Korean J. Math. **19** (2011), No. 1, pp. 53–59

CONVERGENCE OF INTEGRABLE SEMIGROUPS

Young S. Lee

Abstract. We study some properties of integrable semigroup and its generator, and then we establish convergence of integrable semigroups on the norming dual pairs.

1. Introduction

Let X and Y be Banach spaces and let $\langle\cdot,\ \cdot\rangle$ be a bilinear form on $X \times Y$ which separates points, i.e. $\langle x,\ y\rangle = 0$ for all $x \in X$ implies $y = 0$ and $\langle x,\ y\rangle = 0$ for all $y \in Y$ implies $x = 0$.

A norming dual pair is a triple $(X,\ Y,\ \langle\cdot,\ \cdot\rangle)$ satisfying

$$||x|| = \sup\{|\langle x,\ y\rangle| : y \in Y,\ ||y|| \le 1\}$$

and

$$||y|| = \sup\{|\langle x,\ y\rangle| : x \in X,\ ||x|| \le 1\}.$$

We will write $(X,\ Y)$ instead of $(X,\ Y,\ \langle\cdot,\ \cdot\rangle)$ if the duality pairing is understood. Note that if $(X,\ Y)$ is a norming dual pair then Y is isometrically isomorphic to a closed subspace of X^*, and so we can identify Y as a closed subspace of X^*. For more information about the dual pair, see [2, 3].

We define a locally convex topology on X. For a bounded subset $M \subset Y$, $p_M(x) = \sup_{y \in M} |\langle x,\ y\rangle|$ defines a seminorm on X. Let $\mathcal{M}$ be a collection of bounded subsets of Y. Then the collection of seminorms $\{p_M : M \in \mathcal{M}\}$ defines a locally convex topology on X if and only if $\mathcal{M}$ is separating, i.e. for every $x \in X$ there exists $M \in \mathcal{M}$ such that $p_M(x) \neq 0$. In this case $\tau_{\mathcal{M}}$ denotes the locally convex topology on X induced by $\{p_M : M \in \mathcal{M}\}$.

Received February 14, 2011. Revised March 11, 2011. Accepted March 15, 2011.
2000 Mathematics Subject Classification: 47D60.
Key words and phrases: dual pairs, integrable semigroups, pseudoresolvent, generators, convergence.

A locally convex topology τ on X is called consistent if $(X,\ \tau)' = Y$, i.e. every τ-continuous linear functional ϕ on X is of the form $\phi(x) = \langle x,\ y\rangle$ for some y in Y. Every consistent topology is of form $\tau_{\mathcal{M}}$ for some separating collection $\mathcal{M}$ of bounded subsets of Y, and there exists a coarsest consistent topology, namely the weak topology $\sigma(X,\ Y) = \tau_{\mathcal{M}}$, where $\mathcal{M}$ is a collection of all finite subsets of Y (see [3]).

If τ is a locally convex topology on X, $L(X,\ \tau)$ is the space of all τ-continuous linear operators on X. If τ is a norm topology, we write $L(X)$ for $L(X,\ ||\cdot||)$.

In this paper, we study convergence of integrable semigroups on norming dual pairs. We introduce integrable semigroup which may have no continuity properties. And then we establish Trotter-Kato type convergence theorem, i.e. the convergence of generators in some sense implies the convergence of integrable semigroups.

2. Convergence Theorem

First, we give a definition and some properties of integrable semigroups.

Definition 1. Let $(X,\ Y)$ be a norming dual pair. A semigroup on $(X,\ Y)$ is a family of operators $\{T(t) : t \geq 0\} \subset L(X,\ \sigma)$ such that $T(t+s) = T(t)T(s)$ for all $s,\ t \geq 0$ and $T(0) = I$, the identity operator on X. A semigroup is said to be exponentially bounded if there exist $M \geq 1$ and $\omega \in R$ such that $||T(t)|| \leq Me^{\omega t}$ for all $t \geq 0$.

An exponentially bounded semigroup is said to be *integrable* if for each λ with $Re\lambda > \omega$, there exists an operator $R(\lambda) \in L(X,\ \sigma)$ such that

$$\langle R(\lambda)x,\ y\rangle = \int_0^\infty e^{-\lambda t}\langle T(t)x,\ y\rangle dt$$

for all $x \in X$ and $y \in Y$.

By the semigroup property of $\{T(t) : t \geq 0\}$, $\{R(\lambda) : Re\lambda > \omega\}$ is a pseudoresolvent (see [1]). Hence there exists a unique multivalued operator $\mathcal{A}$ such that $R(\lambda) = (\lambda - \mathcal{A})^{-1}$ and the kernel and the range of $R(\lambda)$ are independent of λ (see [4]). If $R(\lambda)$ is injective, then $\mathcal{A}$ is single valued. In this case, we say that $\{T(t) : t \geq 0\}$ has a generator A and $R(\lambda) = (\lambda - A)^{-1}$. In general, $R(\lambda)$ may not be injective because we did not require any continuity of $T(t)x$.

We now state some properties of integrable semigroups.

LEMMA 2. *Let $\{T(t) : t \geq 0\}$ be an integrable semigroup on the norming dual pair $(X,\ Y)$ with the generator A. Then*

(i) *for $x \in X$ and $t \geq 0$ $\int_0^t T(s)xds \in D(A)$ and $T(t)x - x = A\int_0^t T(s)xds$.*
(ii) *$(\lambda R(\lambda) - I)\int_0^t T(s)xds = (T(t) - I)R(\lambda)x$ for all $x \in X$ and $t \geq 0$.*
(iii) *The following statements are equivalent.*
  (a) *$x \in \overline{D(A)}$.*
  (b) *$T(t)x$ is continuous at 0.*
  (c) *$\lim_{\lambda\to\infty} \lambda R(\lambda)x = x$.*

Proof. (i) is Proposition 2.4 in [3].
(ii) By (i), Lemma 4.8 and Proposition 5.3 in [2], we have

$$\begin{aligned}(\lambda R(\lambda) - I)\int_0^t T(s)xds &= AR(\lambda)\int_0^t T(s)xds \\ &= A\int_0^t T(s)R(\lambda)xds \\ &= T(t)R(\lambda)x - R(\lambda)x.\end{aligned}$$

(iii) By Proposition 2.4 in [3] , $T(t)x$ is continuous for each $x \in D(A)$. By the continuity of $T(t)x$ for $x \in D(A)$ and exponential boundedness of $||T(t)||$, (a) implies (b). Suppose that $T(t)x$ is continuous at 0. Then for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that $||T(t)x - x|| < \varepsilon$ for all $0 \leq t < \delta$. Let $y \in Y$ and $||y|| \leq 1$.

Then

$$\begin{aligned}&\langle \lambda R(\lambda)x - x,\ y\rangle \\ &\quad= \int_0^\infty \lambda e^{-\lambda t}\langle T(t)x - x,\ y\rangle dt \\ &\quad= \int_0^\delta \lambda e^{-\lambda t}\langle T(t)x - x,\ y\rangle dt + \int_\delta^\infty \lambda e^{-\lambda t}\langle T(t)x - x,\ y\rangle dt.\end{aligned}$$

Hence we have

$$\begin{aligned}
&|\langle \lambda R(\lambda)x - x,\ y\rangle| \\
\le\ & \int_0^\delta |\lambda e^{-\lambda t}\langle T(t)x - x,\ y\rangle| dt + \int_\delta^\infty |\lambda e^{-\lambda t}\langle T(t)x - x,\ y\rangle| dt \\
\le\ & \varepsilon \int_0^\delta \lambda e^{-\lambda t} dt + \int_\delta^\infty \lambda e^{-\lambda t}(Me^{\omega t} + 1)||x|| dt \\
=\ & \varepsilon(1 - e^{-\lambda\delta}) + (\frac{\lambda M}{\lambda - \omega} e^{-(\lambda-\omega)\delta} + e^{-\lambda\delta})||x||.
\end{aligned}$$

Since $(X,\ Y)$ is a norming dual pair, we have $\lim_{\lambda\to\infty} \lambda R(\lambda)x = x$. Therefore (b) implies (c) since $R(\lambda)x \in D(A)$, (c) implies (a). □

Note that if the generator A is densely defined, then an integrable semigroup is a C_0 semigroup, by Lemma 2 (iii). We now consider the continuity of semigroups.

Definition 3. Let $\{T(t) : t \ge 0\}$ be a semigroup on $(X,\ Y)$ and let τ be a locally convex topology on X. $\{T(t) : t \ge 0\}$ is said to be τ-continuous (at 0) if for every $x \in X$ $T(t)x$ is τ-continuous (at 0) in $t \ge 0$.

Recall that a semigroup $\{T(t) : t \ge 0\}$ on a locally convex space $(X,\ \tau)$ is said to be equicontinuous if for every τ-continuous seminorm p, there exists a τ-continuous seminorm q such that $p(T(t)x) \le q(x)$ for all $x \in X$ and $t \ge 0$. A semigroup $\{T(t) : t \ge 0\}$ is said to be locally equicontinuous if $\{T(t) : 0 \le t \le t_0\}$ is equicontinuous for each $t_0 > 0$. (See [5].)

In general, τ-continuity at 0 does not imply τ-continuity. Since τ-continuity at 0 implies sequential τ-density of $D(A)$, $\{T(t) : t \ge 0\}$ is τ-continuous if it is locally τ-equicontinuous by Proposition 3.3 in [3].

Theorem 4. *Let $\{T(t) : t \ge 0\}$ be an equicontinuous integrable semigroup on the norming dual pair $(X,\ Y)$ and let τ be a consistent locally convex topology on X. Then $\{T(t) : t \ge 0\}$ is τ-continuous at 0 if and only if $\tau - \lim_{\lambda\to\infty} \lambda R(\lambda)x = x$ for all $x \in X$. Moreover, $R(\lambda)$ is injective and so $\{T(t) : t \ge 0\}$ has a generator A such that $D(A)$ is sequentially τ-dense in X.*

Proof. The necessary condition is given in Theorem 2.10 in [3].

Since τ is consistent, there exists a separating collection $\mathcal{M}$ of bounded subsets of Y such that $\tau = \tau_{\mathcal{M}}$. Let $x \in X$ and $S \in \mathcal{M}$. Then there exists $\{x_n\}$ in $D(A)$ such that $\tau - \lim_{n\to\infty} x_n = x$. By Lemma 2, $T(t)x_n$ is continuous at 0 and so $T(t)x_n$ is τ-continuous at 0. For $y \in S$, we have

$$\begin{aligned}&|\langle T(t)x - x,\ y\rangle| \\ &\quad\leq\ |\langle T(t)x - T(t)x_n,\ y\rangle| + |\langle T(t)x_n - x_n,\ y\rangle| + |\langle x_n - x,\ y\rangle|.\end{aligned}$$

By the equicontinuity of $\{T(t) : t \geq 0\}$, there exists n such that

$$|\langle T(t)(x - x_n),\ y\rangle| + |\langle x_n - x,\ y\rangle| < \varepsilon/2.$$

Since $T(t)x_n$ is continuous at 0, there exists $\delta > 0$ such that $|\langle T(t)x_n - x_n,\ y\rangle| < \varepsilon/2$ for all $0 \leq t < \delta$. Since $S \in \mathcal{M}$ is arbitrary, $T(t)x$ is τ-continuous at 0 for all $x \in X$. □

Now we can prove the following convergence theorem for integrable semigroups.

Theorem 5. *Let τ be a consistent locally convex topology on X. Let $\{T_n(t) : t \geq 0\}$ and $\{T(t) : t \geq 0\}$ be integrable semigroups on $(X,\ Y)$ with the generators A_n and A, respectively satisfying $||T_n(t)|| \leq Me^{\omega t}$ and $||T(t)|| \leq Me^{\omega t}$ for all $t \geq 0$.*

Suppose that $\{T_n(t) : t \geq 0\}$ are τ-equicontinuous, uniformly in n, i.e. for any τ-continuous seminorm p on X, there exists a τ-continuous seminorm q on X such that $p(T_n(t)x) \leq q(x)$ for all $t \geq 0$, $x \in X$, and $n = 1,\ 2,\ \cdots$. Suppose that $\tau - \lim_{n\to\infty} R_n(\lambda)x = R(\lambda)x$ for all $x \in X$. Then

$$\tau - \lim_{n\to\infty} T_n(t)x = T(t)x$$

for all $x \in \overline{D(A)}$, and the convergence is uniform on bounded t-intervals.

Proof. Since τ is consistent, $\tau = \tau_{\mathcal{M}}$ for some separating collection $\mathcal{M}$ of bounded subsets of Y. Let $x \in D(A)$ and $S \in \mathcal{M}$. Then $x = R(\lambda)z$ for some $z \in X$ and for $y \in S$

$$\begin{aligned}&\langle T_n(t)x - T(t)x,\ y\rangle \\ &\quad= \langle T_n(t)R(\lambda)z - T(t)R(\lambda)z,\ y\rangle \\ &\quad= \langle T_n(t)R(\lambda)z - T_n(t)R_n(\lambda)z,\ y\rangle + \langle T_n(t)R_n(\lambda)z - R_n(\lambda)z,\ y\rangle \\ &\qquad+\langle R_n(\lambda)z - R(\lambda)z,\ y\rangle + \langle R(\lambda)z - T(t)R(\lambda)z,\ y\rangle.\end{aligned}$$

Since $\{T_n(t) : t \geq 0\}$ are τ-equicontinuous, uniformly in n, there exists a continuous seminorm q on X such that

$$\begin{aligned} |\langle T_n(t)R(\lambda)z - T_n(t)R_n(\lambda)z,\ y\rangle| &\leq p_S(T_n(t)(R(\lambda)z - R_n(\lambda)z)) \\ &\leq q(R(\lambda)z - R_n(\lambda)z). \end{aligned}$$

Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} R_n(\lambda)x = R(\lambda)x$ for all $x \in X$, there exists n_0 such that

$$|\langle T_n(t)R(\lambda)z - T_n(t)R_n(\lambda)z,\ y\rangle| + |\langle R_n(\lambda)z - R(\lambda)z,\ y\rangle| < \varepsilon/2$$

for all $n > n_0$. By Lemma 2, we have

$$\begin{aligned} &\langle T_n(t)R_n(\lambda)z - R_n(\lambda)z,\ y\rangle \\ &= \langle(\lambda R_n(\lambda) - I)\int_0^t T_n(s)zds,\ y\rangle \\ &= \int_0^\infty \lambda e^{-\lambda r}\langle T_n(r)\int_0^t T_n(s)zds - \int_0^t T_n(s)zds,\ y\rangle dr \\ &= \int_0^\infty \lambda e^{-\lambda r}\langle\int_0^t T_n(r+s)zds - \int_0^t T_n(s)zds,\ y\rangle dr \\ &= \int_0^\infty \lambda e^{-\lambda r}\langle\int_t^{t+r} T_n(s)zds - \int_0^r T_n(s)zds,\ y\rangle dr. \end{aligned}$$

Hence for $0 \leq t \leq T$ we have

$$\begin{aligned} &|\langle T_n(t)R_n(\lambda)z - R_n(\lambda)z,\ y\rangle| \\ &\leq \int_0^\infty \lambda e^{-\lambda r}(\int_t^{t+r} ||T_n(s)z||ds + \int_0^r ||T_n(s)z||ds)||y||dr \\ &\leq \int_0^\infty \lambda e^{-\lambda r}M(\int_t^{t+r} e^{\omega s}ds + \int_0^r e^{\omega s}ds)||z||||y||dr \\ &\leq M||z||||y||\int_0^\infty \lambda e^{-\lambda r}(e^{\omega(t+r)}r + e^{\omega r}r)dr \\ &= M||z||||y||\int_0^\infty \lambda e^{-(\lambda-\omega)r}r(e^{\omega t}+1)dr \\ &\leq M||z||||y||\frac{\lambda}{(\lambda-\omega)^2}(e^{\omega T}+1) \to 0 \end{aligned}$$

as $\lambda \to \infty$. By the similar argument for $|\langle R(\lambda)z - T(t)R(\lambda)z,\ y\rangle|$, there exists λ_0 such that

$$|\langle T_n(t)R_n(\lambda)z - R_n(\lambda)z,\ y\rangle| + |\langle R(\lambda)z - T(t)R(\lambda)z,\ y\rangle| < \varepsilon/2$$

for all $\lambda > \lambda_0$. Thus we have $|\langle T_n(t)x - T(t)x,\ y\rangle| < \varepsilon$ for all $n \geq n_0$ and $y \in S$. Since $S \in \mathcal{M}$ is arbitrary, the result follows. □

Remark 6. In addition to assumptions of Theorem 5, suppose that $\{T(t) : t \geq 0\}$ is continuous at 0. By Theorem 4, the above convergence theorem holds for all $x \in X$.

References

[1] W. Arendt, *Vector-valued Laplace transforms and Cauchy problems*, Israel J. Math. **59** (1987), 327 – 352.

[2] M. Kunze, *A general Pettis integral and application to transition semigroups*, Preprint, ArXiv: 0901.

[3] M. Kunze, *Continuity and equicontinuity of semigroups on norming dual pairs*, Semigroup Forum **79** (2009), 540 – 560.

[4] M. Hasse, *The functional calculus for sectorial operators*, Oper. Theory Adv. Appl. **169**, Birkhäser, 2006.

[5] K. Yosida, *Functional analysis*, Spinger-Verlag, Berlin, 1980.

Department of Mathematics
Seoul Women's University
Seoul, 139-774, Korea
E-mail: younglee@swu.ac.kr