [7] introduced the $\sim_n$ notion of conjugacy in semigroups. If S is a semigroup and let $x, y \in S$. Then,

$$x \sim_n y \Leftrightarrow \exists\ p, q \in S^1 \text{ such that } xp = py, yq = qx, x = pyq \text{ and } y = qxp.$$

This relation is an equivalence relation in any semigroup and does not get reduced to a universal relation in a semigroup with zero.

The aim of this paper is to introduce a new definition of conjugacy in an arbitrary semigroup. The new conjugacy is an equivalence relation $\sim_r$ (the r-conjugacy) on any semigroup S.

J.Araujo et.al in [6] characterized $\sim_c$ conjugacy in constant rich subsemigroups of $\mathcal{P}(T)$ (the semigroup of partial maps on a non empty set T) with the help of rp-hom of their digraphs. In this paper we prove similar results for $\sim_r$ notion of conjugacy for any subsemigroup of $\mathcal{P}(T)$ without the assumption of constant rich on S.

2. The notion $\sim_r$

If S is a semigroup and let $x, y \in S$. Then,

$$x \sim_r y \Leftrightarrow \exists\ p, q, u, v \in S^1 \text{ such that } xp = py, yq = qx, x = pyu \text{ and } y = qxv.$$

THEOREM 2.1. *If S is a semigroup then*

(1) $\sim_r$ is an equivalence relation in any semigroup.
(2) $[0]_r = \{0\}$.
(3) If S is a group then $\sim_r$ reduces to the usual notion of conjugacy.

Proof. (1) Let $x \sim_r y$ then there exists $p, q, u, v \in S^1$ such that $xp = py, yq = qx, x = pyu$ and $y = qxv$.

(i) **Reflexivity:** We take $p = q = u = v = 1$ and we get the required.
(ii) **Symmetry:** This condition is by definition of notion.
(iii) **Transitivity:** Let $x \sim_r y$ and $y \sim_r z$ then there exists p_1, q_1, u_1, v_1 and p_2, q_2, u_2, v_2 such that $xp_1 = p_1y, yq_1 = q_1x, x = p_1yu_1$ and $y = q_1xv_1$ and $xp_2 = p_2y, yq_2 = q_2x, x = p_2yu_2$ and $y = q_2xv_2$. Now $xp_1p_2 = p_1yp_2 = p_1p_2z$, $zq_2q_1 = q_2yq_1 = q_2q_1x$, $x = p_1yu_1 = p_1p_2zu_2u_1$ and $z = q_2yv_2 = q_2q_1xv_1v_2$. Hence $x \sim_r z$.

(2) Let $x \neq 0$ and let $x \sim_r 0$ then there exists $p, q, u, v \in S^1$ such that $xp = p0, 0q = qx, x = p0u$ and $0 = qxv$. This means $x = 0$. So we get $[0]_r = \{0\}$.

(3) Let $x \sim_r y$ then there exists $p, q, u, v \in S^1$ such that $xp = py, yq = qx, x = pyu$ and $y = qxv$. From $xp = py$ we can pre-multiply by p^{-1} both sides to get $y = g^{-1}xg$ which is the usual notion of conjugacy. □

THEOREM 2.2. *Let S be a semigroup then $\sim_n \subseteq \sim_r \subseteq \sim_c \subseteq \sim_o$.*

Proof. Let $x, y \in S^1$ and let $x \sim_n y$ then there exists $p, q \in S^1$ such that $xp = py, yq = qx, x = pyq$ and $y = qxp$. we can take $u = q$ and $v = p$ so $x \sim_r y$. Thus $\sim_n \subseteq \sim_r$. Next we prove $\sim_r \subseteq \sim_c$. Let $x \sim_r y$ then there exist $p, q, u, v \in S^1$ such that $xp = py, yq = qx, x = pyu$ and $y = qxv$. If $x = 0$ then $y = 0$ since $[0]_r = 0$ and so $x \sim_c y$. Suppose $x \neq 0$ and let $m \in S^1$ be such that $mx \neq 0$. Then $(mx)p \neq 0$ since if $(mx)p = 0$ then $mpy = 0$ which further implies $mpyu = 0$ which implies $mx = 0$ which is a contradiction. Hence $(mx)p \neq 0$. Similarly, if $m \in S^1$ is such that $my \neq 0$ then $(my)q \neq 0$. So, $p \in \mathbb{P}^1(x)$ and $q \in \mathbb{P}^1(y)$. Hence $x \sim_c y$. Since $\sim_c \subseteq \sim_o$ is obvious. Hence we get the required result. □

3. $\sim_r$ notion of conjugacy through digraphs in $\mathcal{P}(T)$

Definition 3.1. Let T be any set and R be a binary relation on T then $\Gamma = (T, R)$ is called a *directed graph* (or a *digraph*). Any $p \in T$ is called a *vertex* and any $(p, q) \in R$ is called an *arc* of Γ.
For example, Let $T = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 3)\}$. Then the digraph Γ is as under,

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet}\ \overset{4}{\bullet}\ .$$

Definition 3.2. A vertex $p \in T$ for which there exists no q in T such that $(p, q) \in R$ is called a *terminal vertex* of Γ. A vertex $p \in T$ is said to be initial vertex if there is no $q \in T$ for which $(q, p) \in R$ while as a vertex $p \in T$ is said to be a non initial vertex if $(q, p) \in R$ for some $q \in T$.

For any $\sigma \in \mathcal{P}(T)$, $\Gamma(\sigma) = (T, R_\sigma)$ represents a digraph, where for all $p, q \in T, (p, q) \in R_\sigma$ if and only if $p \in \mathrm{dom}(\sigma)$ and $p\sigma = q$. For example, If $T = \{1, 2, 3\}$ and $R_\sigma = \{(1, 2), (2, 1)\}$. Then the digraph $\Gamma(\sigma)$ is represented as

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{1}{\bullet}\ .$$

For a non-empty set T, we fix an element $\diamond \notin T$. For $\sigma \in \mathcal{P}(T)$ and $t \in T$, we will write $t\sigma = \diamond$, if and only if $t \notin \mathrm{dom}(\sigma)$. we also assume that $\diamond\sigma = \diamond$. With this notation it makes sense to write $s\sigma = t\tau$ or $s\sigma \neq t\tau$ $(\sigma, \tau \in \mathcal{P}(T), s, t \in T)$ even when $s \notin \mathrm{dom}(\sigma)$ or $t \notin \mathrm{dom}(\tau)$. For any $\sigma \in \mathcal{P}(T)$ span(σ) represents $\mathrm{dom}(\sigma) \cup \mathrm{im}(\sigma)$.

For semigroups U and S, we write $U \leq S$ to mean that U is a subsemigroup of S.

Definition 3.3. Let $\Gamma_1 = (T_1, R_1)$ and $\Gamma_2 = (T_2, R_2)$ be digraphs. A mapping α from T_1 to T_2 is called a *homomorphism* from Γ_1 to Γ_2 if for all $p, q \in T_1, (p, q) \in R_1$ implies $(p\alpha, q\alpha) \in R_2$. A partial mapping α from T_1 to T_2 is called a *partial homomorphism* from Γ_1 to Γ_2 if for all $p, q \in \mathrm{dom}(\alpha)$, $(p, q) \in R_1$ implies $(p\alpha, q\alpha) \in R_2$.

Definition 3.4. A partial homomorphism α from T_1 to T_2 is called a *restrictive partial homomorphism* from Γ_1 to Γ_2 if it satisfies the following conditions:

(a) If $(p, q) \in R_1$, then $p, q \in \mathrm{dom}(\alpha)$ and $(p\alpha, q\alpha) \in R_2$.
(b) If p is a terminal vertex in Γ_1 and $p \in \mathrm{dom}(\alpha)$, then $p\alpha$ is a terminal vertex in Γ_2.

We say that Γ_1 is *rp-homomorphic* to Γ_2 if there is an rp-homomorphism from Γ_1 to Γ_2.

Throughout this paper by an rp-hom we shall mean an rp-homomorphism between any two digraphs and by hom we shall mean a homomorphism.

The next theorem provides necessary and sufficient condition for two elements of subsemigroup of $\mathcal{P}(T)$ to be $\sim_r$ related.

Theorem 3.5. *Let $S \leq \mathcal{P}(T)$ and $\sigma, \tau \in S$. Then $\sigma \sim_r \tau$ if and only if there are $\alpha, \beta, \phi, \psi \in S^1$ for which α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with $q\alpha\phi = q$ for every non initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non initial vertex k of $\Gamma(\tau)$.*

Proof. Suppose $\sigma \sim_r \tau$ in S. If $\sigma = 0$ then $\tau = 0$ and so $\alpha = \mathrm{id}_T \in S^1$ is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and $\beta = \mathrm{id}_T \in S^1$ is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$. Next suppose $\sigma \neq 0$ and let $\sigma \sim_r \tau$ in S then $\sigma\alpha = \alpha\tau, \tau\beta = \beta\sigma, \sigma = \alpha\tau\phi$ and $\tau = \beta\sigma\psi$ for some $\alpha, \beta, \phi, \psi \in S^1$. Let $(p, q) \in \sigma$. Then $p\alpha\tau\phi = q$, $(p\alpha)\tau\phi = q$ which implies $p \in \mathrm{dom}\alpha$. Again

$$q\alpha\phi = (p\sigma)\alpha\phi = p\sigma\alpha\phi = p\alpha\tau\phi = p\sigma = q \tag{3.1}$$

which implies $q \in \mathrm{dom}\alpha$. Next $(p\alpha)\tau = p\alpha\tau = p\sigma\alpha = q\alpha$ (by 3.1) $\neq \diamond$, $(p\alpha, q\alpha) \in \Gamma(\tau)$. Again let p be a terminal vertex of $\Gamma(\sigma)$ and $p \in \mathrm{dom}\alpha$ then as $\sigma\alpha = \alpha\tau$, $(p\alpha)\tau = p\alpha\tau = p\sigma\alpha = \diamond\alpha = \diamond$. Thus $p\alpha$ is a terminal vertex in $\Gamma(\tau)$. So α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$. Again by using $\tau\beta = \beta\sigma$ and $\tau = \beta\sigma\psi$ we can prove the β is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$.

Conversely let there are $\alpha, \beta, \phi, \psi \in S^1$ for which α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with $q\alpha\phi = q$ for every non initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non initial vertex k of $\Gamma(\tau)$. We show $\sigma \sim_r \tau$ in S. Let $p \in T$. The following cases arise.
Case 1: Suppose $p \notin \mathrm{dom}\sigma$, then $p\sigma = \diamond$. Then $p(\sigma\alpha) = (p\sigma)\alpha = \diamond\alpha = \diamond$. If $p \notin \mathrm{dom}\alpha$ then $p(\alpha\tau) = (p\alpha)\tau = \diamond$. So, $\sigma\alpha = \alpha\tau$. Also $p\alpha\tau\phi = \diamond$, so $\sigma = \alpha\tau\phi$. If $p \in \mathrm{dom}\alpha$ then as p is a terminal vertex of $\Gamma(\sigma)$ and since α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ we have $p(\alpha\tau) = \diamond$ so $p\alpha\tau\phi = \diamond$ i.e, $\sigma = \alpha\tau\phi$ in this case.
Case 2: Suppose $p \in \mathrm{dom}\sigma$ and let $q = p\sigma$. Then by definition of rp-hom $p, q \in \mathrm{dom}\alpha$ and $(p\alpha)\tau = q\alpha$. Hence $p(\sigma\alpha) = (p\sigma)\alpha = q\alpha$ and $p(\alpha\tau) = (p\alpha)\tau = q\alpha$. So, $\sigma\alpha = \alpha\tau$. Also, $p\alpha\tau\phi = p\sigma\alpha\phi = q\alpha\phi = q$ as $q\alpha\phi = q$ for any non initial vertex q of $\Gamma(\sigma)$. So, $\sigma = \alpha\tau\phi$.
By symmetry β is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$ such that $\tau\beta = \beta\sigma$ and $\tau = \beta\sigma\psi$. Thus $\sigma \sim_r \tau$. This proves the Theorem. □

If $\sigma, \tau \in \mathcal{T}(T)$ (the semigroup of full transformations on T). Then every rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ is hom. So we have the following corollary.

Corollary 3.6. *Let $S \leq \mathcal{T}(T)$ and $\sigma, \tau \in S$. Then $\sigma \sim_r \tau$ if and only if there are $\alpha, \beta, \phi, \psi \in S^1$ such that α is a hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is a hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with $q\alpha\phi = q$ for every non initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non initial vertex k of $\Gamma(\tau)$.*

4. $\sim_r$ notion of conjugacy through connected partial transformations

Definition 4.1. Let $\cdots, p_{-2}, p_{-1}, p_0, p_1, p_2, \cdots$ be pairwise distinct elements of T. The following maps introduced by J. Araujo et al in [6] are very important for our study.

(1) A $\sigma \in \mathcal{P}(T)$ is called a *cycle* of length k if $\sigma = (p_0 p_1 p_2 \cdots p_{k-1})$ where $(k \geq 1)$. i.e., $p_j = p_{j-1}\sigma$, $j = 1, 2, \cdots, k$ and $p_0 = p_{k-1}\sigma$ and we write it as

$$p_0 \to p_1 \to p_2 \to \cdots \to p_{k-1} \to p_0.$$

(2) A $\sigma \in \mathcal{P}(T)$ is called a *right ray* if $\sigma = [p_0\ p_1\ p_2\ \cdots >$. i.e., $p_j = p_{j-1}\sigma$, $j \geq 1$ and we write it as

$$p_0 \to p_1 \to p_2 \to \cdots .$$

(3) A $\sigma \in \mathcal{P}(T)$ is called a *double ray* if $\sigma = < \cdots p_{-1}\ p_0\ p_1\ \cdots >$. i.e., $p_j = p_{j-1}\sigma$, $j \in \mathbb{Z}$ and we write it as

$$\cdots \to p_{-1} \to p_0 \to p_1 \to p_2 \to \cdots .$$

(4) A $\sigma \in \mathcal{P}(T)$ is called a *left ray*, if $\sigma = < \cdots p_2\ p_1\ p_0]$. i.e., $p_j\sigma = p_{j-1}$, $j \geq 1$ and we write it as

$$\cdots \to p_2 \to p_1 \to p_0.$$

(5) A $\sigma \in \mathcal{P}(T)$ is called a *chain* of length k if $\sigma = [p_0\ p_1\ p_2\ \cdots\ p_k]$. i.e., $p_j = p_{j-1}\sigma$, $j = 1, 2, \cdots, k$ and we write it as

$$p_0 \to p_1 \to p_2 \to \cdots \to p_k.$$

These are called as *basic partial maps.*

Definition 4.2. Any element $\kappa \neq 0$ in $\mathcal{P}(T)$ is said to be *connected* if for some non negative integers m, n, $p\kappa^m = q\kappa^n \neq \diamond$ for all $p, q \in \text{span}(\kappa)$.

For example, Let $T = \{1, 2, 3, 4, 5\}$. Define $\kappa \in \mathcal{P}(T)$ by $\kappa = \{(1,2), (2,3), (3,4)\}$, then the diagraph of κ is as under

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet} \longrightarrow \overset{4}{\bullet}.$$

Then κ is connected.

Definition 4.3. For $\sigma, \tau \in \mathcal{P}(T)$, if $\text{dom}(\tau) \subseteq \text{dom}(\sigma)$ and $p\tau = p\sigma$ for every $p \in \text{dom}(\tau)$ then τ is said to be *contained* in σ written as $\tau \subseteq \sigma$. They are *disjoint* if $\text{dom}(\sigma) \cap \text{dom}(\tau) = \emptyset$ and *completely disjoint* if $\text{span}(\sigma) \cap \text{span}(\tau) = \emptyset$.

For example, $[p\ q\ r\ s \cdots >$ and $[a\ b\ c\ p]$ in $\mathcal{P}(\mathbb{Z})$ are disjoint while as $[a\ b\ \cdots >$ and $[u\ v]$ are completely disjoint.

Definition 4.4. Let C be a set of pairwise disjoint elements of $\mathcal{P}(T)$. Then, for $x \in T$

$$x\Big(\bigcup_{\kappa \in C} \kappa\Big) = \begin{cases} x\kappa \text{ if } x \in \text{dom}(\kappa) \text{ for some } \kappa \in C \\ \diamond \text{ otherwise.} \end{cases}$$

is called the *join* of the elements of C denoted by $\bigcup_{\kappa \in C} \kappa$.

Definition 4.5. Let $\sigma \in \mathcal{P}(T)$ and let ν be a basic partial map with $\nu \subset \sigma$ then ν is *maximal* in σ if $x \notin \text{dom}(\nu)$ implies $x \notin \text{dom}(\sigma)$ and $x \notin \text{im}(\nu)$ implies $x \notin \text{im}(\sigma)$ for every $x \in \text{span}(\nu)$.

For example, Let $\sigma = [p\ q\ r\ s \cdots > \cup [a\ b\ c\ p] \in \mathcal{P}(\mathbb{Z})$. Then σ contains infinitely many right rays. For example, $[c\ p\ q\ r \cdots >$ but only two of them namely $[p\ q\ r\ s \cdots >$ and $[a\ b\ c\ p\ q\ r\ s \cdots >$ are maximal in σ.

Proposition 4.6. [6, Proposition 4.5] *Let $\sigma \in \mathcal{P}(T)$ with $\sigma \neq 0$. Then there exists a unique set C of pairwise completely disjoint, connected maps contained in σ such that $\sigma = \bigcup_{\kappa \in C} \kappa$.*

The componenents of C in Proposition 4.6 are called as connected *components* of σ. Through out this paper by c-component we shall mean a connected component.

Example 4.7. Let $T = \{1,2,3,4,5\}$ and $\sigma \in \mathcal{P}(T)$ be defined as $\sigma = \{(1,2),(2,3),(4,5)\}$. Clearly σ has a unique representation in terms of of pairwise completely disjoint, connected maps contained in σ. i.e., $\sigma = \cup_{\sigma_i \in C} \sigma_i$ where $C = \{\sigma_1, \sigma_2\}$ and $\sigma_1 = \{(1,2),(2,3)\}$ and $\sigma_2 = \{(4,5)\}$.

For any c-component κ of $\sigma \in \mathcal{P}(T)$, α_κ denotes the restriction of σ on Span(κ).

Lemma 4.8. *Let $S \leq \mathcal{P}(T)$ and $\sigma, \tau \in S$ with $\sigma \sim_r \tau$ then there exists $\alpha, \beta \in S^1$ such that dom(α) = span(σ) and dom(β) = span(τ).*

Proof. Let $\sigma \sim_r \tau$ then there exists $\alpha, \beta, \phi, \psi \in S^1$ such that $\sigma\alpha = \alpha\tau, \tau\beta = \beta\sigma, \sigma = \alpha\tau\phi$ and $\tau = \beta\sigma\psi$. By Theorem 3.5, α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$. We have to show that $\mathrm{dom}(\alpha) = \mathrm{span}(\sigma)$. Let $x \in \mathrm{span}(\sigma)$ which means $x \in \mathrm{dom}(\sigma) \cup \mathrm{im}(\sigma)$. If $x \in \mathrm{dom}(\sigma)$ then there exists $y \in T$ such that $(x,y) \in \sigma$. Since α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$. Therefore $x, y \in \mathrm{dom}(\alpha)$. So in this case $\mathrm{span}(\sigma) \subseteq \mathrm{dom}(\alpha)$. Similarly if $x \in \mathrm{im}(\sigma)$ then $\mathrm{span}(\sigma) \subseteq \mathrm{dom}(\alpha)$. Next we have to show $\mathrm{dom}(\alpha) \subseteq \mathrm{span}(\sigma)$. Since $\sigma = \alpha\tau\phi$, implies $\mathrm{dom}(\alpha) = \mathrm{dom}(\sigma) \subseteq \mathrm{span}(\sigma)$ implies $\mathrm{dom}(\alpha) \subseteq \mathrm{span}(\sigma)$. By similar arguments we can show that $\mathrm{dom}(\beta) = \mathrm{span}(\tau)$. ☐

The next Proposition is the interconnection of c-components and $\sim_r$ notion of conjugacy.

Proposition 4.9. *Let $S \leq \mathcal{P}(T)$ and $\sigma, \tau \in S$. Then, $\sigma \sim_r \tau$ if and only if*

(1) For every c-component κ of σ there exist a c-component λ of τ and an rp-hom $\alpha_\kappa \in \mathcal{P}(T)$ from $\Gamma(\kappa)$ to $\Gamma(\lambda)$ with dom(α_κ) = span(κ). Similar holds from τ to σ.

(2) $\cup_{\kappa \in C} \alpha_\kappa \in S^1$, where C is the collection of c-components of σ. Similar holds for τ.

(3) There is $\alpha, \beta, \phi, \psi \in S^1$ such that $q\alpha\phi = q$ for any non initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non initial vertex k of $\Gamma(\tau)$.

Proof. If $\sigma = 0$ then $\tau = 0$ and the result follows trivially. Suppose $\sigma \neq 0$ then $\tau \neq 0$ and let $\sigma \sim_r \tau$, then there is $\alpha, \beta, \phi, \psi \in S^1$ such that $\sigma\alpha = \alpha\tau, \tau\beta = \beta\sigma, \sigma = \alpha\tau\phi$ and $\tau = \beta\sigma\psi$ and so by Theorem 3.5, α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$. By Lemma 4.8 $\mathrm{dom}\alpha = \mathrm{span}(\sigma)$. By Proposition 4.6 $\sigma = \cup_{\kappa \in C}\kappa$ and $\tau = \cup_{\lambda \in C'}\lambda$ for some sets C, C' of pairwise completely disjoint connected maps contained in σ and τ respectively. Let κ be a c-component of σ and let $p \in \mathrm{span}(\kappa)$, since α is an rp-hom this means $p\alpha \in \lambda$ for some c-component λ of τ. We claim that $(\mathrm{span}(\kappa))\alpha \subseteq \mathrm{span}(\lambda)$. Let $z \in \mathrm{span}(\kappa)$ then by definition of connectedness there exists $r, s \geq 0$ such that $p\sigma^r = p\kappa^r = z\kappa^s = z\sigma^s \neq \diamond$. Since $\sigma \sim_r \tau$ we have $(z\alpha)\tau^s = (z\sigma^s)\alpha = (p\sigma^r)\alpha = (p\alpha)\tau^r \neq \diamond$ which implies $p\alpha$ and $z\alpha$ are in the span of same c-component of τ. So $z\alpha \in \mathrm{span}(\lambda)$. The claim has been proved. Let $\alpha_\kappa = \alpha|_{span(\kappa)}$. Then α_κ is an rp-hom from $\Gamma(\kappa)$ to $\Gamma(\lambda)$ with $\mathrm{dom}(\alpha_\kappa) = \mathrm{span}(\kappa)$.

Also $\cup_{\kappa\in C}\alpha_\kappa = \alpha \in S^1$ (by the definition of α_κ) and by $\mathrm{dom}(\alpha) = \mathrm{span}(\sigma)$). Similar holds from τ to σ.

Conversely, suppose that (1), (2) and (3) are satisfied. Let $\alpha = \bigcup_{\kappa\in C}\alpha_\kappa$. Note that α is well defined since α_κ and $\alpha_{\kappa'}$ are disjoint if $\kappa \neq \kappa'$. Suppose $(q, z) \in \sigma$. Then $q, z \in \mathrm{span}(\kappa)$ for some c-component κ of σ. Thus $q, z \in \mathrm{dom}(\alpha_\kappa)$ and $q\alpha = q\alpha_\kappa \to z\alpha_\kappa = z\alpha$, implying $q\alpha \xrightarrow{\tau} z\alpha$.
Suppose q is a terminal vertex in $\Gamma(\sigma)$ and $q \in \mathrm{dom}(\sigma)$. Then there is a unique c-component κ of σ such that q is a terminal vertex in $\Gamma(\kappa)$. Then $q\alpha = q\alpha_\kappa$ is a terminal vertex in $\Gamma(\lambda)$ and so a terminal vertex in $\Gamma(\tau)$. Hence α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$. Further $\mathrm{dom}(\alpha) = \mathrm{span}(\sigma)$, (by the definition of α) and $\alpha \in S^1$ (by (2)). By symmetry, we can similarly prove for β. Then by condition (3) and by Theorem 3.5 we have $\sigma \sim_r \tau$. □

Definition 4.10. Let X be a nonempty subset of the set $\mathbb{Z}_+$ of positive integers. Then X is partially ordered by the relation $|$(divides). Order the elements of X according to usual less than relation as $x_1 < x_2 < x_3 \cdots$, we define a subset $\mathrm{sac}(X)$ of X as follows : for every integer n, $1 \le n < |X| + 1$,

$$\mathrm{sac}(X) = \{x_n \in X\text{: for all } i < n, x_n \text{ is not a multiple of } x_i\}.$$

The set $\mathrm{sac}(X)$ is a *maximal anti-chain* of the poset $(X, |)$. We will call $\mathrm{sac}(X)$, the standard anti-chain of X.

For example, if $X = \{2, 4, 7\}$ then $\mathrm{sac}(X) = \{2, 7\}$.

Let σ be in $\mathcal{P}(T)$ such that σ contains a cycle. Let X denote the set of lengths of cycles in σ. The standard anti-chain of $(X, |)$ will be called the *cycle set* of σ and denoted by $\mathrm{cs}(\sigma)$.

Definition 4.11. A connected partial map κ is said to be of rro type (right rays only) if it has a maximal right ray but no cycles, double rays, left rays or maximal chains, and is of cho type (chains only) if it has a maximal chain but no cycles or rays.

Lemma 4.12. [6, Lemma 4.11] *Let $\kappa \in \mathcal{P}(T)$ such that κ contains a maximal left ray or it is of cho type. Then κ contains a unique terminal vertex.*

Definition 4.13. Let $\kappa \in \mathcal{P}(T)$ be connected such that κ has a maximal left ray or is of cho type. The unique terminal vertex of κ established by Lemma 4.12 will be called the root of κ.

A relation R on a non empty set E is called well founded if every nonempty subset $D \subseteq E$ contains an R-minimal element that is, $p \in D$ exists such that there is no $q \in D$ with $(q, p) \in R$. Let R be a well-founded relation on a set E. Then there is a unique function π defined on E having values as ordinals so that

$$\pi(p) = \sup\{\pi(q) + 1 : (q, p) \in R\}.$$

for every $p \in E$ is called the rank of p in $< E, R >$ [11, Theorem 2.27] which proves (1) and (2). The condition (3) follows from Theorem 3.5.

Let $\kappa \in \mathcal{P}(T)$ be connected of rro type or cho then $\pi_\kappa(p)$ denotes the rank of p under the relation κ.

EXAMPLE 4.14. Let $T = \{x, y, c, \cdots, x_1, y_1, z_1 \cdots\}$ and let $\kappa = [x, y, z, \cdots >\in \mathcal{P}(T)$. Then $\pi(x) = 0, \pi(y) = 1$ and so on.

Let $< u_q >_{q\geq 0}$ and $< v_q >_{q\geq 0}$ be sequences of ordinals. Then we say that $< v_q >$ *dominates* $< u_q >$ if

$$v_{q+r} \geq u_q \text{ for every } q \geq 0 \text{ and for some } r \geq 0.$$

Let $\kappa \in \mathcal{P}(T)$ be connected of rro type and $\mu = [p_0 p_1 p_2 \cdots >$ be a maximal right ray in κ. We denote by $< \mu_q^\kappa >_{q\geq 0}$ the sequence of ordinals with

$$\mu_q^\kappa = \pi_\kappa(p_q) \text{ for every } q \geq 0.$$

For example, let $T = \{p_0, p_1, p_2, \cdots, q_0, q_1, q_2, \cdots\}$ and let $\kappa = [p_0 p_1 p_2 p_3 \cdots > \cup [q_0 p_2] \cup [q_1 q_2 p_2] \cup [q_3 q_4 q_5 p_2] \cup [q_6 q_7 q_8 q_9 p_2] \cup \cdots \in \mathcal{P}(T)$ and $\mu = [p_0 p_1 p_2 \cdots >\in \kappa$, then the sequence $< \mu_q^\kappa >$ is $< 0, 1, \omega, \omega+1, \omega+2, \omega+3, \cdots >$.

The next results (Proposition 4.15 to Proposition 4.20) are from Araujo et.al [6] and are required to prove the Theorem 4.23.

PROPOSITION 4.15. *Let $\kappa, \lambda \in \mathcal{P}(T)$ be connected such that κ has a cycle $(p_0 p_1 \cdots p_{k-1})$. Then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$ if and only if λ has a cycle $(q_0 q_1 ... q_{m-1})$ such that $m|k$.*

LEMMA 4.16. *Let $\kappa, \lambda \in \mathcal{P}(T)$ be connected such that λ has a cycle $(q_0\ q_1\ ...\ q_{m-1})$. Suppose κ has a double ray or is of rro type. Then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$.*

LEMMA 4.17. *Let κ, $\lambda \in \mathcal{P}(T)$ be connected. Suppose that λ has a double ray and κ either has a double ray or has rro type. Then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$.*

LEMMA 4.18. *Let $\kappa, \lambda \in \mathcal{P}(T)$ be connected. Suppose that λ has a maximal left ray and κ either has a maximal left ray or is of cho type. Then $\Gamma(\kappa)$ is rp-hom $\Gamma(\lambda)$.*

PROPOSITION 4.19. *Let $\kappa, \lambda \in \mathcal{P}(T)$ be connected of cho type with roots p_0 and q_0 respectively. Then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$ if and only if $\pi(x_0) \leq \pi(y_0)$.*

PROPOSITION 4.20. *Let $\kappa, \lambda \in \mathcal{P}(T)$ be connected of rro type. Then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$ if and only if there are maximal right ray μ in κ and η in λ such that $< \eta_n^\lambda >$ dominates $< \mu_n^\kappa >$.*

LEMMA 4.21. *Let κ, $\lambda \in \mathcal{P}(T)$ be connected with κ being of rro type and suppose $\kappa \sim_r \lambda$, then λ cannot have a maximal left ray or is of cho type.*

Proof. Let $\kappa \sim_r \lambda$ then by Theorem 3.5 there exists α which is an rp-hom from $\Gamma(\kappa)$ to $\Gamma(\lambda)$. Let $[a_0 a_1 a_2 \cdots >$ be a right ray in κ. Suppose to the contrary that λ has a maximal left ray or is of cho type. Let b_0 be the root of λ. By definition of connectedness there exists $k \geq 0$ such that $(a_0\alpha)\lambda^k = b_0$. As $\kappa \sim_r \lambda$, $\kappa\alpha = \alpha\lambda$ and so $(a_0\alpha)\lambda^{k+1} = (a_0\kappa^{k+1})\alpha = a_{k+1}\alpha$. But $(a_0\alpha)\lambda^{k+1} = (a_0\alpha)\lambda^k\lambda = b_0\lambda = \diamond$ and so $a_{k+1}\alpha = \diamond$ which is a contradiction. Hence the result follows. $\square$

LEMMA 4.22. *Let $\sigma, \tau \in \mathcal{P}(T)$ such that $\sigma \sim_r \tau$. If σ contains a cycle of length r, then τ has a cycle of length s such that $s|r$.*

Proof. Let σ contains a cycle of length r and let $\sigma \sim_r \tau$. By Proposition 4.6, $\sigma = \cup_{\kappa \in C} \kappa$ where C is the set of pairwise completely disjoint connected maps contained in σ and $\tau = \cup_{\lambda \in C'} \lambda$ where C' is the set of pairwise completely disjoint connected maps contained in τ. By Proposition 4.9, for a c-component κ of σ containing a cycle of length r there exists a c-component λ of τ such that $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$. Then by Proposition 4.15, λ has a cycle of length s such that $s|r$. So if σ contains a cycle of length r, then τ has a cycle of length s such that $s|r$. □

By Theorem 3.5 and the above discussed results we now prove a result on $\sim_r$ notion of conjugacy in sybsemigroups of $\mathcal{P}(T)$.

THEOREM 4.23. *Let σ, $\tau \in \mathcal{P}(T)$. Then $\sigma \sim_r \tau$ in S if and only if $\sigma = \tau = 0$ or $\sigma, \tau \neq 0$ and the following conditions hold*:

(1) *There is $\alpha, \beta, \phi, \psi \in S^1$ such that $q\alpha\phi = q$ for any non initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non initial vertex k of $\Gamma(\tau)$;*
(2) *cs(σ) = cs(τ);*
(3) *σ contains a double ray but no cycle if and only if τ contains a double ray but no cycle;*
(4) *If σ contains a c-component κ of rro type but no cycles or double rays then τ contains a c-component λ of rro type but no cycles or double rays and $<\eta_p^\lambda>$ dominates $<\mu_p^\kappa>$ for some maximal right rays μ in κ and η in λ;*
(5) *If τ contains a c-component λ of rro type but no cycles or double rays then σ contains a c-component κ of rro type but no cycles or double rays and $<\mu_p^\kappa>$ dominates $<\eta_p^\lambda>$ for some maximal right rays η in λ and μ in κ;*
(6) *σ contains a maximal left ray if and only if τ contains a maximal left ray;*
(7) *If σ contains a c-component κ of cho type with root p_0 but no maximal left rays then τ contains a c-component λ of cho type with root q_0 but no maximal left rays, and $\pi_\kappa(p_0) \leq \pi_\lambda(q_0)$;*
(8) *If τ contains a c-component λ of cho type with root q_0 but no maximal left ray then σ contains a c-component κ of cho type with root p_0 but no maximal left rays, and $\pi_\lambda(q_0) \leq \pi_\kappa(p_0)$.*

Proof. Let $\sigma \sim_r \tau$ in S. Suppose $\sigma = \tau = 0$ then condition (1) to (8) holds trivially. Suppose $\sigma, \tau \neq 0$ then by Theorem 3.5 there exists $\alpha, \beta, \phi, \psi \in S^1$ such that α is an rp-hom from $\Gamma(\sigma)$ to $\Gamma(\tau)$ and β is an rp-hom from $\Gamma(\tau)$ to $\Gamma(\sigma)$ with $q\alpha\phi = q$ for any non initial vertex q of $\Gamma(\sigma)$ and $k\beta\psi = k$ for every non initial vertex k of $\Gamma(\tau)$. By Lemma 4.8, we can assume that dom(α) = span(σ).

(2) Suppose σ has a cycle. Then, by Lemma 4.22, τ also has a cycle. Let $r \in$ cs(σ). Then σ has a cycle of length r, and so again by Lemma 4.22, τ has a cycle of length s such that $s|r$. By the definition of cs(τ), there is $s_1 \in$ cs(τ) such that $s_1|s$. Thus τ has a cycle of length s_1 and so by Lemma 4.22, σ has a cycle of length r_1 such that $r_1|s_1$. Hence $r_1|s_1|s|r$. Since cs(σ) is an anti-chain, $r_1|r$ and $r \in$ cs(σ) implies $r_1 = r$. Thus $r = r_1 = s_1$ and so $r \in$ cs(τ). We have proved that cs(σ) $\subseteq$ cs(τ). The converse follows by symmetry. Hence cs(τ) = cs(σ). If neither σ nor τ has a cycle, then cs(σ) = cs(τ) = ϕ.
(3) Suppose σ has a double ray but no cycles. Then by Lemma 4.22 τ cannot have a cycle. Also since $\Gamma(\sigma)$ is rp-hom to $\Gamma(\tau)$, so we have $< \cdots p_{-1}\, p_0\, p_1\, \cdots >$. The elements $\cdots p_{-1}\alpha,\ p_0\alpha,\ p_1\alpha, \cdots$ are pairwise distinct (since otherwise τ would

have a cycle), and so $<\cdots p_{-1}\alpha\ p_0\alpha\ p_1\alpha\cdots>$ is a double ray in τ. The converse is true by symmetry.

(4) Suppose σ has a c-component κ of rro type but no cycle or a double ray. By Proposition 4.9, there is a c-component κ of τ so that $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$. By (1) and (2), λ does not have a cycle nor a double ray. By Lemma 4.21, λ does not have a maximal left ray or is of cho type. Hence λ is rro type. By Proposition 4.20, there are maximal right rays μ in κ and η in λ such that $<\eta_p^{\lambda}>$ dominates $<\mu_p^{\kappa}>$.

(5) It follows by symmetry of (3).

(6) Suppose σ has a maximal left ray $<\cdots p_2\ p_1\ p_0]$. Then since $\Gamma(\sigma)$ is rp-hom to $\Gamma(\tau)$ we have $\cdots \xrightarrow{\tau} p_2\alpha \xrightarrow{\tau} p_1\alpha \xrightarrow{\tau} p_0\alpha$ and $p_0\alpha$ is a teriminal vertex in $\Gamma(\tau)$, which implies that $<\cdots p_2\alpha\ p_1\alpha\ p_0\alpha]$ is a maximal left ray in τ. The converse is true by symmetry.

(7) Suppose σ has a c-component κ of cho type with root p_0 but no maximal left ray. By Proposition 4.9 and its proof, there is a c-component λ of τ such that $\alpha_\kappa = \alpha|(\mathrm{span}(\kappa))$ is an rp-hom from $\Gamma(\kappa)$ to $\Gamma(\lambda)$. Since p_0 is a teriminal vertex in κ, $q_0 = p_0\alpha_\kappa$ is a terminal vertex in λ. Since τ has no maximal left ray (by(3)), λ is of cho type and q_0 is the root of λ. Therefore by Proposition 4.20, $\pi_\kappa(p_0) \le \pi_\lambda(q_0)$.

(8) Proof follows by symmetry of (6).

Conversely, if $\sigma = \tau = 0$, then $\sigma \sim_r \tau$. Suppose $\sigma, \tau \neq 0$ and all conditions from (1) to (8) hold. Let κ be a c-component of σ. We will prove that $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$ for some c-component λ of τ. The result then follows by Proposition 4.9.
Suppose κ has a cycle of length r, since by (1), $cs(\sigma) = cs(\tau)$, τ has a cycle υ of length s such that $s|r$. Let κ be the c-component of τ containing υ. Then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$ by Proposition 4.15.
Suppose κ has a double ray. If some c-component λ of τ has a cycle, then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$ by Lemma 4.16. Suppose τ does not have a cycle. Then, by (1) and (2), both σ and τ have a double ray but not a cycle. Let λ be a c-component of τ containing a double ray. Then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$ by Lemma 4.17.
Suppose κ is of rro type. If τ has some c-component λ with a cycle or a double ray, then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$ by Lemma 4.16 and Lemma 4.17. Suppose τ does not have a cycle or a double ray. Then by (3), there is a c-component λ in τ of rro type such that $<\eta_p^{\lambda}>$ dominates $<\mu_p^{\kappa}>$ for some maximal right rays μ in κ and η in λ. Hence $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$ by Proposition 4.20.
Suppose κ has a maximal left ray. Then by (5) there is some c-component λ of τ has a maximal left ray. Then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$ by Lemma 4.18.
Suppose κ is of cho type with root p_0. If τ has some c-component λ having a maximal left ray then $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$ by Lemma 4.18. Suppose τ does not have a maximal left ray. Then by (5), σ does not have a maximal left ray, and so by (6), there is a c-component λ in τ of cho type with root q_0 such that $\pi_\kappa(p_0) \le \pi_\kappa(q_0)$. Hence $\Gamma(\kappa)$ is rp-hom to $\Gamma(\lambda)$, by Proposition 4.20.
We have proved that for every c-component κ of σ there exists a c-component λ of τ and an rp-hom $\alpha_\kappa \in \mathcal{P}(T)$ from $\Gamma(\kappa)$ to $\Gamma(\lambda)$. We may assume that for every c-component κ of σ, $\mathrm{dom}(\alpha_\kappa) = \mathrm{span}(\kappa)$. Hence $\Gamma(\kappa)$ is rp-hom to $\Gamma(\tau)$ by Proposition 4.9. By symmetry, $\Gamma(\tau)$ is rp-hom to $\Gamma(\sigma)$. Then by (8) and Theorem 3.5 we get $\sigma \sim_r \tau$. $\square$

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