Korean J. Math. **30** (2022), No. 1, pp. 161–173
http://dx.doi.org/10.11568/kjm.2022.30.1.161

FIXED POINT THEOREM VIA MEIR-KEELER CONTRACTION IN RECTANGULAR M_b-METRIC SPACE

Mohammad Asim* and Meenu

Abstract. In this paper, we present a fixed point theorem for Meir-Keeler contraction in the framework of Rectangular M_b-metric Space. Our main result improves some existing results in literature. An example is also adopted to exhibit the utility of our main result.

1. Introduction

Fixed point theory is one of the most powerful tools of modern mathematics and is considered a core subject of non-linear analysis. Fixed point theory is applied to many areas of current interest such as Functional Analysis, Operator Theory, Approximation Theory, succession approximation, integrative equations, variational inequalities, and several others, with topological considerations playing a crucial role. The strength of fixed point theory lies in its applications scattered throughout the existing literature, even in diverse fields such as Biology, Chemistry, Physics, Engineering, Game theory, Economics. Moreover, the metric fixed point theory has been a flourishing area of research for many mathematicians. In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem known as the "Banach Contraction Principle" [9] which is one of the most crucial and fruitful results of analysis and is considered as the main source of metric fixed point theory. Since then, this pioneering work has been studied and generalized in many different directions (see [1, 4, 6, 12–17, 21, 23–25, 30, 36]).

One of them was put forwarded by Meir and Keeler in 1969, in which they restricted the theorem to weakly uniformly strict contraction and proved Banach's theorem, i.e.,

Theorem 1.1. [29] *Let (X, d) be a complete metric space. Define a mapping $T : X \to X$ such that for given $\epsilon > 0$, there exists $\delta > 0$, T satisfy the following condition:*

$$\epsilon \leq d(x, y) < \epsilon + \delta \implies d(Tx, Ty) < \epsilon.$$

Then T has a unique fixed point.

Later on, the above Theorem 1.1 has been generalized by several authors in all possible ways. One may refer to [20], [31], [7], [27], [22], [32]. On the other hand,

Received January 31, 2022. Revised March 18, 2022. Accepted March 18, 2022.
2010 Mathematics Subject Classification: 47H10, 54H25.
Key words and phrases: Fixed point, Meir-Keeler contraction, rectangular M_b-metric space.
∗ Corresponding author.
© The Kangwon-Kyungki Mathematical Society, 2022.
This is an Open Access article distributed under the terms of the Creative commons Attribution Non-Commercial License (http://creativecommons.org/licenses/by-nc/3.0/) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.

various types of generalized metric spaces are proposed, i.e., partial metric space [28], b-metric space [11], partial b-metric space [37], rectangular metric space [10], rectangular b-metric space [19], M-metric space [3], M_b-metric space [33], rectangular M-metric space [34], and much more.

Now, we recall the definition of partial metric space as follows:

Definition 1.2. [28] Let X be a non-empty set. A mapping $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X if (for all $x, y, z \in X$):

1. $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
2. $p(x, x) \leq p(x, y)$,
3. $p(x, y) = p(y, x)$,
4. $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is said to be a partial metric space.

In 2000, Branciari [10] generalized the idea of metric space by replacing the triangular inequality with more general inequality, namely, quadrilateral inequality (namely, involving four points instead of three) for introducing the notion of rectangular metric spaces as follows:

Definition 1.3. [10] Let X be a non-empty set. A mapping $r : X \times X \to \mathbb{R}^+$ is said to be a rectangular metric on X if, r satisfies the following (for all $x, y \in X$ and all distinct $u, v \in X \setminus \{x, y\}$,):

1. $r(x, y) = 0$ if and only if $x = y$,
2. $r(x, y) = r(y, x)$,
3. $r(x, y) \leq r(x, u) + r(u, v) + r(v, y)$.

Then the pair (X, r) is said to be a rectangular metric space.

In 2014, Shukla [38] introduced partial rectangular metric spaces as generalization of rectangular metric spaces as follows:

Definition 1.4. [38] Let X be a non-empty set. A mapping $p_r : X \times X \to \mathbb{R}^+$ is said to be a partial rectangular metric on X if, for all $x, y \in X$ and all distinct $u, v \in X \setminus \{x, y\}$, it satisfies the following axioms:

($1p_r$) $x = y \Leftrightarrow p_r(x, x) = p_r(x, y) = p_r(y, y)$,
($2p_r$) $p_r(x, x) \leq p_r(x, y)$,
($3p_r$) $p_r(x, y) = p_r(y, x)$,
($4p_r$) $p_r(x, y) \leq p_r(x, u) + p_r(u, v) + p_r(v, y) - p_r(u, u) - p_r(v, v)$.

The pair (X, p_r) is said to be a partial rectangular metric space.

In 2014, Asadi et at [3] enlarged the class of partial metric space by introducing M-metric spaces i.e.,

Notation 1. [3] consider these notations:

1. $m_{x,y} = \min\{m(x, x), m(y, y)\}$,
2. $M_{x,y} = \max\{m(x, x), m(y, y)\}$.

Definition 1.5. [3] Let X be a non-empty set and $m : X \times X \to [0, \infty)$ be a mapping then m is a m-metric if it satisfies the following conditions:

1. $m(x, x) = m(y, y) = m(x, y) \Leftrightarrow x = y$,
2. $m_{xy} \leq m(x, y)$,

3. $m(x,y) = m(y,x)$,
4. $(m(x,y) - m_{x,y}) \leq (m(x,z) - m_{x,z}) + (m(z,y) - m_{z,y})$

for all $x, y, z \in X$. The pair (X, m) is known as M-metric space.

In 2016, Nabil Mlaiki et al. [33] proposed the generalization of M-metric Space, i.e. M_b metric space and proved some fixed point results which helps to great deal for ensuring the secured communication in computer i.e.,

Notation 2. [33] Consider the following notation

1. $m_{b_{x,y}} = \min\{m_b(x,x), m_b(y,y)\}$,
2. $M_{b_{x,y}} = \max\{m_b(x,x), m_b(y,y)\}$.

Definition 1.6. [33] Let X be a non-empty set and $m_b : X \times X \to [0, \infty)$ be a mapping. Then m_b is a M_b-metric if it satisfies the following conditions:

1. $m_b(x,x) = m_b(y,y) = m_b(x,y) \Leftrightarrow x = y$,
2. $m_{b_{x,y}} \leq m_b(x,y)$,
3. $m_b(x,y) = m_b(y,x)$,
4. There exists a real number $s \geq 1$ such that
   $m_b(x,y) - m_{b_{x,y}} \leq s[(m_b(x,z) - m_{b_{x,z}}) + (m_b(z,y) - m_{b_{z,y}})] - m_b(z,z)$

for all $x, y, z \in X$. Then s is called the coefficient of the M_b -metric space (X, m_b).

In 2019, M. Asim et al. [5] introduced Rectangular M_b-metric space and proved an analogue of Banach contraction principle i.e.,

Notation 3. [5] Following notations are useful:

1. $r_{mb_{x,y}} = \min\{r_{mb}(x,x), r_{mb}(y,y)\}$,
2. $R_{mb_{x,y}} = \max\{r_{mb}(x,x), r_{mb}(y,y)\}$.

Definition 1.7. [5] Let X be a non-empty set and $r_{mb} : X \times X \to [0, \infty)$ be a mapping then r_{mb} is a rectangular M_b-metric if it satisfies the following conditions:

1. $r_{mb}(x,x) = r_{mb}(x,y) = r_{mb}(y,y)$ if and only if $x = y$,
2. $r_{mb_{x,y}} \leq r_{mb}(x,y)$,
3. $r_{mb}(x,y) = r_{mb}(y,x)$,
4. there exists a real number $s \geq 1$ such that

$$\begin{aligned} r_{mb}(x,y) - r_{mb_{x,y}} \leq\ & s[(r_{mb}(x,u) - r_{mb_{x,u}}) + (r_{mb}(u,v) - r_{mb_{u,v}}) \\ & + (r_{mb}(v,y) - r_{mb_{v,y}})] - r_{mb}(u,u) - r_{mb}(v,v) \end{aligned}$$

for all $x, y \in X$ and all distinct $u, v \in X \backslash \{x, y\}$. The pair (X, r_{mb}) is called rectangular M_b-metric space.

Let us recall the fixed point theorem proved in [5]:

Theorem 1.8. [5] *Let (X, r_{mb}) be a rectangular M_b-metric space with coefficient $s \geq 1$. Define a mapping $T : X \to X$ such that it satisfies the following conditions:*

1. *$r_{mb}(Tx, Ty) \leq k r_{mb}(x,y)$ for all $x, y \in X$, where $k \in [0, \frac{1}{s})$*
2. *(X, r_{mb}) is r_{mb}-complete.*

Then T has a unique fixed point ξ such that $r_{mb}(\xi, \xi) = 0$.

Now, let us recall some important definitions for further discussion.

Definition 1.9. [32] Let $\tau_{r_{mb}}$ be the topology generated on X by r_{mb}-metric. A open-r_{mb} ball in $(X, \tau_{r_{mb}})$ be defined as

$$B_{r_{mb}}(x, \epsilon) = \{y \in X \mid r_{mb}(x, y) - r_{mb_{x,y}} < \rho\}.$$

Let $B = \{B_{r_{mb}}(x, \rho) \mid x \in X, \rho > 0\}$ be the family of open r_{mb}-balls describing the base of topology.

Definition 1.10. [32] Let X be a non-empty set and define a self-mapping $T : X \to X$ and a mapping $\alpha : X \times X \to [0, \infty)$ such that

$$\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1$$

for all $x, y \in X$. Then T is said to be α-admissible.

Definition 1.11. [32] Let $T : X \to X$ be an α-admissible mapping $\alpha : X \times X \to [0, \infty)$ such that

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, z) \geq 1 \text{ then } \alpha(x, z) \geq 1$$

for all $x, y, z \in X$. Then T is said to be triangular α-admissible

Definition 1.12. [32] Let (X, m_b) is a M_b-metric space with coefficient $s \geq 1$. Define an α-admissible mapping $T : X \to X$ such that for every $\epsilon > 0$ there exists $\delta > 0$, we have

$$\epsilon \leq \beta(m_b(x, y))M(x, y) < \epsilon + \delta$$

this implies that $\alpha(x, y)m_b(Tx, Ty) < \epsilon$, where

$$M(x, y) = \max(m_b(x, y), m_b(Tx, x), m_b(Ty, y)) \text{ for all } x, y \in \mathbf{N}$$

and $\beta : [0, \infty) \to [0, \frac{1}{s})$ is the given mapping. Then T is known as generalized Meir-Keeler contraction of Type(I).

Definition 1.13. [32] Let (X, m_b) is a M_b-metric space with coefficient $s \geq 1$. Define a α-admissible mapping $T : X \to X$ such that for every $\epsilon > 0$ there exist $\delta > 0$, we have

$$\epsilon \leq \beta(m_b(x, y))N(x, y) < \epsilon + \delta$$

this implies that $\alpha(x, y)m_b(Tx, Ty) < \epsilon$, where

$$N(x, y) = \max\left(m_b(x, y), \frac{1}{2}[m_b(Tx, x) + m_b(Ty, y)]\right) \text{ for all } x, y \in \mathbf{N}$$

and $\beta : [0, \infty) \to [0, \frac{1}{s})$ is the given mapping. Then T is known as generalized Meir-Keeler contraction of Type(II).

Remark 1.14. 1. Let T be a generalized Meir-Keeler contraction of type(I) then

$$\alpha(x, y)m_b(Tx, Ty) < \beta(m_b(x, y))M(x, y)$$

for all $x, y \in X$ and $M(x, y) > 0$.
2. $N(x, y) \leq M(x, y)$ for all $x, y \in X$.

In this paper, we establish some of the fixed point theorem for a Meir-Keeler type contraction in rectangular M_b-metric spaces. Also, we extend and improve some existing results in the literature of fixed point theory.

2. Main Result

Now, we prove fixed point theorem for Meir-Keeler type contraction in Rectangular M_b-metric space.

THEOREM 2.1. *Let (X, r_{mb}) is a rectangular M_b-metric space with coefficient s. Define a triangular α-admissible mapping $T : X \to X$ such that it satisfies the following conditions:*

1. *there exists $x_0 \in X$ for which $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$.*
2. *Let x_n be a r_{mb} convergent sequence in X, i.e., $\{x_n\} \to z$ as $n \to \infty$. Also, $\alpha(x_n, x_m) \geq 1$ and $\alpha(x_n, x) \geq 1$ for all $n, m \in N$.*
3. *for every $\epsilon > 0$ there exists $\delta > 0$ such that the following condition hold:*

$$2s\epsilon \leq r_{mb}(y, Ty)\frac{1 + r_{mb}(x, Tx)}{1 + M(x, y)} + N(x, y) < s(2\epsilon + \delta)$$

then $\alpha(x, y)r_{mb}(Tx, Ty) < \epsilon$.

Then T has a unique fixed point ξ in X.

Proof. Let x_0 be any arbitrary point in X satisfying the first condition and define a sequence $\{x_n\}$ as :

$$x_1 = Tx_0,\ x_2 = T^2x_0,\ x_3 = T^3x_0, \cdots, x_n = T^nx_0, \cdots$$

Without loss of generality, let us suppose that for all $n \in N$, we have $x_{n+1} \neq x_n$. T being α-admissible, therefore $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$. Also $\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$. Proceeding in the same way, we get $\alpha(x_n, x_{n+1}) \geq 1$. We claim that $\lim_{n\to\infty} r_{mb}(x_n, x_{n+1}) = 0$.

$$\begin{aligned} r_{mb}(x_n, x_{n+1}) &= r_{mb}(Tx_{n-1}, Tx_n) \\ &\leq \alpha(x_{n-1}, x_n)r_{mb}(Tx_{n-1}, Tx_n) \\ &< \frac{1}{2s}r_{mb}(x_n, x_{n+1})\frac{1 + r_{mb}(x_{n-1}, x_n)}{1 + M(x_{n-1}, x_n)} + N(x_{n-1}, x_n) \end{aligned}$$

where

$$M(x_{n-1}, x_n) = \max\{r_{mb}(x_{n-1}, x_n), r_{mb}(x_n, x_{n+1})\}.$$

Let if possible, $M(x_{n-1}, x_n) = r_{mb}(x_n, x_{n+1})$ then,

$$\begin{aligned} r_{mb}(x_n, x_{n+1}) = r_{mb}(Tx_{n-1}, Tx_n) &\leq \alpha(x_{n-1}, x_n)r_{mb}(Tx_{n-1}, Tx_n) \\ &< \frac{1}{2s}r_{mb}(x_n, x_{n+1})\frac{1 + r_{mb}(x_{n-1}, x_n)}{1 + r_{mb}(x_n, x_{n+1})} \\ &\quad + \frac{1}{2s}r_{mb}(x_n, x_{n+1}) \\ &< \frac{1}{2s}r_{mb}(x_n, x_{n+1}) + \frac{1}{2s}r_{mb}(x_n, x_{n+1}) \\ &= \frac{1}{s}r_{mb}(x_n, x_{n+1}) \leq r_{mb}(x_n, x_{n+1}), \end{aligned}$$

which is a contradiction. Therfore, by using $M(x_{n-1}, x_n) = r_{mb}(x_n, x_{n+1})$ and $N(x_{n-1}, x_n) \leq M(x_{n-1}, x_n)$, we have

$$\begin{aligned} r_{mb}(x_n, x_{n+1}) &< \frac{1}{2s} r_{mb}(x_n, x_{n+1}) \frac{1 + r_{mb}(x_{n-1}, x_n)}{1 + r_{mb}(x_{n-1}, x_n)} + \frac{1}{2s} r_{mb}(x_{n-1}, x_n) \\ &= \frac{1}{2s} r_{mb}(x_n, x_{n+1}) + \frac{1}{2s} r_{mb}(x_{n-1}, x_n) \\ &\leq \frac{1}{2s} r_{mb}(x_{n-1}, x_n) + \frac{1}{2s} r_{mb}(x_{n-1}, x_n) \\ &= \frac{1}{s} r_{mb}(x_{n-1}, x_n) \\ &\leq \frac{1}{s^2} r_{mb}(x_{n-2}, x_{n-1}) \\ &\vdots \\ &\leq \frac{1}{s^n} r_{mb}(x_0, x_1), \end{aligned}$$

taking limit $n \to \infty$, we get

$$\lim_{n\to\infty} r_{mb}(x_n, x_{n+1}) = 0.$$

Similarly

$$r_{mb}(x_n, x_n) = r_{mb}(Tx_{n-1}, Tx_{n-1}) \leq \frac{1}{s} r_{mb}(x_{n-1}, x_{n-1}) \leq \cdots \leq \frac{1}{s^n} r_{mb}(x_0, x_0).$$

Taking limit $n \to \infty$, we have

$$\lim_{n\to\infty} r_{mb}(x_n, x_n) = 0.$$

Now, we will prove that if $n \neq m$ then $x_n \neq x_m$. Let if possible, $x_n = x_m$ for some $n > m$, such that

$$x_{n+1} = Tx_n = Tx_m = x_{m+1}.$$

Thus

$$r_{mb}(x_m, x_{m+1}) = r_{mb}(x_n, x_{n+1}) < \frac{1}{s} r_{mb}(x_{n-1}, x_n) < \cdots < \frac{1}{s^{n-m}} r_{mb}(x_m, x_{m+1}),$$

which is the contradiction. Hence $x_n \neq x_m$ for all $n \neq m$.

Let $\epsilon > 0$ and $\delta' = \min\{\delta, \epsilon, 1\}$. As $\lim\limits_{n\to\infty} r_{mb}(x_n, x_{n+1}) = 0$ therefore, there exists $p \in N$ such that

$$r_{mb}(x_m, x_{m+1}) < \frac{\delta'}{4} \text{ for all } m \geq p.$$

Let $\rho = s\left(2\epsilon + \frac{\delta'}{4}\right)$ and define a set

$$B_{r_{mb}}[x_p, \rho] = \{x_i \mid i \geq p, r_{mb}(x_i, x_p) - r_{mb_{x_i,x_p}} < \rho\}.$$

Now, we have to show that T maps $B_{r_{mb}}[x_p, \rho]$ to itself. Let $x_l \in B_{r_{mb}}[x_p, \rho]$ then,

$$r_{mb}(x_l, x_p) - r_{mb_{x_l,x_p}} < \rho$$

If $l = p$, then $Tx_l = Tx_p = x_{p+1} \in B_{r_{mb}}[x_p, \rho]$. Without loss of generality, consider $l > p$.

Case1: Let $2s\epsilon \leq r_{mb}(x_l, x_p)$, such that

$$2s\epsilon < r_{mb}(x_l, x_p) - m_{mb_{x_l,x_p}} < \rho.$$

Also, we know that $r_{mb}(x_l, x_p) \leq N(x_l, x_p)$ and $\epsilon \geq \frac{1}{2s} r_{mb}(x_l, x_p)$. Hence

$$\epsilon \leq \frac{1}{2s} r_{mb}(x_p, x_{p+1}) \frac{1 + r_{mb}(x_l, x_{l+1})}{1 + M(x_l, x_p)} + \frac{1}{2s} N(x_l, x_p).$$

Therefore,

$$\frac{1}{2s} r_{mb}(x_p, x_{p+1}) \frac{1 + r_{mb}(x_l, x_{l+1})}{1 + M(x_l, x_p)} + \frac{1}{2s} N(x_l, x_p) < \epsilon + \frac{\delta'}{4}$$

as a result

$$2s\epsilon \leq r_{mb}(x_p, x_{p+1}) \frac{1 + r_{mb}(x_l, x_{l+1})}{1 + M(x_l, x_p)} + N(x_l, x_p) < s(2\epsilon + \delta').$$

Hence, by using (3) condition of the theorem, we have,

$$r_{mb}(Tx_l, Tx_p) \leq \alpha(x_l, x_p) r_{mb}(Tx_l, Tx_p) < \epsilon.$$

Hence

$$\begin{aligned} r_{mb}(Tx_l, x_p) - r_{mb_{Tx_l, x_p}} &\leq s[(r_{mb}(Tx_l, x_l) - r_{mb_{Tx_l, x_l}}) \\ &\quad +(r_{mb}(x_l, x_{l-1}) - r_{mb_{x_l, x_{l-1}}}) \\ &\quad +(r_{mb}(x_{l-1}, x_p) - r_{mb_{x_{l-1}, x_p}})] \\ &\leq s[r_{mb}(Tx_l, x_l) + r_{mb}(x_l, x_{l-1}) + r_{mb}(x_{l-1}, x_p)] \\ &\leq s[\frac{\delta'}{8} + \frac{\delta'}{8} + \epsilon] \\ &< s\left(\frac{\delta'}{4} + 2\epsilon\right) \\ &< \rho. \end{aligned}$$

Therefore, $x_{l+1} \in B_{r_{mb}}[x_p, \rho]$.

Case2: Let $r_{mb}(x_l, x_p) \leq 2s\epsilon$, such that

$$\begin{aligned} & r_{mb}(Tx_l, x_p) - r_{mb_{Tx_l, x_p}} \\ \leq\ & s[(r_{mb}(Tx_l, Tx_{l-1}) - r_{mb_{Tx_l, Tx_{l-1}}}) + (r_{mb}(Tx_{l-1}, Tx_{l-2}) \\ & -r_{mb_{Tx_{l-1}, Tx_{l-2}}}) + (r_{mb}(Tx_{l-2}, x_p) - r_{mb_{Tx_{l-2}, x_p}})] \\ \leq\ & s[r_{mb}(Tx_l, Tx_{l-1}) + r_{mb}(Tx_{l-1}, Tx_{l-2}) + r_{mb}(Tx_{l-2}, x_p)] \\ \leq\ & s[\alpha(x_l, x_{l-1}) r_{mb}(Tx_l, Tx_{l-1}) + \alpha(x_{l-1}, x_{l-2}) r_{mb}(Tx_{l-1}, Tx_{l-2})] \\ & +s(r_{mb}(Tx_{l-2}, x_p)) \\ <\ & s\left[\frac{1}{2s} r_{mb}(x_{l-1}, x_l) \frac{1 + r_{mb}(x_l, Tx_l)}{1 + M(x_l, x_{l-1})} + \frac{1}{2s} N(x_l, x_{l-1})\right] + \\ & s\left[\frac{1}{2s} r_{mb}(x_{l-2}, x_{l-1}) \frac{1 + r_{mb}(x_{l-1}, Tx_{l-1})}{1 + M(x_{l-1}, x_{l-2})} + \frac{1}{2s} N(x_{l-1}, x_{l-2})\right] \\ & +s(r_{mb}(Tx_{l-2}, x_p)) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}r_{mb}(x_{l-1},x_l)+\frac{r_{mb}(x_{l-1},x_l)r_{mb}(x_l,x_{l+1})}{2(1+r_{mb}(x_l,x_{l-1}))}+\frac{1}{2}N(x_l,x_{l-1})\\
&\quad+\frac{1}{2}r_{mb}(x_{l-2},x_{l-1})+\frac{r_{mb}(x_{l-2},x_{l-1})r_{mb}(x_{l-1},x_l)}{2(1+r_{mb}(x_{l-1},x_{l-2}))}\\
&\quad+\frac{1}{2}N(x_{l-1},x_{l-2})+s\frac{\delta'}{8}\\
&\leq \frac{\delta'}{8}+\frac{r_{mb}(x_{l-1},x_l)r_{mb}(x_l,x_{l+1})}{2(1+r_{mb}(x_l,x_{l-1}))}+\frac{1}{2}N(x_l,x_{l-1})\\
&\quad+\frac{r_{mb}(x_{l-2},x_{l-1})r_{mb}(x_{l-1},x_l)}{2(1+r_{mb}(x_{l-1},x_{l-2}))}+\frac{1}{2}N(x_{l-1},x_{l-2})+s\frac{\delta'}{8}
\end{aligned}$$

also, by using

$$\frac{r_{mb}(x_{l-1},x_l)}{1+r_{mb}(x_l,x_{l-1})}\leq r_{mb}(x_{l-1},x_l)<\frac{\delta'}{8}<1.$$

Therefore,

$$\begin{aligned}
r_{mb}(Tx_l,x_p)-r_{mb_{Tx_l,x_p}} &\leq \frac{\delta'}{8}+\frac{1}{2}r_{mb}(x_{l-1},x_l)+\frac{1}{2}N(x_l,x_{l-1})+\\
&\quad \frac{1}{2}r_{mb}(x_{l-2},x_{l-1})+\frac{1}{2}N(x_{l-1},x_{l-2})+s\frac{\delta'}{8}\\
&< \left[\frac{\delta'}{8}+\frac{\delta'}{8}+s\epsilon\right]+s\frac{\delta'}{8}\\
&\leq s\left(\frac{\delta'}{4}+2\epsilon\right).
\end{aligned}$$

This implies that for all $r>p$, we get

$$r_{mb}(x_r,x_p)-r_{mb_{x_r,x_p}}<s\left(\frac{\delta'}{4}+2\epsilon\right).$$

Let $r>t>s>p$, for $r,s\in N$, we have

$$\begin{aligned}
r_{mb}(x_r,x_s)-r_{mb_{x_r,x_s}} &\leq s[(r_{mb}(x_r,x_t)-r_{mb_{x_r,x_t}})+(r_{mb}(x_r,x_p)-r_{mb_{x_r,x_p}})\\
&\quad+(r_{mb}(x_p,x_s)-r_{mb_{x_p,x_s}})]\\
&\leq s[r_{mb}(x_r,x_t))+r_{mb}(x_r,x_p)+r_{mb}(x_p,x_s)]\\
&< s\left[s\left(\frac{\delta'}{4}+2\epsilon\right)+s\left(\frac{\delta'}{4}+2\epsilon\right)+s\left(\frac{\delta'}{4}+2\epsilon\right)\right].\\
&= \frac{3}{4}s^2(4\epsilon+\delta')\leq 3s^2\epsilon.
\end{aligned}$$

Thus,

$$\lim_{r,s\to\infty} r_{mb}(x_r,x_s)-m_{mb_{x_r,x_s}}$$

exists and is finite. Similarly,

$$\lim_{r,s\to\infty} r_{mb}(x_r,x_s)-m_{mb_{x_r,x_s}}$$

exists and finite. Hence, $\{x_n\}$ is a r_{mb}-Cauchy sequence. X being complete, $\{x_n\}$ is convergent in X. Let $\{x_n\}$ converges to $\xi\in X$ such that

$$\lim_{n\to\infty} r_{mb}(x_n,\xi)-r_{mb_{x_n,\xi}}=0.$$

Now,we claim that ξ is the fixed point of T. For this, let

$$\lim_{n\to\infty} r_{mb}(x_n,\xi) - r_{mb_{x_n,\xi}} = 0.$$

$$\lim_{n\to\infty} r_{mb}(x_{n+1},\xi) - r_{mb_{x_{n+1},\xi}} = 0.$$

$$\lim_{n\to\infty} r_{mb}(Tx_n,\xi) - r_{mb_{Tx_n,\xi}} = 0.$$

$$r_{mb}(T\xi,\xi) - r_{mb_{T\xi,\xi}} = 0.$$

In the same manner, we can show that $r_{mb}(T\xi,\xi) = r_{mb_{T\xi,\xi}}$. Thus, ξ is the fixed point of T. Now, we have to prove that ξ is the unique fixed point of T. For this, let T has two fixed points $\xi,\chi \in X$, i.e., $T\xi = \xi$ and $T\chi = \chi$. Thus

$$\begin{aligned} r_{mb}(\xi,\chi) &= r_{mb}(T\xi,T\chi) \\ &< \alpha(\xi,\chi) r_{mb}(T\xi,T\chi) < \epsilon \\ &= \alpha(\xi,\chi) r_{mb}(\xi,\chi) < \epsilon \end{aligned}$$

This implies that

$$r_{mb}(\xi,\chi) < \epsilon$$

ϵ being arbitrary. Therefore, $r_{mb}(\xi,\chi) = 0$ and hence $\xi = \chi$. □

EXAMPLE 2.2. Let $X = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ and a rectangular M_b-metric is defined on X by

$$r_{mb}(x,y) = \left(\frac{x+y}{2}\right)^2.$$

Hence (X, r_{mb}) is rectangular M_b-metric space with $s = 3$.

Define a mapping $T : X \to X$ is defined by

$$Tx = \frac{x}{3}$$

and $\alpha : X \times X \to [0,\infty)$ by $\alpha(x,y) = \max\{x,y\}$. One can easily see that conditions (1) and (2) of Theorem 2.1 are satisfied. Now for condition (3), we have the following cases (for $\delta > 0$):

Case 1: If $x = 0$ and $y = 1$, then we have

$$\begin{aligned} 2s\epsilon &\le r_{mb}(y,Ty)\frac{1 + r_{mb}(x,Tx)}{1 + M(x,y)} + N(x,y) < s(2\epsilon + \delta) \\ 2 \times 3\epsilon &\le r_{mb}(1,T1)\frac{1 + r_{mb}(0,T0)}{1 + M(0,1)} + N(0,1) < 3(2\epsilon + \delta) \\ &\implies \epsilon \le \frac{29}{312} < \epsilon + \frac{\delta}{2} \\ &\implies \frac{1}{36} < \epsilon \implies \alpha(x,y) r_{mb}(Tx,Ty) < \epsilon. \end{aligned}$$

Case 2: If $x = 0$ and $y = \frac{1}{3}$, then we have

$$\begin{aligned} 6\epsilon &\le r_{mb}\left(\frac{1}{3}, T\frac{1}{3}\right)\frac{1 + r_{mb}(0,T0)}{1 + M(0,\frac{1}{3})} + N\left(0,\frac{1}{3}\right) < 3(2\epsilon + \delta) \\ &\implies \epsilon \le \frac{229}{18360} < \epsilon + \frac{\delta}{2} \\ &\implies \frac{1}{972} < \epsilon \implies \alpha(x,y) r_{mb}(Tx,Ty) < \epsilon. \end{aligned}$$

Case 3: If $x = 0$ and $y = \frac{2}{3}$, then we have

$$\begin{aligned} 6\epsilon &\leq r_{mb}\left(\frac{2}{3}, T\frac{2}{3}\right)\frac{1 + r_{mb}(0, T0)}{1 + M(0, \frac{2}{3})} + N\left(0, \frac{2}{3}\right) < 3(2\epsilon + \delta) \\ &\Longrightarrow \epsilon \leq \frac{242}{5238} < \epsilon + \frac{\delta}{2} \\ &\Longrightarrow \frac{2}{243} < \epsilon \Longrightarrow \alpha(x, y)r_{mb}(Tx, Ty) < \epsilon. \end{aligned}$$

Case 4: If $x = \frac{1}{3}$ and $y = \frac{2}{3}$, then we have

$$\begin{aligned} 6\epsilon &\leq r_{mb}\left(\frac{2}{3}, T\frac{2}{3}\right)\frac{1 + r_{mb}(\frac{1}{3}, T\frac{1}{3})}{1 + M(\frac{1}{3}, \frac{2}{3})} + N\left(\frac{1}{3}, \frac{2}{3}\right) < 3(2\epsilon + \delta) \\ &\Longrightarrow \epsilon \leq \frac{10913}{157464} < \epsilon + \frac{\delta}{2} \\ &\Longrightarrow \frac{1}{54} < \epsilon \Longrightarrow \alpha(x, y)r_{mb}(Tx, Ty) < \epsilon. \end{aligned}$$

Case 5: If $x = \frac{1}{3}$ and $y = 1$, then we have

$$\begin{aligned} 6\epsilon &\leq r_{mb}(1, T1)\frac{1 + r_{mb}(\frac{1}{3}, T\frac{1}{3})}{1 + M(\frac{1}{3}, 1)} + N\left(\frac{1}{3}, 1\right) < 3(2\epsilon + \delta) \\ &\Longrightarrow \epsilon \leq \frac{100}{1053} < \epsilon + \frac{\delta}{2} \\ &\Longrightarrow \frac{4}{81} < \epsilon \Longrightarrow \alpha(x, y)r_{mb}(Tx, Ty) < \epsilon. \end{aligned}$$

Case 6: If $x = \frac{2}{3}$ and $y = 1$, then we have

$$\begin{aligned} 6\epsilon &\leq r_{mb}(1, T1)\frac{1 + r_{mb}(\frac{2}{3}, T\frac{2}{3})}{1 + M(\frac{2}{3}, 1)} + N\left(\frac{2}{3}, 1\right) < 3(2\epsilon + \delta) \\ &\Longrightarrow \epsilon \leq \frac{19309}{118584} < \epsilon + \frac{\delta}{2} \\ &\Longrightarrow \frac{25}{324} < \epsilon \Longrightarrow \alpha(x, y)r_{mb}(Tx, Ty) < \epsilon. \end{aligned}$$

Therefore, the condition (3) is also satisfied for some $\delta > 0$. Thus the example meets all the hypothesis of Theorem 2.1. Hence $\xi = 0$ is a unique fixed point of the mapping T.

If we replace rectangular M_b-metric space by M_b-metric space in Theorem 2.1 so we obtain following corollary due to Mlaiki et al. [32].

Corollary 2.3. *Let (X, r_{mb}) is complete M_b-metric space with coefficient s. Define a triangular α-admissible mapping $T : X \to X$ such that it satisfies the following conditions:*

1. *there exists $x_0 \in X$ for which $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$.*
2. *Let x_n be a r_{mb} convergent sequence in X, i.e., $\{x_n\} \to z$ as $n \to \infty$. Also, $\alpha(x_n, x_m) \geq 1$ and $\alpha(x_n, x) \geq 1$ for all $n, m \in N$.*
3. *for every $\epsilon > 0$ there exists $\delta > 0$ such that the following condition hold:*

$$2s\epsilon \leq r_{mb}(y, Ty)\frac{1 + r_{mb}(x, Tx)}{1 + M(x, y)} + N(x, y) < s(2\epsilon + \delta)$$

then $\alpha(x,y)r_{mb}(Tx,Ty) < \epsilon$.

Then T *has a unique fixed point* ξ *in* X.

The following corollary is a sharpened version of the main result of Samet et al. [35].

COROLLARY 2.4. *Let* (X, r_{mb}) *is complete rectangular* M_b*-metric space with coefficient* s. *Define a triangular* α*-admissible mapping* $T : X \to X$ *such that it satisfies the following conditions:*

1. *there exists* $x_0 \in X$ *for which* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(Tx_0, x_0) \geq 1$.
2. *Let* x_n *be a* r_{mb} *convergent sequence in* X, *i.e.,* $\{x_n\} \to z$ *as* $n \to \infty$. *Also,* $\alpha(x_n, x_m) \geq 1$ *and* $\alpha(x_n, x) \geq 1$ *for all* $n, m \in N$.
3. *for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that the following condition hold:*

$$2s\epsilon \leq r_{mb}(y,Ty)\frac{1 + r_{mb}(x,Tx)}{1 + r_{mb}(x,y)} + r_{mb}(x,y) < s(2\epsilon + \delta)$$

then $\alpha(x,y)r_{mb}(Tx,Ty) < \epsilon$.

Then T *has a unique fixed point* ξ *in* X.

The following corollary is a sharpened version of the Corollary 3.3 of Samet et al. [35].

COROLLARY 2.5. *In Theorem 2.1, if we replace condition* (3) *by*

1. *Assume that for every* $\epsilon > 0$ *there exists* $\delta(\epsilon) > 0$ *such that*

$$2s\epsilon \leq \int_0^{r_{mb}(y,Ty)\frac{1+r_{mb}(x,Tx)}{1+M(x,y)}+N(x,y)} \psi(t)dt < s(2\epsilon + \delta)$$

$$\implies \int_0^{\alpha(x,y)r_{mb}(Tx,Ty)} \psi(t)dt < \epsilon$$

where $\psi : [0, \infty) \to [0, \infty)$ *be a locally integrable function such that* $\int_0^t \psi(u)du > 0$ *for all* $t > 0$.

Then T *has a unique fixed point* ξ *in* X.

References

[1] A.H. Ansari, O. Ege and S. Radenovic, Some fixed point results on complex valued G_b-metric spaces, *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales,* Seria A. Matematicas, **112** (2), (2018), 463-472.

[2] M. Arshad, J. Ahmad and E. Karapınar, Some Common Fixed Point Results in Rectangular Metric Spaces. *International Journal of Analysis*, **2013**, (2013), 1-7.

[3] M. Asadi, E. Karapinar and P. Salimi, New extension of p-metric spaces with some fixed-point results on M-metric spaces, *J. Inequal. Appl.*, **2014**, 18, (2014), 1-9.

[4] M. Asim, M. Imdad and S. Radenovic : Fixed point results in extended rectangular b-metric spaces with an application. *U.P.B. Sci. Bull., Series A*, **81** (2), (2019), 1-8.

[5] M. Asim, A R. Khan and M. Imdad, Rectangular M_b-metric space and fixed point results, *J. of Mathematical Analysis* , **10**, (2019), 10-18.

[6] M. Asim, S. Mujahid, and I. Uddin, Meir-Keeler Contraction In Rectangular M-Metric Space, *Topol. Algebra Appl*, **9**, (2021), 96–104.

[7] H. Aydi, R. Banković, I. Mitrović and M. Nazam, Nemytzki-Edelstein-Meir-Keeler Type Results in b-metric Spaces, *Discrete Dynamics in Nature and Society*, 4745764, 7, (2018), 1-7.

[8] I. A. Bakhtin, The contraction mapping principle in almost metric spaces, Funct. Anal., Gos. Ped. Inst. Unianowsk, **30**, (1989), 26-37.

[9] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrals, *Fund. Math*, 3, (1922), 133-181.
[10] A. Branciari, A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces, *Publ. Math.,* **57**, (2000), 31-37.
[11] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis*, **1** (1), (1993), 5-11.
[12] H.S. Ding, M. Imdad, S. Radenovic, J. Vujakovic, On some fixed point results in b-metric, rectangular and b-rectangular metric spaces, *Arab Journal of Mathematical Sciences,* **22** (2), (2016), 151-164.
[13] O. Ege, Complex valued rectangular b-metric spaces and an application to linear equations, *Journal of Nonlinear Science and Applications,* **8** (6), (2015), 1014-1021.
[14] O. Ege, Complex valued G_b-metric spaces, *Journal of Computational Analysis and Applications,* **21** (2), (2016), 363-368.
[15] O. Ege, Some fixed point theorems in complex valued Gb-metric spaces, *Journal of Nonlinear and Convex Analysis,* **18** (11), (2017), 1997-2005.
[16] O. Ege and I. Karaca, Common fixed point results on complex valued Gb-metric spaces, *Thai Journal of Mathematics,* **16** (3), (2018), 775-787.
[17] O. Ege, C. Park, A.H. Ansari, A different approach to complex valued G_b-metric spaces, *Advances in Difference Equations,* **2020**:152, https://doi.org/10.1186/s13662-020-02605-0, (2020), 1-13.
[18] M. R. Frechet, Sur quelques points du calcul fonctionnel, *Rendic. Circ. Mat. Palermo.* 22, (1906), 1-74.
[19] R George, S. Radenovic, K P Reshma and S Shukla, Rectangular b-metric space and cont Rectangular b-metric space and contraction principles, *J. Nonlinear Sci. Appl* **1** (8), (2015), 1005-1013.
[20] N. Gholamian and M. Khanehgir: Fixed points of generalized Meir-Keeler contraction mappings in b-metric-like spaces, *Fixed Point Theory and Applications*, **2016** (34), (2016).
[21] A. Gholidahneh, S. Sedghi, O. Ege, Z. D. Mitrovic and M. de la Sen, The Meir-Keeler type contractions in extended modular b-metric spaces with an application, *AIMS Mathematics,* **6** (2), (2021), 1781-1799.
[22] S. Gülyaz and E. Karapınar and İ. M. Erhan : Generalized α-Meir-Keeler contraction mappings on Branciari b-metric spaces, *Filomat*, **17** (31), (2017), 5445-5456.
[23] M. Iqbal, A. Batool, O. Ege and M. de la Sen, Fixed point of almost contraction in b-metric spaces, *Journal of Mathematics,* **2020**, 3218134, (2020), 1-6.
[24] M. Iqbal, A. Batool, O. Ege and M. de la Sen, Fixed point of generalized weak contraction in b-metric spaces, *Journal of Function Spaces,* **2021**, 8, (2021), 1-8.
[25] Z. Kadelburg and S. Radenovic, Pata-type common fixed point results in b-metric and b-rectangular metric spaces, *Journal of Nonlinear Analysis and Applications,* **8**, (2015), 944-954.
[26] T. Kamran, M. Samreen and Q. U. L. Ain, A generalization of b-Metric space and some fixed point theorems, *Mathematics*, **5** (2), 19, (2017), doi.org/10.3390/math5020019.
[27] E. Karapinar, S. Czerwik and H. Aydi, (α, φ)-Meir-Keeler Contraction Mappings in Generalized b-Metric Spaces, *Journal of Function Spaces*, **2018** (4), 3264620, (2018).
[28] S. G. Matthews, Partial metric topology, *Annals of the New York Academy of Sciences*, **728**, (1994), 183-197.
[29] A. Meira and E. Keeler A Theorem on Contraction Mappings, *J. of Mathematical Analysis and Applications*, **28**, (1969), 326-369.
[30] Z. D. Mitrovic, A note on a Banachs fixed point theorem in b-rectangular metric space and b-metric space, *Mathematica Slovaca,* **68** (5), (2018), 1113-1116.
[31] Z. D. Mitrovi and S. Radenovi, On Meir-Keeler contraction in Branciari b-metric spaces, *Trans. A. Razmadze Math. Inst.*, **173**, (2019), 83–90.
[32] N. Mlaiki, N. Souayah, K. Abodayeh and T. Abdeljawad, Meir-Keeler contraction mappings in M_b-metric Spaces, *J. Computational Analysis and applications*, **4** (27) ,(2019).
[33] N. Mlaiki, A. Zarrad, N. Souayah, A. Mukheimer and T. Abdeljawed, Fixed Point Theorems in M_b-metric spaces, *J. Math. Anal.*, **7**, (2016), 1-9.

[34] N. Y. Özgür, N. Mlaiki, N. Tas and N. Souayah, A new generalization of metric spaces: rectangular M-metric spaces, *Mathematical Sciences*, **12**, (2018), 223-233.
[35] B. Samet, C. Vetro and H. Yazidi, A fixed point theorem for a Meir-Keeler type contraction through rational expression, *J. Nonlinear Sci. Appl.*, **6**, (2013), 162-169.
[36] T. L. Shateri, O. Ege and M. de la Sen, Common fixed point on the bv(s)-metric space of function-valued mappings, *AIMS Mathematics,* **6** (1), (2021), 1065-1074.
[37] S. Shukla, Partial b metric Spaces and Fixed Point Theorems, *Mediterranean Journal of Mathematics,* **11** (2), (2014), 703–711.
[38] S. Shukla, Partial Rectangular Metric Spaces and Fixed Point Theorems, *The Scientific World Journal,* **7**, (2014), dx.doi.org/10.1155/2014/756298.

Mohammad Asim
Department of Mathematics, Faculty of Science,
Shree Guru Gobind Singh Tricentenary University, Gurugram, Haryana, India.
E-mail: mailtoasim27@gmail.com

Meenu
Department of Mathematics, Faculty of Science,
Shree Guru Gobind Singh Tricentenary University, Gurugram, Haryana, India.
E-mail: meenudahiya805@gmail.com