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COEFFICIENT BOUNDS FOR A SUBCLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH DZIOK-SRIVASTAVA OPERATOR

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Abstract. In this article, we represent and examine a new subclass of holomorphic and bi-univalent functions defined in the open unit disk $\mathfrak{U}$, which is associated with the Dziok-Srivastava operator. Additionally, we get upper bound estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in the new class and improve some recent studies.

1. Introduction

Let $\mathcal{A}$ be a family of functions of the form

$$\mathfrak{f}(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

which are holomorphic in the open unit disk $\mathfrak{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also, we let $\mathcal{S}$ to denote the class of functions $\mathfrak{f} \in \mathcal{A}$ which are univalent in $\mathfrak{U}$.

The Koebe one-quarter theorem [4] ensures that the image of $\mathfrak{U}$ under every univalent function $\mathfrak{f} \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. So every function $\mathfrak{f} \in \mathcal{S}$ has an inverse $\mathfrak{f}^{-1}$, which is defined by

$$\mathfrak{f}^{-1}(\mathfrak{f}(z)) = z \quad z \in \mathfrak{U},$$

and

$$\mathfrak{f}(\mathfrak{f}^{-1}(w)) = w \quad \text{for } |w| < r_0(\mathfrak{f}) \text{ such that } r_0(\mathfrak{f}) \geq \frac{1}{4},$$

where

$$\mathfrak{f}^{-1}(w) = w - a_2^2 w + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots .$$

A function $\mathfrak{f} \in \mathcal{A}$ is said to be bi-univalent in $\mathfrak{U}$ if both $\mathfrak{f}$ and $\mathfrak{f}^{-1}$ are univalent in $\mathfrak{U}$. Let Σ denote the class of bi-univalent functions in $\mathfrak{U}$ given by (1.1).
Lewin [10] enquired the class Σ of bi-univalent functions and established that $|a_2| < 1.51$ for the functions belonging to Σ. Afterward, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$. Kedzierawski [9] proved this conjecture for a special case when the function $\mathfrak{f}$ and $\mathfrak{f}^{-1}$ are starlike functions. Tan [15] obtained the bound for $|a_2|$ namely

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$|a_2| \leq 1.485$ which is the best-known estimate for functions in the class Σ. Recently, their relevance to research the bi-univalent functions class Σ (see [7, 8, 11–13, 16, 17]) and get non-sharp bounds on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The coefficient estimate problem i.e. bound of $|a_j|$ $(j \in \mathbb{N} - \{1, 2\})$ for each $\mathfrak{f} \in \Sigma$ given by [1] is still an open problem.

The Hadamard product of two analytic functions

$$\mathfrak{f}(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad \text{and} \quad \mathfrak{h}(z) = z + \sum_{j=2}^{\infty} b_j z^j,$$

is defined as

$$(\mathfrak{f} * \mathfrak{h})(z) = (\mathfrak{h} * \mathfrak{f})(z) = z + \sum_{j=2}^{\infty} b_j a_j z^j.$$

For the complex parameters $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ with $\mathfrak{c} \neq 0, -1, -2, -3, ...$, the Gaussian hypergeometric function ${}_2\mathcal{F}_1(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}; z)$ is defined as

$${}_2\mathcal{F}_1(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}; z) = \sum_{j=0}^{\infty} \frac{(\mathfrak{a})_j (\mathfrak{b})_j}{(\mathfrak{c})_j} \frac{z^j}{j!} = 1 + \sum_{j=2}^{\infty} \frac{(\mathfrak{a})_{j-1} (\mathfrak{b})_{j-1}}{(\mathfrak{c})_{j-1}} \frac{z^{j-1}}{(j-1)!} \quad z \in \mathfrak{U},$$

where $(\tau)_j$ is the Pochhammer symbol (or the shifted factorial) defined as follows:

$$(\varkappa)_j = \frac{\Gamma(\varkappa + j)}{\Gamma(\varkappa)} = \begin{cases} 1 & j = 0 \\ \varkappa(\varkappa + 1)(\varkappa + 2)...(\varkappa + j - 1) & j = 1, 2, 3, \ldots \end{cases}$$

the generalized hypergeometric function ${}_\mathfrak{q}\mathcal{F}_\mathfrak{s}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}; z)$, $(\mathfrak{q} \leq \mathfrak{s} + 1, z \in \mathfrak{U})$ is defined by the following infinite series:

$$\begin{aligned} {}_\mathfrak{q}\mathcal{F}_\mathfrak{s}(\mathfrak{a}_1, ..., \mathfrak{a}_\mathfrak{q}; \mathfrak{b}_1, ..., \mathfrak{b}_\mathfrak{s}; z) &= \sum_{j=0}^{\infty} \frac{(\mathfrak{a}_1)_j ... (\mathfrak{a}_\mathfrak{q})_j}{(\mathfrak{b}_1)_j ... (\mathfrak{b}_\mathfrak{s})_j} \frac{z^j}{j!} \\ &= 1 + \sum_{j=2}^{\infty} \frac{(\mathfrak{a}_1)_{j-1} ... (\mathfrak{a}_\mathfrak{q})_{j-1}}{(\mathfrak{b}_1)_{j-1} ... (\mathfrak{b}_\mathfrak{s})_{j-1}} \frac{z^{j-1}}{(j-1)!} \end{aligned}$$

correspondingly a function $\mathfrak{h}(\mathfrak{a}_1, ..., \mathfrak{a}_\mathfrak{q}; \mathfrak{b}_1, ..., \mathfrak{b}_\mathfrak{s}; z)$ is defined by

$$\mathfrak{h}(\mathfrak{a}_1, ..., \mathfrak{a}_\mathfrak{q}; \mathfrak{b}_1, ..., \mathfrak{b}_\mathfrak{s}; z) = z {}_\mathfrak{q}\mathcal{F}_\mathfrak{s}(\mathfrak{a}_1, ..., \mathfrak{a}_\mathfrak{q}; \mathfrak{b}_1, ..., \mathfrak{b}_\mathfrak{s}; z), \quad z \in \mathfrak{U}.$$

Dziok and Srivastava [5](see also [6]) considered a linear operator

$$\mathcal{H}(\mathfrak{a}_1, ..., \mathfrak{a}_\mathfrak{q}; \mathfrak{b}_1, ..., \mathfrak{b}_\mathfrak{s}; z) : \mathcal{A} \to \mathcal{A}$$

defined by the following Hadamard product:

$$\mathcal{H}(\mathfrak{a}_1, ..., \mathfrak{a}_\mathfrak{q}; \mathfrak{b}_1, ..., \mathfrak{b}_\mathfrak{s})\mathfrak{f}(z) = \mathfrak{h}(\mathfrak{a}_1, ..., \mathfrak{a}_\mathfrak{q}; \mathfrak{b}_1, ..., \mathfrak{b}_\mathfrak{s}) * \mathfrak{f}(z) \quad \mathfrak{q} \leq \mathfrak{s} + 1, \ z \in \mathfrak{U}.$$

If $\mathfrak{f} \in \mathcal{A}$ is given by (1.1), then we have

$$\mathcal{H}(\mathfrak{a}_1, ..., \mathfrak{a}_\mathfrak{q}; \mathfrak{b}_1, ..., \mathfrak{b}_\mathfrak{s})\mathfrak{f}(z) = z + \sum_{j=2}^{\infty} \Gamma_j[\mathfrak{a}_1; \mathfrak{b}_1] a_j z^j$$

where

$$\Gamma_j[\mathfrak{a}_1; \mathfrak{b}_1] = \frac{(\mathfrak{a}_1)_{j-1} ... (\mathfrak{a}_\mathfrak{q})_{j-1}}{(\mathfrak{b}_1)_{j-1} ... (\mathfrak{b}_\mathfrak{s})_{j-1}} \frac{1}{(j-1)!} \quad j \in \mathbb{N}.$$

To make the notation simple, we write

$$\mathcal{H}_{\mathfrak{q},\mathfrak{s}}[\mathfrak{a}_1;\mathfrak{b}_1;z]=\mathcal{H}(\mathfrak{a}_1,...,\mathfrak{a}_\mathfrak{q};\mathfrak{b}_1,...,\mathfrak{b}_\mathfrak{s})\mathfrak{f}(z).$$

The linear operator $\mathcal{H}_{\mathfrak{q},\mathfrak{s}}[\mathfrak{a}_1;\mathfrak{b}_1;z]$ is a generalization of many other linear operators considered earlier.

In the present article, we innovate a new subclass of the bi-univalent functions which are defined by the Dziok-Srivastava operator also we get upper bound estimates on the coefficients $|a_2|$ and $|a_3|$ by applying the methods used earlier by Srivastava et al. [14] (see also [8]). Our results generalize and improve those in related studies of several earlier authors.

2. The subclass ${}_\Sigma\mathcal{H}^{\Theta,\Upsilon}_{\mathfrak{q},\mathfrak{s}}[\mathfrak{a}_1;\mathfrak{b}_1;\xi]$

In this section, we represent and examine the general subclass ${}_\Sigma\mathcal{H}^{\Theta,\Upsilon}_{\mathfrak{q},\mathfrak{s}}[\mathfrak{a}_1;\mathfrak{b}_1;\xi]$.

Definition 2.1. Let the analytic functions $\Theta,\Upsilon:\mathfrak{U}\to\mathbb{C}$ be so constrained that

$$\min\{\mathfrak{Re}(\Theta(z)),\mathfrak{Re}(\Upsilon(z))\}>0,\quad z\in\mathfrak{U}\text{ and }\Theta(0)=1=\Upsilon(0).\tag{2.1}$$

We say that a function $\mathfrak{f}\in\ {}_\Sigma\mathcal{H}^{\Theta,\Upsilon}_{\mathfrak{q},\mathfrak{s}}[\mathfrak{a}_1;\mathfrak{b}_1;\xi]$, $(\xi\geq 1)$, if the following conditions satisfy

$$\mathfrak{f}\in\Sigma\quad\text{and}\quad(1-\xi)\frac{\mathcal{H}_{\mathfrak{q},\mathfrak{s}}[\mathfrak{a}_1;\mathfrak{b}_1;z]}{z}+\xi(\mathcal{H}_{\mathfrak{q},\mathfrak{s}}[\mathfrak{a}_1;\mathfrak{b}_1;z])'\in\Theta(\mathfrak{U}),\quad z\in\mathfrak{U},\tag{2.2}$$

and

$$(1-\xi)\frac{\mathfrak{g}(w)}{w}+\xi\mathfrak{g}'(w)\in\Upsilon(\mathfrak{U}),\quad w\in\mathfrak{U},\tag{2.3}$$

where the function $\mathfrak{g}(w)$ is given by

$$\begin{aligned}\mathfrak{g}(w)&=\mathcal{H}^{-1}_{\mathfrak{q},\mathfrak{s}}[\mathfrak{a}_1;\mathfrak{b}_1;z]\\&=w-\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1]a_2w^2+(2(\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1])^2a_2-\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]a_3)w^3+\cdots.\end{aligned}\tag{2.4}$$

Remark 2.2. There are different options of the functions $\Theta(z)$ and $\Upsilon(z)$ which would provide interesting subclasses of the analytic function class $\mathcal{A}$.

1. If we take

$$\Theta(z)=\Upsilon(z)=\left(\frac{1+z}{1-z}\right)^\lambda\quad z\in\mathfrak{U},\ 0<\lambda\leq 1,$$

then the functions $\Theta(z)$ and $\Upsilon(z)$ satisfy the hypotheses of Definition 2.1. Clearly, if $\mathfrak{f}\in\ {}_\Sigma\mathcal{H}^{\Theta,\Upsilon}_{\mathfrak{q},\mathfrak{s}}[\mathfrak{a}_1;\mathfrak{b}_1;\xi]$, then we have

$$\left|arg\left((1-\xi)\frac{\mathcal{H}_{\mathfrak{q},\mathfrak{s}}[\mathfrak{a}_1;\mathfrak{b}_1;z]}{z}+\xi(\mathcal{H}_{\mathfrak{q},\mathfrak{s}}[\mathfrak{a}_1;\mathfrak{b}_1;z])'\right)\right|<\frac{\lambda\pi}{2}\quad z\in\mathfrak{U},\ \xi\geq 1,$$

and

$$\left|arg\left((1-\xi)\frac{\mathfrak{g}(w)}{w}+\xi\mathfrak{g}'(w)\right)\right|<\frac{\lambda\pi}{2}\quad w\in\mathfrak{U},\ \xi\geq 1.$$

2. If we take
$$\Theta(z)=\Upsilon(z)=\frac{1+(1-2\delta)z}{1-z}\quad z\in\mathfrak{U},\ 0\leq\delta<1,$$
then the functions $\Theta(z)$ and $\Upsilon(z)$ satisfy the hypotheses of Definition 2.1. Clearly, if $\mathfrak{f}\in\ {}_{\Sigma}\mathcal{H}_{\mathfrak{q},\mathfrak{s}}^{\Theta,\Upsilon}[\mathfrak{a}_1;\mathfrak{b}_1;\xi]$, then we have
$$\mathfrak{Re}\left[(1-\xi)\frac{\mathcal{H}_{\mathfrak{q},\mathfrak{s}}[\mathfrak{a}_1;\mathfrak{b}_1;z]}{z}+\xi(\mathcal{H}_{\mathfrak{q},\mathfrak{s}}[\mathfrak{a}_1;\mathfrak{b}_1;z])'\right]>\delta\quad z\in\mathfrak{U},\ \xi\geq1,\ 0\leq\delta<1,$$
and
$$\mathfrak{Re}\left[(1-\xi)\frac{\mathfrak{g}(w)}{w}+\xi\mathfrak{g}'(w)\right]>\delta,\quad w\in\mathfrak{U},\ \xi\geq1,\ 0\leq\delta<1.$$
3. For $\mathfrak{q}=2,\mathfrak{s}=1,\mathfrak{a}_1=\mathfrak{a}_2=\mathfrak{b}_1=\xi=1$ and $\Theta(z)=\Upsilon(z)=\left(\frac{1+z}{1-z}\right)^{\lambda}$, we have
$${}_{\Sigma}\mathcal{H}_{1,2}^{\Theta,\Upsilon}[1;1;1]=\mathcal{H}_{\Sigma}^{\lambda},$$
where the class $\mathcal{H}_{\Sigma}^{\lambda}$ was studied by Srivastava et al [14].
4. For $\mathfrak{q}=2,\mathfrak{s}=1,\mathfrak{a}_1=\mathfrak{a}_2=\mathfrak{b}_1=\xi=1$ and $\Theta(z)=\Upsilon(z)=\frac{1+(1-2\delta)z}{1-z}$, we have
$${}_{\Sigma}\mathcal{H}_{1,2}^{\Theta,\Upsilon}[1;1;1]=H_{\Sigma}(\delta),$$
where the class $\mathcal{H}_{\Sigma}(\delta)$ was studied by Srivastava et al [14].
5. For $\Theta(z)=\Upsilon(z)=\left(\frac{1+z}{1-z}\right)^{\lambda}$ we have
$${}_{\Sigma}\mathcal{H}_{\mathfrak{q},\mathfrak{s}}^{\Theta,\Upsilon}[\mathfrak{a}_1;\mathfrak{b}_1;\xi]=\mathcal{H}_{\mathfrak{q},\mathfrak{s}}^{\Sigma}[\mathfrak{a}_1;\mathfrak{b}_1;\lambda;\xi],$$
where the class $\mathcal{H}_{\mathfrak{q},\mathfrak{s}}^{\Sigma}[\mathfrak{a}_1;\mathfrak{b}_1;\lambda;\xi]$ was introduced and studied by M. K. Aouf [2].

3. Coefficient Estimates

For proof of the theorem, we need the following lemma.

LEMMA 3.1. [4] *If $\phi\in\mathcal{P}$, then $|\phi_j|\leqslant 2$ for each j, where $\mathcal{P}$ is the class of all functions $\phi(z)$ analytic in $\mathfrak{U}$ for which $\mathfrak{Re}(\phi(z))>0,\phi(z)=1+\phi_1z+\phi_2z^2+\cdots$ for $z\in\mathfrak{U}$.*

THEOREM 3.2. *Let $\mathfrak{f}(z)$ given by the Taylor Maclaurin series expansion* (1.1) *be in the class ${}_{\Sigma}\mathcal{H}_{\mathfrak{q},\mathfrak{s}}^{\Theta,\Upsilon}[\mathfrak{a}_1;\mathfrak{b}_1;\xi],(\xi\geq1)$. Then,*
$$|a_2|\leq\min\left\{\sqrt{\frac{|\Theta'(0)|^2+|\Upsilon'(0)|^2}{2(\xi+1)^2|\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1]|^2}},\sqrt{\frac{|\Theta''(0)|+|\Upsilon''(0)|}{4(2\xi+1)|\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1]|^2}}\right\},\tag{3.1}$$
and
$$|a_3|\leq\min\left\{\frac{|\Theta'(0)|^2+|\Upsilon'(0)|^2}{2(\xi+1)^2|\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]|}+\frac{|\Theta''(0)|+|\Upsilon''(0)|}{4(2\xi+1)|\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]|},\frac{|\Theta''(0)|}{2(2\xi+1)|\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]|}\right\}.$$

Proof. First of all, it follows from the conditions (2.2) and (2.3) that,
$$(1-\xi)\frac{\mathcal{H}_{\mathfrak{q},\mathfrak{s}}[\mathfrak{a}_1;\mathfrak{b}_1;z]}{z}+\xi(\mathcal{H}_{\mathfrak{q},\mathfrak{s}}[\mathfrak{a}_1;\mathfrak{b}_1;z])'=\Theta(z)\quad z\in\mathfrak{U},\tag{3.2}$$
and
$$(1-\xi)\frac{\mathfrak{g}(w)}{w}+\xi\mathfrak{g}'(w)=\Upsilon(w)\quad w\in\mathfrak{U},\tag{3.3}$$

where the function $\mathfrak{g}(w)$ is given by (2.4), respectively, $\Theta(z)$ and $\Upsilon(w)$ satisfy in (2.1). Also, the functions $\Theta(z)$ and $\Upsilon(w)$ have the following Taylor-Maclaurin series expansions:

$$\begin{aligned} \Theta(z) &= 1 + \Theta_1 z + \Theta_2 z^2 + \cdots, \\ \Upsilon(w) &= 1 + \Upsilon_1 w + \Upsilon_2 w^2 + \cdots. \end{aligned} \tag{3.4}$$

Now, by comparing the series expansions (3.4) by the coefficients (3.2) and (3.3), we get

$$(\xi+1)\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1]a_2 = \Theta_1 \tag{3.5}$$

$$(2\xi+1)\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]a_3 = \Theta_2 \tag{3.6}$$

$$-(\xi+1)\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1]a_2 = \Upsilon_1 \tag{3.7}$$

$$(2\xi+1)(2(\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1])^2 a_2^2 - \Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]a_3) = \Upsilon_2. \tag{3.8}$$

From (3.5) and (3.7), we obtain

$$\begin{aligned} \Theta_1 &= -\Upsilon_1 \\ \Theta_1^2 + \Upsilon_1^2 &= 2(\xi+1)^2(\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1])^2 a_2^2. \end{aligned} \tag{3.9}$$

Also, From (3.6) and (3.8), we find that

$$\Theta_2 + \Upsilon_2 = 2(2\xi+1)(\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1])^2 a_2^2. \tag{3.10}$$

Therefore, we find from the equations (3.9) and (3.10) that

$$|a_2|^2 \le \frac{|\Theta'(0)|^2 + |\Upsilon'(0)|^2}{2(\xi+1)^2|\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1]|^2}$$

and

$$|a_2|^2 \le \frac{|\Theta''(0)| + |\Upsilon''(0)|}{4(2\xi+1)|\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1]|^2}.$$

So we get the requested estimate on the coefficient $|a_2|$ as asserted in (3.1). Next, in order to find the bound on the coefficient $|a_3|$, we subtract (3.8) from (3.6). We thus get

$$\Theta_2 - \Upsilon_2 = 2(2\xi+1)(\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]a_3 - (\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1])^2 a_2^2). \tag{3.11}$$

Upon substituting the value of a_2^2 from (3.9) into (3.11), it follows that

$$a_3 = \frac{\Theta_1^2 + \Upsilon_1^2}{2(\xi+1)^2\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]} + \frac{\Theta_2 - \Upsilon_2}{2(2\xi+1)\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]}.$$

We thus find that

$$|a_3| \le \frac{|\Theta'(0)|^2 + |\Upsilon'(0)|^2}{2(\xi+1)^2|\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]|} + \frac{|\Theta''(0)| + |\Upsilon''(0)|}{4(2\xi+1)|\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]|}.$$

On the other hand, upon substituting the value of a_2^2 from (3.10) into (3.11), it follows that

$$a_3 = \frac{\Theta_2}{(2\xi+1)\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]}.$$

Consequently, we have

$$|a_3| \le \frac{|\Theta''(0)|}{2(2\xi+1)|\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]|}.$$

This completes the proof of Theorem 3.2. □

4. Corollaries and Consequences

By setting $\Theta(z) = \Upsilon(z) = \left(\frac{1+z}{1-z}\right)^{\lambda}$, $\xi = 1, \mathfrak{q} = 2$ and $\mathfrak{s} = \mathfrak{a}_1 = \mathfrak{a}_2 = \mathfrak{b}_1 = 1$ in Theorem 3.2. we get the following result.

COROLLARY 4.1. *Let the function $\mathfrak{f}(z)$ given by the Taylor-Maclaurin series expansion* (1.1) *be in the bi-univalent function class $\mathcal{H}_{\Sigma}^{\lambda}$. Then*

$$|a_2| \leq \frac{\sqrt{2}\lambda}{\sqrt{3}} \quad \textit{and} \quad |a_3| \leq \frac{2\lambda^2}{3}.$$

REMARK 4.2. Corollary 4.1 is an development of the following estimates obtained by Srivastava et al. [14].

COROLLARY 4.3. [14] *Let the function $\mathfrak{f}(z)$ given by the Taylor-Maclaurin series expansion* (1.1) *be in the bi-univalent function class $\mathcal{H}_{\Sigma}^{\lambda}$. Then*

$$|a_2| \leq \frac{\sqrt{2}\lambda}{\sqrt{\lambda+2}} \quad \textit{and} \quad |a_3| \leq \frac{(3\lambda+2)\lambda}{3}.$$

By setting $\Theta(z) = \Upsilon(z) = \left(\frac{1+z}{1-z}\right)^{\lambda}$ in Theorem 3.2, we get the following consequence.

COROLLARY 4.4. *Let the function $\mathfrak{f}(z)$ given by the Taylor-Maclaurin series expansion* (1.1) *be in the bi-univalent function class ${}_{\Sigma}\mathcal{H}_{\mathfrak{q},\mathfrak{s}}^{\Theta,\Upsilon}[\mathfrak{a}_1;\mathfrak{b}_1;\xi], (\eta \geq 1)$. Then*

$$|a_2| \leq \min\left\{\frac{2\lambda}{|\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1]|(\xi+1)}\ ,\ \frac{2\lambda}{|\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1]|\sqrt{2(2\xi+1)}}\right\},$$

and

$$|a_3| \leq \frac{2\lambda^2}{|\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]|(2\xi+1)}.$$

Thus, Corollary 4.4 is an improvement of the following estimates obtained by Auof [2].

COROLLARY 4.5. [2] *Let the function $\mathfrak{f}(z)$ given by the Taylor-Maclaurin series expansion* (1.1) *be in the bi-univalent function class $\mathcal{H}_{\mathfrak{q},\mathfrak{s}}^{\Sigma}[\mathfrak{a}_1;\mathfrak{b}_1;\lambda;\xi](\xi \geq 1)$. Then*

$$|a_2| \leq \frac{2\lambda}{|\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1]|\sqrt{(\xi+1)^2+\lambda(1+2\xi-\xi^2)}}$$

and

$$|a_3| \leq \frac{4\lambda^2}{|\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]|(\xi+1)^2} + \frac{2\lambda}{|\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]|(2\xi+1)}.$$

REMARK 4.6. For the coefficient $|a_2|$ with conditions $0 < \lambda \leq 1, \xi \geq 1+\sqrt{2}$

$$\frac{2\lambda}{|\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1]|(\xi+1)} \leq \frac{2\lambda}{|\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1]|\sqrt{(\xi+1)^2+\lambda(1+2\xi-\xi^2)}},$$

and with conditions $0 < \lambda \leq 1, 1 \leq \xi < 1+\sqrt{2}$

$$\frac{2\lambda}{|\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1]|\sqrt{2(2\xi+1)}} \leq \frac{2\lambda}{|\Gamma_2[\mathfrak{a}_1;\mathfrak{b}_1]|\sqrt{(\xi+1)^2+\lambda(1+2\xi-\xi^2)}}.$$

Otherwise, for the coefficient $|a_3|$, we make the following investigations:

$$\begin{aligned}\frac{2\lambda^2}{|\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]|(2\xi+1)} &\leq \frac{2\lambda}{|\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]|(2\xi+1)} \\ &\leq \frac{4\lambda^2}{|\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]|(\xi+1)^2} + \frac{2\lambda}{|\Gamma_3[\mathfrak{a}_1;\mathfrak{b}_1]|(2\xi+1)}.\end{aligned}$$

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