Korean J. Math. **30** (2022), No. 2, pp. 231–238
http://dx.doi.org/10.11568/kjm.2022.30.2.231

INEQUALITIES FOR A POLYNOMIAL WHOSE ZEROS ARE WITHIN OR OUTSIDE A GIVEN DISK

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Abstract. In this paper we prove some results by using a simple but elegant techniques to improve and strengthen some generalizations and refinements of two widely known polynomial inequalities and thereby deduce some useful corollaries.

1. Introduction

Let $\mathbb{P}_n$ be the space of complex polynomials $P(z) := \sum_{j=1}^{n} c_j z^j$ of degree at most n. For each real number $k > 0$, we define the following:

$$D_k := \{z \in \mathbb{C} : |z| = k\}$$
$$D_k^- := \{z \in \mathbb{C} : |z| < k\}$$
$$D_k^+ := \{z \in \mathbb{C} : |z| > k\}$$

To be brief, we shall denote D_1, D_1^-, D_1^+ simply by D, D^-, D^+ respectively.
For every $P \in \mathbb{P}_n$ and P' as its derivative one form of the classical Bernstein inequality [2] for polynomials can be

$$\max_{z\in D} |P'(z)| \leq n \max_{z\in D} |P(z)|. \tag{1}$$

An improved form of this inequality due to Frappier, Rahman and Rusheweyh [3] states that, if $P(z)$ is a polynomial of degree n, then

$$\max_{z\in D} |P'(z)| \leq n \max_{1\leq k\leq 2n} |P(e^{\frac{ik\pi}{n}})|. \tag{2}$$

Clearly (2) represents a refinement of (1), since the maximum of $|P(z)|$ for $z \in D$ may be larger than the maximum of $|P(z)|$ taken over the $(2n)^{th}$ roots of unity, as is shown by the simpler example $P(z) = z^n + ia,\ a > 0$. Its worth mentioning that equality holds in (1) if and only if P has all its zeros at the origin. Dependence of inequalities on location of zeros made it prerequisite to learn the behaviour of inequality (1) while

Received January 11, 2022. Revised April 25, 2022. Accepted April 25, 2022.
2010 Mathematics Subject Classification: 30A10, 30C10, 30D15.
Key words and phrases: Polynomials, Maximum Modulus Principle, Inequalities in the complex domain, Zeros.
† This work was supported by DST-INSPIRE.
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restricting ourselves to the class of polynomials having zeros in a given region. Among various forms, we mention following two results of Malik [7] which stand out in terms of their impact in the journey carried out in this direction :
If $P \in \mathbb{P}_n$ is such that it does not vanish in the open disk D_k^-, then for $k \geq 1$

$$\max_{z\in D} |P'(z)| \leq \frac{n}{1+k} \max_{z\in D} |P(z)| \tag{3}$$

and in case it does not vanish in the open disk D_k^+, then for $k \leq 1$

$$\max_{z\in D} |P'(z)| \geq \frac{n}{1+k} \max_{z\in D} |P(z)|. \tag{4}$$

For $k = 1$, inequality (3) reduces to a result conjectured by Erdös and latter proved by Lax [6] , whereas inequality (4) reduces to a result proved by Turán [8]. In this direction the following result analogous to inequality (2) was proved by Aziz [1].

Theorem 1.1. *If $P(z)$ is a polynomial of degree n having no zeros in the disk D^-, then for every real α*

$$\max_{z\in D} |P'(z)| \leq \frac{n}{2}\{M_\alpha^2 + M_{\alpha+\pi}^2\}^{\frac{1}{2}} \tag{5}$$

where

$$M_\alpha = \max_{1\leq k\leq n} |P(e^{\frac{i(\alpha+2k\pi)}{n}})| \tag{6}$$

and $M_{\alpha+\pi}$ is obtained from (6) by replacing α by $\alpha + \pi$.

It was Dubinin [4] who improved on Tuŕan's result [8] and proved the following:

Theorem 1.2. *If $P(z) := \sum_{j=0}^{n} c_j z^j = c_n \prod_{j=1}^{n}(z - z_j),\ |z_j| \leq 1,\ j = 1, 2, \ldots, n$ is a polynomial of degree n, then the following inequality holds at each point z on the circle D such that $P(z) \neq 0$,*

$$\max_{z\in D} |P'(z)| \geq \left[\frac{n}{2} + \frac{1}{2}\frac{|c_n| - |c_0|}{|c_n| + |c_0|}\right] \max_{z\in D} P(z). \tag{7}$$

The following Lemma which is due to Aziz [1]:

Lemma 1.3. *If $P(z)$ is a polynomial of degree n and $P^*(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then for $|z| = 1$ and for every real α,*

$$|P'(z)|^2 + |(P^*(z))'|^2 \leq \frac{n^2}{2}(M_\alpha^2 + M_{\alpha+\pi}^2), \tag{8}$$

where M_α is defined by (6).

2. Main Results

In this paper we prove some results which besides the above two theorems refine some other polynomial inequalities. In fact we prove :

Theorem 2.1. *If $P(z) := \sum_{j=0}^{n} c_j z^j = c_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having no zeros in the disk D_k^-, $k \geq 1$, then for each point z on D_k such that $P(z) \neq 0$ and for every given real α,*

$$\max_{z\in D} |P'(z)| \leq$$

$$\frac{1}{2}\left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|P(z)|^2}{M_\alpha^2 + M_{\alpha+\pi}^2} - \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)(1+k)}\left\{n + \frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right\}\right]^{\frac{1}{2}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}$$

where M_α and $M_{\alpha+\pi}$ are defined by (6).

Proof. Suppose that $P(z) \neq 0$ for $z \in D_k$. Since $P(z) = c_n \sum_{j=1}^{n}(z - z_j)$, therefore

$$Re\frac{zP'(z)}{P(z)} = Re\sum_{j=1}^{n} \frac{z}{z - z_j}, |z_j| \geq k \geq 1.$$

Now for $z \neq z_j$ we have

$$\begin{aligned} Re\frac{z}{z - z_j} &= Re\frac{e^{i\theta}}{e^{i\theta} - r_j e^{i\theta_j}}, \ |r_j| \geq k \geq 1, \forall j = 1, 2, \ldots, n \\ &= Re\frac{1 - r_j e^{i(\theta - \theta_j)}}{1 - 2r_j cos(\theta - \theta_j) + r_j^2} \\ &= \frac{1 - r_j cos(\theta - \theta_j)}{1 - 2r_j cos(\theta - \theta_j) + r_j^2} \\ &\leq \frac{1}{1 + r_j} \\ &= \frac{1}{1 + |z_j|}. \end{aligned}$$

Therefore,

$$Re\frac{zP'(z)}{P(z)} \leq \sum_{j=1}^{n} \frac{1}{1 + |z_j|}. \tag{9}$$

Also if $P^*(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then we have for $z \in D$,

$$|(P^*(z))'| = |nP(z) - zP'(z)|.$$

This gives for $z \in D$

$$\begin{aligned} \left|\frac{z(P^*(z))'}{P(z)}\right|^2 &= \left|n - z\frac{P'(z)}{P(z)}\right|^2 \\ &= n^2 + \left|\frac{zP'(z)}{P(z)}\right|^2 - 2nRe\left(\frac{zP'(z)}{P(z)}\right). \\ &\geq n^2 + \left|z\frac{P'(z)}{P(z)}\right|^2 - 2n\left(\sum_{j=1}^{n} \frac{1}{1 + |z_j|}\right). \end{aligned} \tag{10}$$

This gives

$$|(P^*(z))'|^2 \geq n^2|P(z)|^2 + |zP'(z)|^2 - 2n|P(z)|^2\left(\sum_{j=1}^{n}\frac{1}{1+|z_j|}\right).$$

Equivalently for $|z| = 1$

$$|P'(z)|^2 \leq |(P^*(z))'|^2 - n^2|P(z)|^2 + 2n|P(z)|^2\left(\sum_{j=1}^{n}\frac{1}{1+|z_j|}\right).$$

Therefore

$$2|P'(z)|^2 \leq |P'(z)|^2 + |(P^*(z))'|^2 - n\left\{n - 2\sum_{j=1}^{n}\frac{1}{1+|z_j|}\right\}|P(z)|^2. \tag{11}$$

Now using Lemma 1.3 in (11), we get

$$2|P'(z)|^2 \leq \frac{n^2}{2}\left\{(M_\alpha^2 + M_{\alpha+\pi}^2)\right\} - n\left(n - 2\sum_{j=1}^{n}\frac{1}{1+|z_j|}\right)|P(z)|^2,$$

which gives,

(12)

$$\begin{aligned}
&4|P'(z)|^2 \leq n^2(M_\alpha^2 + M_{\alpha+\pi}^2) + 4n|P(z)|^2\sum_{j=1}^{n}\frac{1}{1+|z_j|} - 2n^2|P(z)|^2\\
&= \left[n^2 + \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)}\sum_{j=1}^{n}\frac{1}{1+|z_j|} - \frac{2n^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)}|P(z)|^2\right](M_\alpha^2 + M_{\alpha+\pi}^2)\\
&= \left[n^2 - \frac{2n^2(k-1)|P(z)|^2}{(k+1)(M_\alpha^2 + M_{\alpha+\pi}^2)} - \frac{4n^2|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)(k+1)} + \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)}\sum_{j=1}^{n}\frac{1}{1+|z_j|}\right\}\right](M_\alpha^2 + M_{\alpha+\pi}^2)\\
&= \left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)} - \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)}\left\{\frac{n}{1+k} - \frac{1}{1+k}\sum_{j=1}^{n}\frac{k+1}{1+|z_j|}\right\}\right](M_\alpha^2 + M_{\alpha+\pi}^2)\\
&\leq \left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)} - \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)}\left\{\frac{n}{1+k} - \frac{1}{1+k}\sum_{j=1}^{n}\frac{k-|z_j|}{k+|z_j|}\right\}\right](M_\alpha^2 + M_{\alpha+\pi}^2)\\
&= \left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)} - \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)}\left\{\frac{n}{1+k} - \frac{1}{1+k}\sum_{j=1}^{n}\frac{1-\frac{|z_j|}{k}}{1+\frac{|z_j|}{k}}\right\}\right](M_\alpha^2 + M_{\alpha+\pi}^2).
\end{aligned}$$

We have by a simple application of principle mathematical induction,

$$\sum_{j=1}^{n}\frac{1-c_j}{1+c_j} \leq \frac{1-\prod_{j=1}^{n}c_j}{1+\prod_{j=1}^{n}c_j}\ \forall\ n \in \mathbb{N} \text{ and } c_j \geq 1,\ j = 1, 2, \ldots, n.$$

Using this fact in (12), as $\frac{|z_j|}{k} \geq 1$, and then using Vitali's formula, we get

$$|P'(z)|$$

$$\leq \frac{1}{2}\left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)} - \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)(1+k)}\left\{n - \frac{1-\prod_{j=1}^n \frac{|z_j|}{k}}{1+\prod_{j=1}^n \frac{|z_j|}{k}}\right\}\right]^{\frac{1}{2}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)} - \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)(1+k)}\left\{n - \frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right\}\right]^{\frac{1}{2}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)} - \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)(1+k)}\left\{n + \frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right\}\right]^{\frac{1}{2}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}.$$

This completes the proof of theorem. □

For $k = 1$, Theorem 2.1 reduces to the following:

Corollary 2.2. *If $P(z) := c_n\prod_{j=1}^n(z - z_j)$, $|z_j| \geq 1$, $j = 1, 2, \ldots, n$ is a polynomial of degree n then for each point z on D such that $P(z) \neq 0$ and every given real α*

$$\max_{z\in D}|P'(z)| \leq \frac{1}{2}\left[n^2 - \frac{2n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)}\left\{n + \frac{|c_0| - |c_n|}{|c_0| + |c_n|}\right\}\right]^{\frac{1}{2}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}, \tag{13}$$

where M_α and $M_{\alpha+\pi}$ are defined by (6).

Remark 2.3. Since $\frac{|c_0|-|c_n|}{|c_0|+|c_n|} \geq 0$, therefore Corollary 2.2 is an improvement over Theorem 1.1.

Remark 2.4. We have

$$\left(1 - \sqrt{\left|\frac{k^n c_n}{c_0}\right|}\right)^2 \geq 0,$$

therefore

$$\sqrt{\left|\frac{k^n c_n}{c_0}\right|} + \left|\frac{k^n c_n}{c_0}\right|^{\frac{3}{2}} \geq 2\left|\frac{k^n c_n}{c_0}\right|.$$

Equivalently

$$1 - \left|\frac{k^n c_n}{c_0}\right| \geq 1 + \left|\frac{k^n c_n}{c_0}\right| - \sqrt{\left|\frac{k^n c_n}{c_0}\right|} - \left|\frac{k^n c_n}{c_0}\right|^{\frac{3}{2}},$$

or

$$\frac{1 - |\frac{k^n c_n}{c_0}|}{1 + |\frac{k^n c_n}{c_0}|} \geq \frac{\sqrt{|c_0|} - \sqrt{k^n|c_n|}}{\sqrt{|c_0|}}$$

which gives,

$$\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|} \geq \frac{\sqrt{|c_0|} - \sqrt{k^n|c_n|}}{\sqrt{|c_0|}}$$

Therefore from Theorem 2.1, we get:

Corollary 2.5. *If $P(z) := \sum_{j=0}^{n} c_j z^j = c_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having no zeros in the disk D_k^-, $k \geq 1$, then for each point z on D_k such that $P(z) \neq 0$ and for every given real α,*

$$\max_{z\in D}|P'(z)| \leq$$
$$\frac{1}{2}\left[n^2 - \frac{2n^2(k-1)}{k+1}\frac{|P(z)|^2}{M_\alpha^2 + M_{\alpha+\pi}^2} - \frac{4n|P(z)|^2}{(M_\alpha^2 + M_{\alpha+\pi}^2)(1+k)}\left\{n + \frac{\sqrt{|c_0|} - \sqrt{k^n|c_n|}}{\sqrt{|c_0|}}\right\}\right]^{\frac{1}{2}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}$$

where M_α and $M_{\alpha+\pi}$ are defined by (6).

We next prove the following result which is a generalization of Theorem 1.2.

Theorem 2.6. *Suppose $P(z) := \sum\limits_{j=0}^{n} c_j z^j = c_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n having no zeros in the disk D_k^+, $k \leq 1$, then*

$$\max_{z\in D}|P'(z)| \geq \left[\frac{n}{1+k} + \frac{k}{1+k}\left\{\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right\}\right]\max_{z\in D}|P(z)|.$$

The result is sharp and equality holds for the polynomial $P(z) = \left(\frac{z+k}{1+k}\right)^n$.

Proof. Since $P(z)$ has no zeros in D_k^+, therefore, we can write $P(z) := \sum\limits_{j=1}^{n} c_j z^j = c_n \sum\limits_{j=1}^{n}(z - z_j)$, where $|z_j| \leq k \leq 1, \forall j = 1, 2, \ldots, n$. This gives, for the points $z \in D_k$, such that $P(z) \neq 0$

$$Re\left(\frac{zP'(z)}{P(z)}\right) = Re\sum_{j=1}^{n}\frac{z}{z - z_j}.$$

Hence for $z \in D$, we have

$$\begin{aligned}
\left|\frac{P'(z)}{P(z)}\right| &\geq Re\left(\frac{zP'(z)}{P(z)}\right)\\
&= Re\sum_{j=1}^{n}\frac{z}{z - z_j}\\
&\geq \frac{1}{1+|z_j|}\\
&= \frac{n}{1+k} - \sum_{j=1}^{n}\left(-\frac{1}{k+1} - \frac{1}{1+|z_j|}\right)\\
&= \frac{n}{1+k} + \sum_{j=1}^{n}\frac{k - |z_j|}{(k+1)(1+|z_j|)}\\
&\geq \frac{n}{1+k} + \frac{1}{1+k}\sum_{j=1}^{n}\frac{k - |z_j|}{k + |z_j|}.
\end{aligned} \tag{14}$$

From (14), we get

$$\max_{z\in D}|P'(z)| \geq \left[\frac{n}{1+k} + \frac{1}{1+k}\sum_{j=1}^{n}\frac{k-|z_j|}{k+|z_j|}\right]\max_{z\in D}|P(z)| \tag{15}$$
$$= \left[\frac{n}{1+k} + \frac{1}{1+k}\sum_{j=1}^{n}\frac{1-\frac{|z_j|}{k}}{1+\frac{|z_j|}{k}}\right]\max_{z\in D}|P(z)|.$$

We have by a simple application of principle of mathematical induction, $\sum_{j=1}^{n}\frac{1-c_j}{1+c_j} \geq \frac{1-\prod_{j=1}^{n}c_j}{1+\prod_{j=1}^{n}c_j}$ $\forall\ n\in\mathbb{N}$ and $c_j \leq 1$.
Using this fact in (15), as $\frac{|z_j|}{k} \leq 1$, and then using Vitali's formula, we get

$$\max_{z\in D}|P'(z)| \geq \left[\frac{n}{1+k} + \frac{1}{1+k}\left\{\frac{1-\prod_{j=1}^{n}\frac{|z_j|}{k}}{1-\prod_{j=1}^{n}\frac{|z_j|}{k}}\right\}\right]\max_{z\in D}|P(z)|.$$
$$= \left[\frac{n}{1+k} + \frac{1}{1+k}\left\{\frac{k^n|c_n|-|c_0|}{k^n|c_n|+|c_0|}\right\}\right]\max_{z\in D}|P(z)|.$$

This completes the proof of theorem . □

Remark 2.7. Theorem 2.6 is in fact a refinement of the result due to Malik (inequality (4)) and also generalises a result due to Dubinin [4].

It is easy to verify that

$$\frac{k^n|c_n|-|c_0|}{k^n|c_n|+|c_0|} \geq \frac{\sqrt{k^n|c_n|}-\sqrt{|c_0|}}{\sqrt{k^n|c_n|}},$$

therefore, from Theorem 2.6 we have

Corollary 2.8. *Suppose $P(z) := \sum\limits_{j=0}^{n} c_j z^j = c_n\prod_{j=1}^{n}(z-z_j)$ is a polynomial of degree n having no zeros in the disk D_k^+, $k \leq 1$ then*

$$\max_{z\in D}|P'(z)| \geq \left[\frac{n}{1+k} + \frac{k}{1+k}\left\{\frac{\sqrt{k^n|c_n|}-\sqrt{|c_0|}}{\sqrt{k^n|c_n|}}\right\}\right]\max_{z\in D}|P(z)|.$$

For $k=1$, it reduces to a result due to Dubinin [5].

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