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SUPERSTABILITY OF THE p-RADICAL TRIGONOMETRIC FUNCTIONAL EQUATION

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Abstract. In this paper, we solve and investigate the superstability of the p-radical functional equations

$$f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)=\lambda f(x)g(y),$$
$$f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)=\lambda g(x)f(y),$$

which is related to the trigonometric(Kim's type) functional equations, where p is an odd positive integer and f is a complex valued function. Furthermore, the results are extended to Banach algebras.

1. Introduction

In 1940, the stability problem of the functional equation was conjectured by Ulam [22]. In 1941, Hyers [13] obtained a partial answer for the case of additive mapping in this problem.

Thereafter, the stability of the functional equation was improved by Bourgin [8] in 1949, Aoki [3] in 1950, Th. M. Rassias [21] in 1978 and Găvruta [12] in 1994.

In 1979, Baker*et al.* [7] announced the *superstability* as the new concept as follows: If f satisfies $|f(x+y)-f(x)f(y)|\le\epsilon$ for some fixed $\epsilon>0$, then either f is bounded or f satisfies the exponential functional equation $f(x+y)=f(x)f(y)$.

D'Alembert [1] in 1769 (see Kannappen's book [15]) introduced the cosine functional equation

$$f(x+y)+f(x-y)=2f(x)f(y), \tag{C}$$

and which superstability was proved by Baker [6] in 1980.

Baker's result was generalized by Badora [4] in 1998 to a noncommutative group under the Kannappen condition [14]: $f(x+y+z)=f(x+z+y)$, and it again was improved by Badora and Ger [5] in 2002 under the condition $|f(x+y)+f(x-y)-2f(x)f(y)|\le\varphi(x)$ or $\varphi(y)$.

The cosine (d'Alembert) functional equation (C) was generalized to the following:

$$f(x+y)+f(x-y)=2f(x)g(y), \tag{W}$$
$$f(x+y)+f(x-y)=2g(x)f(y), \tag{K}$$

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in which (W) is called the Wilson equation, and (K) arised by Kim was appeared in Kannappen and Kim's paper ([16]).

The superstability of the cosine (C), Wilson (W) and Kim (K) function equations were founded in Badora, Ger, Kannappan and Kim ([4, 5, 16, 17]).

In 2009, Eshaghi Gordji and Parviz [11] introduced the radical functional equation related to the quadratic functional equation

$$f\left(\sqrt{x^2+y^2}\right) = f(x)+f(y). \tag{R}$$

In [19], Kim introduced the trigonometric functional equation as the Pexider-type's as following:

$$f(x+y)-f(x-y) = 2f(x)f(y), \tag{$_-ff$}$$
$$f(x+y)-f(x-y) = 2g(x)f(y). \tag{$_-gf$}$$
$$f(x+y)-f(x-y) = 2f(x)g(y), \tag{$_-fg$}$$
$$f(x+y)-f(x-y) = \lambda f(x)f(y), \tag{$_-ff^\lambda$}$$
$$f(x+y)-f(x-y) = \lambda f(x)g(y), \tag{$_-fg^\lambda$}$$
$$f(x+y)-f(x-y) = \lambda g(x)f(y). \tag{$_-gf^\lambda$}$$

Recently, Almahalebi*et al.* [2] obtained the superstability in Hyer's sense for the p-radical functional equations related to Wilson equation and Kim's equation.

The aim of this paper is to solve and investigate the superstability in Gavurta's sense for the p-radical functional equations related to Kim's equation. as following:

$$f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) = \lambda f(x)f(y), \tag{$_-ff_r^\lambda$}$$
$$f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) = \lambda f(x)g(y), \tag{$_-fg_r^\lambda$}$$
$$f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) = \lambda g(x)f(y), \tag{$_-gf_r^\lambda$}$$
$$f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) = \lambda g(x)g(y). \tag{$_-gg_r^\lambda$}$$

In this paper, let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{C}$ be the field of complex numbers. We may assume that f is a nonzero function, ε is a nonnegative real number, $\varphi : \mathbb{R} \to \mathbb{R}_+$ is a given nonnegative function and p is an odd positive integer.

Let us denoted the equations

$$f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) = 2f(x)f(y), \tag{$_-ff_r$}$$
$$f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) = 2g(x)f(y). \tag{$_-gf_r$}$$
$$f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = 2f(x)f(y), \tag{ff_r}$$
$$f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = 2f(x)g(y), \tag{fg_r}$$
$$f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = \lambda f(x)f(y), \tag{ff_r^λ}$$
$$f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = \lambda f(x)g(y). \tag{fg_r^λ}$$

2. Superstability of the p-radical equations $(_{g}f_r^{\lambda})$ and $(_{f}g_r^{\lambda})$.

In this section, we find a solution and investigate the superstability of p-radical functional equations $(_{g}f_r^{\lambda})$ and $(_{f}g_r^{\lambda})$ related to the functional equations $(_{g}f^{\lambda})$ and $(_{f}g^{\lambda})$ arised by Kim.

In the following lemmas, we find solutions of the functional equations $(_{f}f_r^{\lambda})$, $(_{f}g_r^{\lambda})$ and $(_{g}f_r^{\lambda})$, which confirm are easy.

Lemma 1. *A function $f : \mathbb{R} \to \mathbb{C}$ satisfies $(_{f}f_r^{\lambda})$ if and only if $f(x) = F(x^p)$ for all $x \in \mathbb{R}$, where F is a solution of $(_{f}f^{\lambda})$. In particular, for the case $\lambda = 2$, a function $f : \mathbb{R} \to \mathbb{C}$ satisfies (ff_r) if and only if $f(x) = \cos(x^p)$ for all $x \in \mathbb{R}$, namely, F is a solution of* (C).

Lemma 2. *A function $f, g : \mathbb{R} \to \mathbb{C}$ satisfies $(_{f}g_r^{\lambda})$ if and only if $f(x) = F(x^p)$ and $g(x) = G(x^p)$, where F and G are solutions of $(_{f}g^{\lambda})$. In particular, for the case $\lambda = 2$, a function $f, g : \mathbb{R} \to \mathbb{C}$ satisfies (fg_r) if and only if $f(x) = F(x^p) = \sin(x^p)$ and $g(x) = G(x^p) = \cos(x^p)$, where F and G are solutions of equation (W).*

Lemma 3. *A function $f, g : \mathbb{R} \to \mathbb{C}$ satisfies the functional equation $(_{g}f_r^{\lambda})$ if and only if $f(x) = F(x^p)$ and $g(x) = G(x^p)$, where F and G are solutions of $(_{g}f^{\lambda})$. In particular, for the case $\lambda = 2$, a function $f, g : \mathbb{R} \to \mathbb{C}$ satisfies $(_{g}f_r)$ if and only if $f(x) = F(x^p)$ and $g(x) = G(x^p)$, where F and G are solutions of $(_{g}f)$.*

Now we investigate the superstability of the p-radical trigonometric functional equations $(_{g}f_r^{\lambda})$ and $(_{f}g_r^{\lambda})$.

Theorem 1. *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p + y^p}\right) - f\left(\sqrt[p]{x^p - y^p}\right) - \lambda g(x)f(y)| \leq \begin{cases} (i) \ \ \varphi(x) \\ (ii) \ \ \varphi(y) \ \text{ and } \ \varphi(x). \end{cases} \tag{2.1}$$

Then

(i) either f is bounded or g satisfies (ff_r^{λ}),

(ii) either g is bounded or g satisfies (ff_r^{λ}), and f and g satisfy $(_{g}f_r^{\lambda})$ and (fg_r^{λ}).

Proof. (i) Assume that f is unbounded. Then we can choose $\{y_n\}$ such that $0 \neq |f(y_n)| \to \infty$ as $n \to \infty$.

Putting $y = y_n$ in (2.1) and dividing both sides by $\lambda f(y_n)$, we have

$$\left| \frac{f\left(\sqrt[p]{x^p + y_n^p}\right) - f\left(\sqrt[p]{x^p - y_n^p}\right)}{\lambda f(y_n)} - g(x) \right| \leq \frac{\varphi(x)}{\lambda f(y_n)}. \tag{2.2}$$

As $n \to \infty$ in (2.2), we get

$$g(x) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x^p + y_n^p}\right) - f\left(\sqrt[p]{x^p - y_n^p}\right)}{\lambda f(y_n)} \tag{2.3}$$

for all $x \in \mathbb{R}$.

Replacing y by $\sqrt[p]{y^p + y_n^p}$ and $\sqrt[p]{y^p - y_n^p}$ in (2.1), we obtain

$$|f\left(\sqrt[p]{x^p + (y^p + y_n^p)}\right) - f\left(\sqrt[p]{x^p - (y^p + y_n^p)}\right) - \lambda g(x)f(\sqrt[p]{y^p + y_n^p})| \leq \varphi(x), \tag{2.4}$$

$$|f\left(\sqrt[p]{x^p+(y^p-y_n^p)}\right)-f\left(\sqrt[p]{x^p-(y^p-y_n^p)}\right)-\lambda g(x)f(\sqrt[p]{y^p-y_n^p})|\le\varphi(x), \quad (2.5)$$

for all $x, y, y_n \in \mathbb{R}$.

By (2.4) - (2.5), we obtain

$$\begin{aligned}|f&\left(\sqrt[p]{x^p+(y^p+y_n^p)}\right)-f\left(\sqrt[p]{x^p+(y^p-y_n^p)}\right)+f\left(\sqrt[p]{x^p-(y^p-y_n^p)}\right)\\&-f\left(\sqrt[p]{x^p-(y^p+y_n^p)}\right)-\lambda g(x)[f(\sqrt[p]{y^p+y_n^p})-f(\sqrt[p]{y^p-y_n^p})]|\le 2\varphi(x)\end{aligned}$$

for all $x, y, y_n \in \mathbb{R}$.

This implies that

$$\begin{aligned}&|\frac{f\left(\sqrt[p]{(x^p+y^p)+y_n^p}\right)-f\left(\sqrt[p]{(x^p+y^p)-y_n^p}\right)}{\lambda f(y_n)} \quad (2.6)\\&+\frac{f\left(\sqrt[p]{(x^p-y^p)+y_n^p}\right)-f\left(\sqrt[p]{(x^p-y^p)-y_n^p}\right)}{\lambda f(y_n)}\\&-\lambda g(x)\frac{f(\sqrt[p]{y^p+y_n^p})-f(\sqrt[p]{y^p-y_n^p})}{\lambda f(y_n)}|\le\frac{2\varphi(x)}{\lambda f(y_n)}\end{aligned}$$

for all $x, y, y_n \in \mathbb{R}$.

Letting $n \to \infty$ in (2.6), by applying (2.3), g satisfies the desired result (ff_r^λ).

(ii) First, we show that if f is bounded, then g is also bounded.

If f is bounded, then we choose $y_0 \in \mathbb{R}$ such that $f(y_0) \neq 0$, and then by (2.1) we can obtain

$$\begin{aligned}|g(x)|-&\left|\frac{f\left(\sqrt[p]{x^p+y_0^p}\right)-f\left(\sqrt[p]{x^p-y_0^p}\right)}{\lambda f(y_0)}\right|\\&\le\left|\frac{f\left(\sqrt[p]{x^p+y_0^p}\right)-f\left(\sqrt[p]{x^p-y_0^p}\right)}{\lambda f(y_0)}-g(x)\right|\le\frac{\varphi(y_0)}{\lambda|f(y_0)|} \quad (2.7)\end{aligned}$$

and it follows that g is also bounded on $\mathbb{R}$.

That is, if g is unbounded, then so is f. Hence, by (i), g also satisfies (ff_r^λ).

Let g be unbounded. Then f is also unbounded. So we can choose sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{R}$ such that $g(x_n) \neq 0$ and $|g(x_n)| \to \infty$, $f(y_n) \neq 0$ and $|f(y_n)| \to \infty$ as $n \to \infty$.

For the case $\varphi(y)$ in (ii) of (2.1), taking $x = x_n$, we deduce

$$\lim_{n\to\infty}\frac{f\left(\sqrt[p]{x_n^p+y^p}\right)-f\left(\sqrt[p]{x_n^p-y^p}\right)}{\lambda g(x_n)}=f(y) \quad (2.8)$$

for all $y \in \mathbb{R}$.

Replacing x by $\sqrt[p]{x_n^p+x^p}$ and $\sqrt[p]{x_n^p-x^p}$ in (2.1), we have

$$\begin{aligned}|f&\left(\sqrt[p]{(x_n^p+x^p)+y^p}\right)-f\left(\sqrt[p]{(x_n^p+x^p)-y^p}\right)-\lambda g(\sqrt[p]{x_n^p+x^p})f(y) \quad (2.9)\\&+f\left(\sqrt[p]{(x_n^p-x^p)+y^p}\right)-f\left(\sqrt[p]{(x_n^p-x^p)-y^p}\right)-\lambda g(\sqrt[p]{x_n^p-x^p})f(y)|\le 2\varphi(y)\end{aligned}$$

for all $x, y \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Consequently,

$$\begin{aligned}|&\frac{f\left(\sqrt[p]{x_n^p+(x^p+y^p)}\right)-f\left(\sqrt[p]{x_n^p-(x^p+y^p)}\right)}{\lambda g(x_n)}\\ &-\frac{f\left(\sqrt[p]{x_n^p+(x^p-y^p)}\right)-f\left(\sqrt[p]{x_n^p-(x^p-y^p)}\right)}{\lambda g(x_n)}\\ &-\frac{\lambda g(\sqrt[p]{x_n^p+x^p})+g(\sqrt[p]{x_n^p-x^p})}{\lambda g(x_n)}f(y)|\le\frac{2\varphi(y)}{\lambda g(x_n)},\end{aligned}\tag{2.10}$$

for all $x, y \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Apply the limit (2.8) in (2.10) with the use of $|g(x_n)| \to \infty$ as $n \longrightarrow \infty$. Since g satisfies (ff_r^λ) by (i), f and g are solutions of $({}_{-}gf_r^\lambda)$,

Finally, replace (x, y) by $(\sqrt[p]{x_n^p+y^p}, x)$ and replace (x, y) by $(\sqrt[p]{x_n^p-y^p}, x)$ for $\varphi(y)$ in (ii) of (2.1), respectively. Let us follows the same procedure as from (2.9) to (2.10). Then

$$\begin{aligned}&|f\left(\sqrt[p]{(x_n^p+y^p)+x^p}\right)-f\left(\sqrt[p]{(x_n^p+y^p)-x^p}\right)-\lambda g(\sqrt[p]{x_n^p+y^p})f(x)\\ &+f\left(\sqrt[p]{(x_n^p-y^p)+x^p}\right)-f\left(\sqrt[p]{(x_n^p-y^p)-x^p}\right)-\lambda g(\sqrt[p]{x_n^p-y^p})f(x)|\le 2\varphi(x).\end{aligned}$$

Hence we have

$$\begin{aligned}|&\frac{f\left(\sqrt[p]{x_n^p+(x^p+y^p)}\right)-f\left(\sqrt[p]{x_n^p-(x^p+y^p)}\right)}{\lambda g(x_n)}\\ &+\frac{f\left(\sqrt[p]{x_n^p+(x^p-y^p)}\right)-f\left(\sqrt[p]{x_n^p-(x^p-y^p)}\right)}{\lambda g(x_n)}\\ &-\frac{\lambda g(\sqrt[p]{x_n^p+y^p})+g(\sqrt[p]{x_n^p-y^p})}{\lambda g(x_n)}f(x)|\le\frac{2\varphi(x)}{\lambda g(x_n)},\end{aligned}\tag{2.11}$$

for all $x, y \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Then, by applying (2.8) and (i)'s result, it follows from (2.11) that f and g are solutions of (fg_r^λ). □

By a similar process of the proof of Theorem 2.1, we can prove the following theorem.

THEOREM 2. *Assume that* $f, g : \mathbb{R} \to \mathbb{C}$ *satisfy the inequality*

$$|f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)-\lambda f(x)g(y)|\le\begin{cases}(i)\ \ \varphi(y)\\ (ii)\ \ \varphi(x)\ \text{ and }\ \varphi(y).\end{cases}\tag{2.12}$$

Then

(i) either f(:odd) is bounded or g satisfies $({}_{-}ff_r^\lambda)$,

(ii) either g(with f:odd) is bounded or g satisfies $({}_{-}ff_r^\lambda)$, and f and g satisfy $({}_{-}fg_r^\lambda)$.

Proof. (i) Let f is unbounded. then let us choose $\{x_n\}$ in $\mathbb{R}$ such that $0 \neq |f(x_n))| \to \infty$ as $n \to \infty$.

Taking $x = x_n$ (with $n \in \mathbb{N}$) in (2.12), dividing both sides by $|\lambda \cdot f(x_n)|$, and passing to the limit as $n \to \infty$, we obtain that

$$g(y) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x_n^p + y^p}\right) - f\left(\sqrt[p]{x_n^p - y^p}\right)}{\lambda f(x_n)} \tag{2.13}$$

for all $y \in \mathbb{R}$.

Replace x by $\sqrt[p]{x_n^p + x^p}$ and $\sqrt[p]{-x_n^p + x^p}$ in (2.12). Thereafter we go through the same procedure as in (2.4) $\sim$ (2.6) of Theorem 1. Then, by oddness of f, we obtain

$$\begin{aligned}
&\Big|\frac{f\left(\sqrt[p]{(x_n^p + x^p) + y^p}\right) + f\left(\sqrt[p]{(-x_n^p + x^p) + y^p}\right)}{\lambda f(x_n)} \\
&\quad - \frac{f\left(\sqrt[p]{(x_n^p + x^p) - y^p}\right) + f\left(\sqrt[p]{(-x_n^p + x^p) - y^p}\right)}{\lambda f(x_n)} \\
&\quad - \lambda\frac{f(\sqrt[p]{x_n^p + x^p}) + f(\sqrt[p]{-x_n^p + x^p})}{\lambda f(x_n)} g(y)\Big| \\
&= \Big|\frac{f\left(\sqrt[p]{x_n^p + (x^p + y^p)}\right) - f\left(\sqrt[p]{x_n^p - (x^p + y^p)}\right)}{\lambda f(x_n)} \\
&\quad - \frac{f\left(\sqrt[p]{x_n^p + (x^p - y^p)}\right) - f\left(\sqrt[p]{x_n^p - (x^p - y^p)}\right)}{\lambda f(x_n)} \\
&\quad - \lambda\frac{f(\sqrt[p]{x_n^p + x^p}) - f(\sqrt[p]{x_n^p - x^p})}{\lambda f(x_n)} g(y)\Big| \le \frac{2\varphi(y)}{\lambda f(x_n)}.
\end{aligned} \tag{2.14}$$

Since the right-hand side of the inequality converges to zero as $n \to \infty$ in (2.14), by (2.13), g satisfies $({}_{-}ff_r^{\lambda})$.

(ii) if f is bounded, then we choose $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$, and then in (2.12) we can obtain

$$\begin{aligned}
&|g(y)| - \left|\frac{f\left(\sqrt[p]{x_0^p + y^p}\right) - f\left(\sqrt[p]{x_0^p - y^p}\right)}{\lambda f(x_0)}\right| \\
&\le \left|\frac{f\left(\sqrt[p]{x_0^p + y^p}\right) - f\left(\sqrt[p]{x_0^p - y^p}\right)}{\lambda f(x_0)} - g(y)\right| \le \frac{\varphi(x_0)}{\lambda |f(x_0)|}
\end{aligned} \tag{2.15}$$

and it follows that g is also bounded on $\mathbb{R}$.

That is, assume g is unbounded, then so is f. Hence, by (i), g satisfies $({}_{-}ff^{\lambda})$.

Let us choose $\{y_n\}$ in $\mathbb{R}$ such that $0 \neq |g(y_n)| \to \infty$ as $n \to \infty$.

As before, for the chosen sequence $\{y_n\}$, we obtain that

$$f(x) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x^p + y_n^p}\right) - f\left(\sqrt[p]{x^p - y_n^p}\right)}{\lambda g(y_n)} \tag{2.16}$$

for all $x \in \mathbb{R}$

Let go through the same procedure as in (2.4) $\sim$ (2.6) of Theorem 1 as above.

First, Replace x by $\sqrt[p]{x^p + y_n^p}$ and $\sqrt[p]{x^p - y_n^p}$ in (2.12), respectively, from replaced $\sqrt[p]{x^p + y_n^p}$ difference to replaced $\sqrt[p]{x^p - y_n^p}$, next divided by $\lambda g(y_n)$.

Then we obtain

$$
\begin{aligned}
&|f\left(\sqrt[p]{x^p+y_n^p+y^p}\right)-f\left(\sqrt[p]{x^p+y_n^p-y^p}\right)-\lambda f(\sqrt[p]{x^p+y_n^p})g(y)\\
&\quad -f\left(\sqrt[p]{x^p-y_n^p+y^p}\right)+f\left(\sqrt[p]{x^p-y_n^p-y^p}\right)+\lambda f(\sqrt[p]{x^p-y_n^p})g(y)|\\
&=|\frac{f\left(\sqrt[p]{x^p+y^p+y_n^p}\right)-f\left(\sqrt[p]{x^p+y^p-y_n^p}\right)}{\lambda g(y_n)}\\
&\quad -\frac{f\left(\sqrt[p]{x^p-y^p+y_n^p}\right)-f\left(\sqrt[p]{x^p-y^p-y_n^p}\right)}{\lambda g(y_n)}\\
&\qquad -\lambda\frac{f(\sqrt[p]{x^p+y_n^p})-f(\sqrt[p]{x^p-y_n^p})}{\lambda g(y_n)}g(y)|\le\frac{2\varphi(x)}{\lambda g(y_n)}.
\end{aligned}
\tag{2.17}
$$

Since the right-hand side of the inequality converges to zero as $n\to\infty$ in (2.17), f and g satisfy the required $({}_-fg_r^\lambda)$ from (2.16) and (2.17). $\square$

The following corollaries follow immediate from Theorems 1 and 2.

Corollary 1. *Assume that $f,g:\mathbb{R}\to\mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)f(y)|\le\varepsilon.$$

Then

(i) either f is bounded or g satisfies (ff_r^λ),

(ii) either g is bounded or g satisfies (ff_r^λ), and f and g satisfy $({}_-gf_r^\lambda)$ and (fg_r^λ).

Corollary 2. *Assume that $f,g:\mathbb{R}\to\mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)-\lambda f(x)g(y)|\le\varepsilon.$$

Then

Then

(i) either f(:odd) is bounded or g satisfies $({}_-ff_r^\lambda)$,

(ii) either g(with f:odd) is bounded or g satisfies $({}_-ff_r^\lambda)$, and f and g satisfy $({}_-fg_r^\lambda)$.

Corollary 3. *Assume that $f,g:\mathbb{R}\to\mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)-\lambda f(x)f(y)|\le\begin{cases}(i)\ \varphi(x),\\(ii)\ \varphi(y),\\(iii)\ \varepsilon.\end{cases}$$

Then either f is bounded or f satisfies $({}_-ff^\lambda)$,

Remark 1. In results, letting $p=1$ or $\lambda=2$, one can obtain (C), (W), (K), $({}_-ff^\lambda)$, $({}_-fg^\lambda)$, $({}_-gf^\lambda)$. Hence they can be applied to stability results of cosine, Wilson, Kim, trigonometric functional equations, etc. See Badora [4], Badora and Ger [5], Baker [6], Fassi, et al. [10], Kannappan and Kim [16], [17,19], and Almahalebi, et al. [2]. Letting $p=2,3,4$ and $\lambda=1,2$, we can obtain the other functional equations. If the obtained results can be extend to them, then it will be applied similarly to stability results.

3. Extension to Banach algebras

In this section, we will extend our main results to Banach algebras.

THEOREM 3. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) - \lambda g(x)f(y)\| \le \begin{cases} (i) & \varphi(x) \\ (ii) & \varphi(y) \ \ and \ \ \varphi(x). \end{cases} \tag{3.1}$$

Let $z^* \in E^*$ be an arbitrary linear multiplicative functional.

(i) If $z^* \circ f$ is unbounded, then g satisfies (ff_r^λ).

(ii) If $z^* \circ g$ is unbounded, then g satisfies (ff_r^λ), and f and g satisfy $({}_{-}gf_r^\lambda)$ and (fg_r^λ).

Proof. Assume that (6) holds and let $z^* \in E^*$ be a linear multiplicative functional. Since $\|z^*\| = 1$, for all $x, y \in \mathbb{R}$, we have

$$\begin{aligned} \varphi(x) &\ge \|f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) - \lambda g(x)f(y)\| \\ &= \sup_{\|z^*\|=1} \left|z^*\big(f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) - \lambda g(x)f(y)\big)\right| \\ &\ge \left|z^*\big(f\left(\sqrt[p]{x^p+y^p}\right)\big) - z^*\big(f\left(\sqrt[p]{x^p-y^p}\right)\big) - \lambda \cdot z^*\big(g(x)\big) \cdot z^*\big(f(y)\big)\right|, \end{aligned}$$

which states that the superpositions $z^* \circ f$ and $z^* \circ g$ yield solutions of the inequality (2.1) in Theorem 1.

Hence we can apply to (i) of Theorem 1.

(i) Since, by assumption, the superposition $z^* \circ f$ is unbounded, an appeal to Theorem 1 shows that the superposition $z^* \circ g$ is a solution of (ff_r^λ), that is,

$$(z^* \circ g)\big(\sqrt[p]{x^p+y^p}\big) + (z^* \circ g)\big(\sqrt[p]{x^p-y^p}\big) = \lambda(z^* \circ g)(x)(z^* \circ g)(y).$$

Since z^* is a linear multiplicative functional, we get

$$z^*\big(g\big(\sqrt[p]{x^p+y^p}\big) + g\big(\sqrt[p]{x^p-y^p}\big) - \lambda g(x)g(y)\big) = 0.$$

Hence an unrestricted choice of z^* implies that

$$g\big(\sqrt[p]{x^p+y^p}\big) + g\big(\sqrt[p]{x^p-y^p}\big) - \lambda g(x)g(y) \in \bigcap\{\ker z^* : z^* \in E^*\}.$$

Since E is a semisimple Banach algebra, $\bigcap\{\ker z^* : z^* \in E^*\} = 0$, which means that g satisfies the claimed equation (ff_r^λ).

(ii) By assumption, the superposition $z^* \circ g$ is unbounded, an appeal to Theorem 1 shows that the results hold.

From a similar process as in (2.15) of Theorem 1, we can show that the unboundedness of the superposition $z^* \circ g$ implies the unboundedness of the superposition $z^* \circ f$.

First, it follows from the above result (i) that g satisfies the claimed equation $({}_{-}ff_r^\lambda)$.

Next, an appeal to Theorem 1 shows that $z^* \circ f$ and $z^* \circ g$ are solutions of the equations $({}_{-}gf_r^\lambda)$ and $({}_{-}fg_r^\lambda)$, that is,

$$(z^* \circ f)\big(\sqrt[p]{x^p+y^p}\big) - (z^* \circ f)\big(\sqrt[p]{x^p-y^p}\big) = \lambda(z^* \circ g)(x)(z^* \circ f)(y),$$
$$(z^* \circ f)\big(\sqrt[p]{x^p+y^p}\big) - (z^* \circ f)\big(\sqrt[p]{x^p-y^p}\big) = \lambda(z^* \circ f)(x)(z^* \circ g)(y).$$

This means by a linear multiplicativity of z^* that the differences

$$\mathcal{D}K^\lambda(x,y) := f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) - \lambda g(x)f(y),$$
$$\mathcal{D}W^\lambda(x,y) := f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) - \lambda f(x)g(y)$$

fall into the kernel of z^*. That is, $z^*\left(\mathcal{D}K^\lambda(z,w)\right) = 0$ and $z^*\left(\mathcal{D}W^\lambda(z,w)\right) = 0$.

Hence an unrestricted choice of z^* implies that

$$\mathcal{D}K^\lambda(x,y),\ \mathcal{D}W^\lambda(x,y) \in \bigcap\{\ker z^* : z^* \in E^*\}.$$

Since the algebra E is semisimple, $\bigcap\{\ker z^* : z^* \in E^*\} = 0$, which means that f and g satisfy the claimed equations $(_{-}gf^\lambda)$ and (fg_r^λ). □

Corollary 4. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality*

$$\left\|f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) - \lambda g(x)f(y)\right\| \le \varepsilon.$$

Let $z^* \in E^*$ be an arbitrary linear multiplicative functional.

(i) If $z^* \circ f$ is unbounded, then g satisfies (ff_r^λ).

(ii) If $z^* \circ g$ is unbounded, then g satisfies (ff_r^λ), and f and g satisfy $(_{-}gf_r^\lambda)$ and (fg_r^λ).

By a same procedure as Theorem 3 , we can prove the next theorem as an extension of Theorem 2.

Theorem 4. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality*

$$\left\|f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) - \lambda f(x)g(y)\right\| \le \begin{cases} (i)\ \ \varphi(y) \\ (ii)\ \ \varphi(x)\ \text{ and }\ \varphi(y). \end{cases} \tag{3.2}$$

Let $z^* \in E^*$ be an arbitrary linear multiplicative functional, f is odd.

(i) If $z^* \circ f$ is unbounded, then g satisfies $(_{-}ff^\lambda)$.

(ii) If $z^* \circ g$ is unbounded, then g satisfies $(_{-}ff_r^\lambda)$, and f and g satisfy $(_{-}fg_r^\lambda)$.

Corollary 5. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality*

$$\left\|f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) - \lambda f(x)g(y)\right\| \le \varepsilon.$$

Let $z^* \in E^*$ be an arbitrary linear multiplicative functional.

(i) If $z^* \circ f$ is unbounded, then g satisfies $(_{-}ff^\lambda)$.

(ii) If $z^* \circ g$ (or $z^* \circ f$) is unbounded, then g satisfies $(_{-}ff^\lambda)$, and f and g satisfy $(_{-}fg_r^\lambda)$.

Corollary 6. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality*

$$\left\|f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) - \lambda f(x)f(y)\right\| \le \begin{cases} (i)\ \ \varphi(x) \\ (ii)\ \ \varphi(y) \\ (iii)\ \ \varepsilon. \end{cases}$$

Then either the superposition $z^* \circ f$ is bounded for each linear multiplicative functional $z^* \in E^*$ or f satisfies $(_{-}ff_r^\lambda)$.

Remark 2. Letting $p = 1$ or $\lambda = 2$, then the considered equations impliy (C), (W), (K), $(_{-}ff)$, $(_{-}gf)$, $(_{-}fg)$. Hence they can be appled to stability results of cosine, Wilson, Kim, trigonometric functional equations combined with the minus (See [2, 4, 5, 16–20]).

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