

SEMI-CONFORMAL L -HARMONIC MAPS AND LIOUVILLE TYPE THEOREM

EMBARKA REMLI[†] AND AHMED MOHAMMED CHERIF^{*}

ABSTRACT. In this paper, we prove that every semi-conformal harmonic map between Riemannian manifolds is L -harmonic map. We also prove a Liouville type theorem for L -harmonic maps.

1. Introduction

Consider a smooth map $\varphi : (M^m, g) \rightarrow (N^n, h)$ between Riemannian manifolds and a smooth positive function $L : M^m \times N^n \times \mathbb{R}_+ \rightarrow (0, \infty)$, $(x, y, r) \mapsto L(x, y, r)$. Then for any compact domain D of M^m , the L -energy functional of φ is defined by

$$(1) \quad E_L(\varphi; D) = \int_D L(x, \varphi(x), e(\varphi)(x)) v^g,$$

where $e(\varphi)$ is the energy density of φ defined by

$$(2) \quad e(\varphi) = \frac{1}{2} \sum_{i=1}^m h(d\varphi(e_i), d\varphi(e_i))$$

and v^g is the volume element. Here $\{e_1, \dots, e_m\}$ is an orthonormal frame on (M^m, g) . A map is called L -harmonic if it is a critical point of the L -energy functional over any compact subset D of M^m . We denote by $\partial_r = \partial/\partial r$, $L' = \partial_r(L)$ and

$$(3) \quad L'_\varphi(x) = L'(x, \varphi(x), e(\varphi)(x)), \quad \forall x \in M^m.$$

THEOREM 1.1 (The first variation of E_L , [9]). *Let $\varphi : (M^m, g) \rightarrow (N^n, h)$ be a smooth map and let $\{\varphi_t\}_{t \in (-\epsilon, \epsilon)}$ be a smooth variation of φ supported in D . Then*

$$(4) \quad \left. \frac{d}{dt} E_L(\varphi_t; D) \right|_{t=0} = - \int_D h(\tau_L(\varphi), v) v^g,$$

where $v = \left. \frac{\partial \varphi_t}{\partial t} \right|_{t=0}$ denotes the variation vector field of φ ,

$$(5) \quad \tau_L(\varphi) = L'_\varphi \tau(\varphi) + d\varphi(\text{grad}^{M^m} L'_\varphi) - (\text{grad}^{N^n} L) \circ \varphi,$$

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^{*} Corresponding author.

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and $\tau(\varphi)$ is the tension field of φ given by

$$(6) \quad \tau(\varphi) = \text{trace}_g \nabla d\varphi.$$

From the first variation formula (4), a map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is L -harmonic if and only if $\tau_L(\varphi) = 0$.

Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds. Let $x \in M^m$, the tangent space at x splits $T_x M^m = H_x \oplus V_x$, where $V_x = \text{Ker } d_x \varphi$ and $H_x = V_x^\perp$ is the orthogonal complement of the vertical space V_x . The map φ is called semi-conformal if for each $x \in M^m$ with $d_x \varphi \neq 0$, the restriction $d_x \varphi : H_x \longrightarrow T_{\varphi(x)} N^n$ is conformal and surjective. Setting $\lambda(x) = 0$ at points x with $d_x \varphi = 0$, we obtain a continuous function $\lambda : M^m \longrightarrow \mathbb{R}_+$ such that for any $X, Y \in H_x$

$$h(d_x \varphi(X), d_x \varphi(Y)) = \lambda^2(x)g(X, Y).$$

The function λ is called the dilation of φ . Note that the generalized conformal maps are discussed in [10]

The purpose of this paper is to provide a proof of the Liouville type theorem for L -harmonic maps from complete noncompact Riemannian manifold (M^m, g) with positive Ricci curvature into a Riemannian manifold (N^n, h) with non-positive sectional curvature, where $L \in C^\infty(M^m \times N^n \times \mathbb{R}_+)$ is a smooth positive function which satisfies some suitable conditions.

2. Semi-conformal L -harmonic maps

Let (M^m, g) be a Riemannian manifold and let N^n the Euclidian space \mathbb{R}^n equipped with the Riemannian metric $h = dy_1^2 + \dots + dy_n^2$. We have the following results.

THEOREM 2.1. *Any semi-conformal harmonic map $\varphi : M^m \longrightarrow \mathbb{R}^n$ is L -harmonic with $L(x, y, r) = F(2y + (n-2)\varphi(x))r$, for all $(x, y, r) \in M^m \times \mathbb{R}^n \times \mathbb{R}_+$, where $F \in C^\infty(\mathbb{R}^n)$ is a smooth positive function.*

Proof. A semi-conformal harmonic map φ is L -harmonic if and only if

$$d\varphi(\text{grad}^{M^m} L'_\varphi) - (\text{grad}^{\mathbb{R}^n} L) \circ \varphi = 0,$$

where $L'_\varphi : M^m \longrightarrow (0, +\infty)$ is a smooth positive function given by

$$L'_\varphi(x) = F(n\varphi(x)).$$

Let us choose $\{e_1, \dots, e_m\}$ to be an orthonormal frame on a domain of M^m such that the vectors $\{e_1, \dots, e_n\}$ are horizontal and the vectors $\{e_{n+1}, \dots, e_m\}$ are vertical, so that $d\varphi(e_i) = \lambda(\tilde{e}_i \circ \varphi)$ for $i = 1, \dots, n$, where $\{\tilde{e}_1, \dots, \tilde{e}_n\}$ is an orthonormal frame on

a domain of \mathbb{R}^n . Then, we get

$$\begin{aligned}
 d\varphi(\text{grad}^{M^m} L'_\varphi) &= \sum_{i=1}^m e_i(L'_\varphi) d\varphi(e_i) \\
 &= n \sum_{i=1}^n d\varphi(e_i)(F) d\varphi(e_i) \\
 &= n \lambda^2 \sum_{i=1}^n (\tilde{e}_i \circ \varphi)(F) (\tilde{e}_i \circ \varphi) \\
 (7) \qquad \qquad \qquad &= n \lambda^2 (\text{grad}^{\mathbb{R}^n} F) \circ \varphi,
 \end{aligned}$$

and the term $(\text{grad}^{\mathbb{R}^n} L) \circ \varphi$ is given by

$$\begin{aligned}
 (\text{grad}^{\mathbb{R}^n} L) \circ \varphi &= \sum_{i=1}^n \left[\frac{\partial L}{\partial y_i} \frac{\partial}{\partial y_i} \right] \circ \varphi \\
 &= 2e(\varphi) \sum_{i=1}^n \left[\frac{\partial F}{\partial y_i} \frac{\partial}{\partial y_i} \right] \circ \varphi \\
 (8) \qquad \qquad \qquad &= 2e(\varphi) (\text{grad}^{\mathbb{R}^n} F) \circ \varphi,
 \end{aligned}$$

since $e(\varphi) = \frac{n}{2}\lambda^2$, we get $(\text{grad}^{\mathbb{R}^n} L) \circ \varphi = n \lambda^2 (\text{grad}^{\mathbb{R}^n} F) \circ \varphi$. \square

Using Theorem 2.1, we can construct many examples for semi conformal L -harmonic maps.

EXAMPLE 2.2. The Hopf construction map $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$, defined by

$$\varphi(x) = (x_1^2 + x_2^2 - x_3^2 - x_4^2, 2x_1x_3 + 2x_2x_4, 2x_2x_3 - 2x_1x_4),$$

is a semi conformal harmonic map with dilation

$$\lambda(x) = 2|x|, \forall x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4,$$

see [3]. According to Theorem 2.1 the map φ is L -harmonic, where L is the form

$$F(x_1^2 + x_2^2 - x_3^2 - x_4^2 + 2y_1, 2x_1x_3 + 2x_2x_4 + 2y_2, 2x_2x_3 - 2x_1x_4 + 2y_3) r,$$

for all $(x, y, r) \in \mathbb{R}^4 \times \mathbb{R}^3 \times \mathbb{R}_+$, where $F \in C^\infty(\mathbb{R}^3)$ is a smooth positive function.

If $n = 1$, we have the following corollary.

COROLLARY 2.3. *A smooth function $\varphi \in C^\infty(M^m)$ is harmonic if and only if it is L -harmonic for $L(x, y, r) = F(2y - \varphi(x))r$, for all $(x, y, r) \in M^m \times \mathbb{R} \times \mathbb{R}_+$ where $F \in C^\infty(\mathbb{R})$ is a smooth positive function.*

Proof. First note that, the function φ is L -harmonic if and only if

$$(9) \qquad \tau_L(\varphi) = L'_\varphi \tau(\varphi) + d\varphi(\text{grad}^{M^m} L'_\varphi) - (\text{grad}^{\mathbb{R}} L) \circ \varphi = 0,$$

where $L'_\varphi(x) = F(\varphi(x))$ for all $x \in M^m$. We compute

$$\begin{aligned}
 d\varphi(\text{grad}^M L'_\varphi) &= \sum_{i=1}^m e_i(L'_\varphi) d\varphi(e_i) \\
 &= \sum_{i=1}^m e_i(F \circ \varphi) e_i(\varphi) \\
 &= \sum_{i=1}^m e_i(\varphi) (F' \circ \varphi) e_i(\varphi) \\
 (10) \qquad &= (F' \circ \varphi) |\text{grad}^{M^m} \varphi|^2.
 \end{aligned}$$

Here $\{e_1, \dots, e_m\}$ is an orthonormal frame in M^m and $e_i(\varphi) = d\varphi(e_i)$. The term $-(\text{grad}^{\mathbb{R}} L) \circ \varphi$ of (9) is given by

$$\begin{aligned}
 -(\text{grad}^{\mathbb{R}} L) \circ \varphi &= -\frac{1}{2} \sum_{i=1}^m e_i(\varphi)^2 [2(F' \circ \varphi)] \\
 (11) \qquad &= -|\text{grad}^{M^m} \varphi|^2 (F' \circ \varphi).
 \end{aligned}$$

Then the proof follows from (9), (10) and (11). □

EXAMPLE 2.4. The harmonic function

$$\varphi : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}, \quad (x_1, x_2) \longmapsto \frac{x_1}{x_1^2 + x_2^2},$$

is L -harmonic with

$$L(x_1, x_2, y, r) = F \left(2y - \frac{x_1}{x_1^2 + x_2^2} \right) r, \quad \forall (x_1, x_2, y, r) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+,$$

where $F \in C^\infty(\mathbb{R})$ is a smooth positive function.

For $n = 2$, we have the following result.

THEOREM 2.5. *A semi-conformal map $\varphi : M^m \longrightarrow N^2$ from a Riemannian manifold to a Riemannian 2-manifold is L -harmonic with*

$$L(x, y, r) = r e^{\alpha(x)\beta(y)}, \quad \forall (x, y, r) \in M^m \times N^2 \times \mathbb{R}_+,$$

where $\alpha \in C^\infty(M^m)$ and $\beta \in C^\infty(N^2)$ if and only if

$$\tau(\varphi) + (\beta \circ \varphi) d\varphi(\text{grad}^M \alpha) = 0.$$

Proof. First, note that the function L'_φ is given by

$$L'_\varphi(x) = e^{\alpha(x)\beta(\varphi(x))}, \quad \forall x \in M^m.$$

Let us choose $\{e_1, \dots, e_m\}$ to be an orthonormal frame on a domain of M^m such that the vectors $\{e_1, e_2\}$ are horizontal and the vectors $\{e_3, \dots, e_m\}$ are vertical, so that $d\varphi(e_i) = \lambda(\tilde{e}_i \circ \varphi)$ for $i = 1, 2$, where $\{\tilde{e}_1, \tilde{e}_2\}$ is an orthonormal frame on a domain

of N^2 . Then we get

$$\begin{aligned} d\varphi(\text{grad}^{M^m} L'_\varphi) &= \sum_{i=1}^m e_i(L'_\varphi) d\varphi(e_i) \\ &= \sum_{i=1}^m e^{\alpha(\beta \circ \varphi)} e_i(\alpha(\beta \circ \varphi)) d\varphi(e_i) \\ &= e^{\alpha(\beta \circ \varphi)} \{(\beta \circ \varphi) d\varphi(\text{grad}^{M^m} \alpha) + \alpha d\varphi(\text{grad}^{M^m}(\beta \circ \varphi))\}. \end{aligned}$$

The term $d\varphi(\text{grad}^{M^m}(\beta \circ \varphi))$ is given by

$$\begin{aligned} d\varphi(\text{grad}^{M^m}(\beta \circ \varphi)) &= \sum_{i=1}^m e_i(\beta \circ \varphi) d\varphi(e_i) \\ &= \sum_{i=1}^2 d\varphi(e_i)(\beta) d\varphi(e_i) \\ &= \sum_{i=1}^2 \lambda^2 (\tilde{e}_i \circ \varphi)(\beta) (\tilde{e}_i \circ \varphi) \\ &= \lambda^2 (\text{grad}^{N^2} \beta) \circ \varphi. \end{aligned}$$

Hence we conclude that

$$d\varphi(\text{grad}^{M^m} L'_\varphi) = e^{\alpha(\beta \circ \varphi)} \{(\beta \circ \varphi) d\varphi(\text{grad}^{M^m} \alpha) + \alpha \lambda^2 (\text{grad}^{N^2} \beta) \circ \varphi\}.$$

Since $e(\varphi) = \lambda^2$, we get

$$\begin{aligned} (\text{grad}^{N^2} L) \circ \varphi &= \sum_{i=1}^2 (\tilde{e}_i \circ \varphi)(L) (\tilde{e}_i \circ \varphi) \\ &= \lambda^2 \sum_{i=1}^2 (\tilde{e}_i \circ \varphi)(\alpha \beta) e^{\alpha(\beta \circ \varphi)} (\tilde{e}_i \circ \varphi) \\ &= \alpha \lambda^2 e^{\alpha(\beta \circ \varphi)} (\text{grad}^{N^2} \beta) \circ \varphi. \end{aligned}$$

Hence the L -tension field of φ is given by

$$\tau_L(\varphi) = e^{\alpha(\beta \circ \varphi)} [\tau(\varphi) + (\beta \circ \varphi) d\varphi(\text{grad}^{M^m} \alpha)].$$

This completes the proof of Theorem 2.5. \square

EXAMPLE 2.6 (The foliation by the circles of Villarceau, [2]). Let M^3 be the manifold $\mathbb{R} \times \mathbb{R}^2 \setminus \{0\}$ and let $\varphi : M^3 \rightarrow \mathbb{R}^2$ be defined by

$$\varphi(x_1, x_2, x_3) = \left(\frac{\left(1 - \frac{|x|^2}{2}\right) x_2 + \sqrt{2} x_1 x_3}{x_2^2 + x_3^2}, \frac{\left(1 - \frac{|x|^2}{2}\right) x_3 - \sqrt{2} x_1 x_2}{x_2^2 + x_3^2} \right).$$

Then φ is semi-conformal and its dilation is given by the function

$$\lambda(x) = \frac{\left(1 - \frac{|x|^2}{2}\right)^2}{(x_2^2 + x_3^2)^2}, \quad \forall x = (x_1, x_2, x_3) \in M^3.$$

The tension field of φ is

$$\tau(\varphi)(x) = \left(-\frac{x_2}{x_2^2 + x_3^2}, -\frac{x_3}{x_2^2 + x_3^2} \right).$$

According to Theorem 2.5, with $\alpha(x) = c_1 \ln(2 + |x|^2) + c_2$ and $\beta(y) = -\frac{1}{c_1}$, where $c_1 \in \mathbb{R}^*$, $c_2 \in \mathbb{R}$, the map φ is L -harmonic with

$$L(x, y, r) = \frac{e^{-\frac{c_2}{c_1}r}}{2 + |x|^2}, \quad \forall (x, y, r) \in M^3 \times \mathbb{R}^2 \times \mathbb{R}_+.$$

3. A Liouville type theorem for L -harmonic maps

THEOREM 3.1. *Let (M^m, g) be a complete noncompact Riemannian manifold with positive Ricci curvature $\text{Ricci}^{M^m} \geq 0$, (N^n, h) a Riemannian manifold with non-positive sectional curvature $\text{Sect}^{N^n} \leq 0$. Consider an L -harmonic map φ from (M^m, g) to (N^n, h) , where $L \in C^\infty(M^m \times N^n \times \mathbb{R}_+)$ is a smooth positive function. Suppose that*

$$L'_\varphi > 0, \quad \text{Hess}^{M^m} L'_\varphi \leq 0, \quad \text{Hess}^{N^n} L \geq 0,$$

$$d\varphi(\text{grad}^{M^m} L'_\varphi)(L) \leq 0, \quad \int_{M^m} L'_\varphi v^g = \infty, \quad \int_{M^m} L'_\varphi |d\varphi|^2 v^g < \infty.$$

Then φ is constant.

We need the following Lemmas to prove Theorem 3.1.

LEMMA 3.2 ([6, 11]). *Let $\varphi : (M^m, g) \rightarrow (N^n, h)$ be a smooth mapping between Riemannian manifolds and let $f \in C^\infty(M^m)$. Then*

$$\langle d\varphi, \nabla^\varphi d\varphi(\text{grad}^{M^m} f) \rangle = \frac{1}{2}(\text{grad}^{M^m} f)(|d\varphi|^2) + \langle d\varphi, d\varphi(\nabla^{M^m} \text{grad}^{M^m} f) \rangle.$$

Here \langle, \rangle denotes the inner product on $T^*M^m \otimes \varphi^{-1}TN^n$.

LEMMA 3.3. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds, and $L \in C^\infty(M^m \times N^n \times \mathbb{R}_+)$ a smooth positive function. Then an L -harmonic map $\varphi : (M^m, g) \rightarrow (N^n, h)$ satisfies*

$$\begin{aligned} \frac{1}{2}\Delta^{M^m} |d\varphi|^2 &= |\nabla d\varphi|^2 + \frac{1}{L'_\varphi{}^2} |d\varphi(\text{grad}^{M^m} L'_\varphi)|^2 - \frac{1}{2L'_\varphi} (\text{grad}^{M^m} L'_\varphi)(|d\varphi|^2) \\ &\quad - \frac{1}{L'_\varphi} \langle d\varphi, d\varphi(\nabla^{M^m} \text{grad}^{M^m} L'_\varphi) \rangle - \frac{1}{L'_\varphi{}^2} d\varphi(\text{grad}^{M^m} L'_\varphi)(L) \\ &\quad + \frac{1}{L'_\varphi} \langle d\varphi, \nabla^\varphi(\text{grad}^{N^n} L) \circ \varphi \rangle + \sum_{i=1}^m h(d\varphi(\text{Ricci}^{M^m} e_i), d\varphi(e_i)) \\ &\quad - \sum_{i,j=1}^m h(R^{N^n}(d\varphi(e_i), d\varphi(e_j))d\varphi(e_j), d\varphi(e_i)), \end{aligned}$$

where $\{e_1, \dots, e_m\}$ be an orthonormal frame on (M^m, g) .

Proof. We start recalling the standard Bochner formula for the smooth map φ . Let $\{e_1, \dots, e_m\}$ be an orthonormal frame on (M^m, g) . Then

$$(12) \quad \begin{aligned} \frac{1}{2} \Delta^{M^m} |d\varphi|^2 &= |\nabla d\varphi|^2 + \langle d\varphi, \nabla^\varphi \tau(\varphi) \rangle + \sum_{i=1}^m h(d\varphi(\text{Ricci}^{M^m} e_i), d\varphi(e_i)) \\ &\quad - \sum_{i,j=1}^m h(R^{N^n}(d\varphi(e_i), d\varphi(e_j)) d\varphi(e_j), d\varphi(e_i)), \end{aligned}$$

where $|\nabla d\varphi|$ and $\langle d\varphi, \nabla^\varphi \tau(\varphi) \rangle$ are defined by

$$|\nabla d\varphi|^2 = \sum_{i,j=1}^m h(\nabla d\varphi(e_i, e_j), \nabla d\varphi(e_i, e_j))$$

and

$$\langle d\varphi, \nabla^\varphi \tau(\varphi) \rangle = \sum_{i=1}^m h(d\varphi(e_i), \nabla_{e_i}^\varphi \tau(\varphi)),$$

respectively. Since φ is L -harmonic, i.e.,

$$\tau_L(\varphi) = L'_\varphi \tau(\varphi) + d\varphi(\text{grad}^{M^m} L'_\varphi) - (\text{grad}^{N^n} L) \circ \varphi = 0$$

and $L'_\varphi > 0$ on M^m , we obtain

$$(13) \quad \tau(\varphi) = -\frac{1}{L'_\varphi} d\varphi(\text{grad}^{M^m} L'_\varphi) + \frac{1}{L'_\varphi} (\text{grad}^{N^n} L) \circ \varphi.$$

Hence we get the following

$$(14) \quad \begin{aligned} \langle d\varphi, \nabla^\varphi \tau(\varphi) \rangle &= \frac{1}{L'_\varphi} |d\varphi(\text{grad}^{M^m} L'_\varphi)|^2 - \frac{1}{L'_\varphi} \langle d\varphi, \nabla^\varphi d\varphi(\text{grad}^{M^m} L'_\varphi) \rangle \\ &\quad - \frac{1}{L'_\varphi} h(d\varphi(\text{grad}^{M^m} L'_\varphi), (\text{grad}^{N^n} L) \circ \varphi) \\ &\quad + \frac{1}{L'_\varphi} \langle d\varphi, \nabla^\varphi (\text{grad}^{N^n} L) \circ \varphi \rangle, \end{aligned}$$

by the Lemma 3.2. Since the second term on the left-hand side of (14) is

$$(15) \quad \begin{aligned} -\frac{1}{L'_\varphi} \langle d\varphi, \nabla^\varphi d\varphi(\text{grad}^{M^m} L'_\varphi) \rangle &= -\frac{1}{2L'_\varphi} (\text{grad}^{M^m} L'_\varphi)(|d\varphi|^2) \\ &\quad - \frac{1}{L'_\varphi} \langle d\varphi, d\varphi(\nabla^{M^m} \text{grad}^{M^m} L'_\varphi) \rangle, \end{aligned}$$

Lemma 3.3 follows by (12), (14) and (15). \square

Proof. of theorem 3.1. By Lemma 3.3, we get

$$\begin{aligned}
\frac{1}{2}L'_\varphi\Delta^{M^m}|d\varphi|^2 &= L'_\varphi|\nabla d\varphi|^2 + \frac{1}{L'_\varphi}|d\varphi(\text{grad}^{M^m} L'_\varphi)|^2 - \frac{1}{2}(\text{grad}^{M^m} L'_\varphi)(|d\varphi|^2) \\
&\quad - \langle d\varphi, d\varphi(\nabla^{M^m} \text{grad}^{M^m} L'_\varphi) \rangle - \frac{1}{L'_\varphi}d\varphi(\text{grad}^{M^m} L'_\varphi)(L) \\
&\quad + \langle d\varphi, \nabla^\varphi(\text{grad}^{N^n} L) \circ \varphi \rangle + L'_\varphi \sum_{i=1}^m h(d\varphi(\text{Ricci}^{M^m} e_i), d\varphi(e_i)) \\
&\quad - L'_\varphi \sum_{i,j=1}^m h(R^{N^n}(d\varphi(e_i), d\varphi(e_j))d\varphi(e_j), d\varphi(e_i)).
\end{aligned}$$

If we denote $\Delta_L^{M^m} \rho \equiv L'_\varphi \Delta^{M^m} \rho + (\text{grad}^{M^m} L'_\varphi)(\rho)$ for all $\rho \in C^\infty(M^m)$, then

$$\begin{aligned}
\frac{1}{2}\Delta_L^{M^m}|d\varphi|^2 &= L'_\varphi|\nabla d\varphi|^2 + \frac{1}{L'_\varphi}|d\varphi(\text{grad}^{M^m} L'_\varphi)|^2 - \langle d\varphi, d\varphi(\nabla^{M^m} \text{grad}^{M^m} L'_\varphi) \rangle \\
&\quad - \frac{1}{L'_\varphi}d\varphi(\text{grad}^{M^m} L'_\varphi)(L) + \langle d\varphi, \nabla^\varphi(\text{grad}^{N^n} L) \circ \varphi \rangle \\
&\quad + L'_\varphi \sum_{i=1}^m h(d\varphi(\text{Ricci}^{M^m} e_i), d\varphi(e_i)) \\
&\quad - L'_\varphi \sum_{i,j=1}^m h(R^{N^n}(d\varphi(e_i), d\varphi(e_j))d\varphi(e_j), d\varphi(e_i)).
\end{aligned}$$

Since $\text{Sect}^{N^n} \leq 0$, $\text{Ricci}^{M^m} \geq 0$, $\text{Hess}^{N^n} L \geq 0$, $\text{Hess}^{M^m} L'_\varphi \leq 0$ and $d\varphi(\text{grad}^{M^m} L'_\varphi)(L) \leq 0$ by (16), we obtain the following inequality

$$(16) \quad \frac{1}{2}\Delta_L^{M^m}|d\varphi|^2 \geq L'_\varphi|\nabla d\varphi|^2.$$

Since $\frac{1}{2}\Delta_L^{M^m}|d\varphi|^2 = |d\varphi|\Delta_L^{M^m}|d\varphi| + L'_\varphi|\text{grad}^{M^m}|d\varphi|^2$, by (16) and the Kato's inequality [4], we get the following

$$(17) \quad |d\varphi|\Delta_L^{M^m}|d\varphi| \geq L'_\varphi(|\nabla d\varphi|^2 - |\text{grad}^{M^m}|d\varphi|^2) \geq 0.$$

Let $\rho : M^m \rightarrow \mathbb{R}$ be a smooth function with compact support. Then

$$\begin{aligned}
\rho^2|d\varphi|\Delta_L^{M^m}|d\varphi| &= \rho^2|d\varphi|\text{div}^{M^m}(L'_\varphi \text{grad}^{M^m}|d\varphi|) \\
&= \text{div}^{M^m}(\rho^2|d\varphi|L'_\varphi \text{grad}^{M^m}|d\varphi|) - L'_\varphi\rho^2|\text{grad}^{M^m}|d\varphi|^2 \\
(18) \quad &\quad - 2L'_\varphi\rho|d\varphi|g(\text{grad}^{M^m}\rho, \text{grad}^{M^m}|d\varphi|).
\end{aligned}$$

From (17), (18) and the Stokes theorem, we deduce

$$\begin{aligned}
0 &\leq - \int_{M^m} L'_\varphi\rho^2|\text{grad}^{M^m}|d\varphi|^2 v^g \\
(19) \quad &\quad - 2 \int_{M^m} L'_\varphi\rho|d\varphi|g(\text{grad}^{M^m}\rho, \text{grad}^{M^m}|d\varphi|)v^g.
\end{aligned}$$

Using the Young inequality [14], we have

$$(20) \quad \begin{aligned} -2g(|d\varphi| \operatorname{grad}^{M^m} \rho, \rho \operatorname{grad}^{M^m} |d\varphi|) &\leq \frac{1}{\epsilon} |d\varphi|^2 |\operatorname{grad}^{M^m} \rho|^2 \\ &+ \epsilon \rho^2 |\operatorname{grad}^{M^m} |d\varphi||^2 \end{aligned}$$

for any $\epsilon > 0$. Substituting (20) in (19), we obtain

$$\begin{aligned} 0 &\leq - \int_{M^m} L'_\varphi \rho^2 |\operatorname{grad}^{M^m} |d\varphi||^2 v^g + \frac{1}{\epsilon} \int_{M^m} L'_\varphi |d\varphi|^2 |\operatorname{grad}^{M^m} \rho|^2 v^g \\ &+ \epsilon \int_{M^m} L'_\varphi \rho^2 |\operatorname{grad}^{M^m} |d\varphi||^2 v^g, \end{aligned}$$

that is,

$$(21) \quad (1 - \epsilon) \int_{M^m} L'_\varphi \rho^2 |\operatorname{grad}^{M^m} |d\varphi||^2 v^g \leq \frac{1}{\epsilon} \int_{M^m} L'_\varphi |d\varphi|^2 |\operatorname{grad}^{M^m} \rho|^2 v^g.$$

Choose the smooth cut-off $\rho = \rho_R$ i.e., $\rho \leq 1$ on M^m , $\rho = 1$ on the ball $B(0, R)$, $\rho = 0$ on $M^m \setminus B(0, 2R)$ and $|\operatorname{grad}^{M^m} \rho| \leq \frac{2}{R}$. Let $0 < \epsilon < 1$. Replacing $\rho = \rho_R$ in (21), we obtain

$$(22) \quad 0 \leq (1 - \epsilon) \int_{M^m} L'_\varphi \rho^2 |\operatorname{grad}^{M^m} |d\varphi||^2 v^g \leq \frac{4}{\epsilon R^2} \int_{M^m} L'_\varphi |d\varphi|^2 v^g.$$

Since $\int_{M^m} L'_\varphi |d\varphi|^2 v^g < \infty$, when $R \rightarrow \infty$, we have

$$\frac{4}{\epsilon R^2} \int_{M^m} L'_\varphi |d\varphi|^2 v^g \rightarrow 0.$$

Thus, by (22), we have $|\operatorname{grad}^{M^m} |d\varphi|| = 0$ i.e., $|d\varphi| = c$ constant. By the assumptions,

$$\frac{c^2}{2} \int_{M^m} L'_\varphi v^g < \infty \quad \text{and} \quad \int_{M^m} L'_\varphi v^g = \infty.$$

Hence $c = 0$, that is, φ is constant map. \square

If $L(x, y, r) = r$ for all $(x, y, r) \in M^m \times N^n \times \mathbb{R}_+$, then we recover the following classical result.

COROLLARY 3.4. [7] *Let (M^m, g) be a complete noncompact Riemannian manifold of infinite volume with positive Ricci curvature and (N^n, h) a Riemannian manifold with non-positive sectional curvature. Then a harmonic map $\varphi : (M^m, g) \rightarrow (N^n, h)$ with finite energy, i.e.,*

$$E(\varphi) = \frac{1}{2} \int_{M^m} |d\varphi|^2 v^g < \infty,$$

is constant.

Let $f : M^m \rightarrow (0, \infty)$ be a smooth function. If $L(x, y, r) = f(x)r$ for all $(x, y, r) \in M^m \times N^n \times \mathbb{R}_+$. We recover the following result.

COROLLARY 3.5. [11,13] *Let (M^m, g) be a complete noncompact Riemannian manifold with positive Ricci curvature and (N^n, h) a Riemannian manifold with non-positive sectional curvature. Let f be a smooth positive function on M^m with non-positive Hessian $\operatorname{Hess}^{M^m} f \leq 0$. If $\operatorname{Vol}_f(M^m)$ is infinite, then an f -harmonic map*

$\varphi : (M^m, g) \longrightarrow (N^n, h)$ with finite f -energy, i.e.,

$$E_f(\varphi) = \frac{1}{2} \int_{M^m} f |d\varphi|^2 v^g < \infty,$$

is constant.

References

- [1] P. Baird, A. Fardoun and S. Ouakkas, *Liouville-type Theorems for Biharmonic Maps between Riemannian Manifolds*, Advances in Calculus of Variations. **3** (1) (2009), 49–68.
- [2] P. Baird, A. Fardoun and S. Ouakkas, *Conformal and semi-conformal biharmonic maps*, Ann Glob Anal Geom. **34** (2008), 403–414.
- [3] P. Baird, J.C. Wood, *Harmonic Morphisms between Riemannian Manifolds*, Clarendon Press, Oxford, 2003.
- [4] P. Berard, *A note on Bochner type theorems for complete manifolds*, Manuscripta Math. **69** (1990), 261–266.
- [5] S. Y. Cheng, *Liouville theorem for harmonic maps*, Geometry of the Laplace operator, Proc. Sym. Pure Math. **36** (1980), 147–151.
- [6] M. Djaa and A. Mohammed Cherif, *On generalized f -harmonic maps and liouville type theorem*, Konuralp Journal of Mathematics. **4** (1) (2016), 33–44.
- [7] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1) (1964), 109–160.
- [8] J. Liu, *Liouville-type Theorems of p -harmonic Maps with free Boundary Values*, Hiroshima Math. **40** (2010), 333–342.
- [9] A. Mohammed Cherif and M. Djaa, *Geometry of energy and bienergy variations between Riemannian Manifolds*, Kyungpook Math. J. **55** (3) (2015), 715–730.
- [10] A. Mohammed Cherif, H. Elhendi and M. Terbeche, *On Generalized Conformal Maps*, Bulletin of Mathematical Analysis and Applications. **4** (4) (2012), 99–108.
- [11] M. Rimoldi and G. Veronelli, *f -Harmonic Maps and Applications to Gradient Ricci Solitons*, arXiv:1112.3637, (2011).
- [12] R. M. Schoen, and S. T. Yau, *Harmonic Maps and the Topology of Stable Hypersurfaces and Manifolds with Non-negative Ricci Curvature*, Comment. Math. Helv. **51** (3) (1976), 333–341.
- [13] D. Xu Wang, *Harmonic Maps from Smooth Metric Measure Spaces*, Internat. J. Math. **23** (9) (2012), 1250095.
- [14] W.C. Young, *On the multiplication of successions of Fourier constants*, Proc. Royal Soc. Lond. **87** (596) (1912), 331–339.

Embarka Remli

Department of Mathematics, University Mustapha Stambouli, Mascara 29000,
Algeria

E-mail: ambarka.ramli@univ-mascara.dz

Ahmed Mohammed Cherif

Department of Mathematics, University Mustapha Stambouli, Mascara 29000,
Algeria

E-mail: a.mohammedcherif@univ-mascara.dz