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A STUDY ON OPERATORS SATISFYING $|T^2| \geq |T^*|^2$

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Abstract. Let $\mathcal{A}^*$ denotes the class of operators satisfying $|T^2| \geq |T^*|^2$. In this paper, we show if the restriction to a non-trivial invariant subspace $\mathcal{M}$ of an operator $T \in \mathcal{A}^*$ is normal, then $\mathcal{M}$ reduces T.

1. Introduction

Let $\mathscr{L}(\mathscr{H})$ denotes the algebra of bounded linear operators on a complex infinite dimensional Hilbert space $\mathscr{H}$. Recall ([1] and [3]) that $T \in \mathscr{L}(\mathscr{H})$ is called *hyponormal* if $T^*T \geq TT^*$, and T is called *paranormal* (resp.$*$-paranormal) if $||T^2x|| \geq ||Tx||^2$ (resp. $||T^2x|| \geq ||T^*x||^2$) for all unit vector $x \in \mathscr{H}$. Following [3] and [5] we say that $T \in \mathscr{L}(\mathscr{H})$ belongs to *class* A if $|T^2| \geq |T|^2$. Recently, B. P. Duggal, I. H. Jeon, I. H. Kim ([2]) consider a following new class of operators; we say that an operator $T \in \mathscr{L}(\mathscr{H})$ belongs to $*$-*class* A if

$$|T^2| \geq |T^*|^2.$$

For brevity, we shall denote classes of hyponormal operators, paranormal operators, $*$-paranormal operators, class A operators, and $*$-class A operators by $\mathcal{H}$, $\mathcal{PN}$, $\mathcal{PN}^*$, $\mathcal{A}$, and $\mathcal{A}^*$, respectively. From [3] and [2], it is well known that

$$\mathcal{H} \subset \mathcal{A} \subset \mathcal{PN} \text{ and } \mathcal{H} \subset \mathcal{A}^* \subset \mathcal{PN}^*.$$

In [2], many results of $*$-paranormal operators were proved. In particular, $*$-paranormal operator have SVEP, the single-valued extension

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property, everywhere. Indeed, more is true: $*$-paranormal operators satisfy (Bishop's) property (β), where $A \in \mathscr{L}(\mathscr{H})$ satisfies property (β) if, for an open subset $\mathcal{U}$ of the complex plane and a sequence $\{f_n\}$ of analytic functions $f_n : \mathcal{U} \longrightarrow \mathscr{H}$, $(A-\lambda)f_n(\lambda)$ converges uniformly to 0 on compact subsets of $\mathcal{U}$ implies f_n converges uniformly to 0 on compact subsets of $\mathcal{U}$ [6]. Since an operator $T \in \mathcal{A}^*$ is $*$-paranormal [2], we can see the following.

PROPOSITION 1.1. *An operator $T \in \mathcal{A}^*$ satisfies (Bishop's) property (β), and so have SVEP.*

In this paper, we show if the restriction to a non-trivial invariant subspace $\mathcal{M}$ of a $*$-class A operator T is normal, then $\mathcal{M}$ reduces T. Also, from this result, we have some corollaries.

2. Results

We begin with the following result showed in [2].

LEMMA 2.1. *If $T \in \mathcal{A}^*$ and $\mathcal{M}$ is an invariant subspace of T, then $T\,|_{\mathcal{M}} \in \mathcal{A}^*$.*

The following is a structural result.

THEOREM 2.2. *Let $\mathcal{M}$ be a non-trivial invariant subspace for an operator $T \in \mathcal{A}^*$ and let $T\,|_{\mathcal{M}}$ be the restriction of T to $\mathcal{M}$. If $T\,|_{\mathcal{M}}$ is normal, then $\mathcal{M}$ reduces T.*

Proof. Let $T = \begin{pmatrix} T\,|_{\mathcal{M}} & A \\ 0 & B \end{pmatrix}$ on $\mathcal{M} \oplus \mathcal{M}^{\perp}$. Then from matrices calculations we have

$$TT^* = \begin{pmatrix} T\,|_{\mathcal{M}} T\,|_{\mathcal{M}}^* + AA^* & AB^* \\ BA^* & BB^* \end{pmatrix}$$

and

$$T^{*2}T^2 = \begin{pmatrix} T\,|_{\mathcal{M}}^{*\,2} T\,|_{\mathcal{M}}^{2} & T\,|_{\mathcal{M}}^{*\,2} T\,|_{\mathcal{M}} A + T\,|_{\mathcal{M}}^{*\,2} AB \\ A^*T\,|_{\mathcal{M}}^{*} T\,|_{\mathcal{M}}^{2} + BA^*T\,|_{\mathcal{M}}^{2} & D \end{pmatrix},$$

where

$$D = (A^*T\,|_{\mathcal{M}}^{*} + BA^*)T\,|_{\mathcal{M}} A + (A^*T\,|_{\mathcal{M}}^{*} + BA^*)AB + B^*B.$$

Let P be the orthogonal projection of $\mathscr{H}$ onto $\mathcal{M}$. Then we have

$$\begin{aligned}
\begin{pmatrix} T\,|_{\mathcal{M}}\,T\,|_{\mathcal{M}}^* + AA^* & 0 \\ 0 & 0 \end{pmatrix} &= PTT^*P \\
&= P|T^*|^2P \\
&\leq P(T^{*2}T^2)^{\frac{1}{2}}P \\
&\leq (PT^{*2}T^2P)^{\frac{1}{2}} \\
&\qquad \text{by Hansen's inequality(cf.[4]))} \\
&= \begin{pmatrix} T\,|_{\mathcal{M}}^{*\,2}T\,|_{\mathcal{M}}^2 & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{2}},
\end{aligned}$$

which implies that

$$T\,|_{\mathcal{M}}\,T\,|_{\mathcal{M}}^* + AA^* \leq T\,|_{\mathcal{M}}^*\,T\,|_{\mathcal{M}}\,.$$

Since $T\,|_{\mathcal{M}}$ is normal, we have that $A = 0$. Hence $\mathcal{M}$ reduces T. □

The following result was proved in [2]. We give a different proof using Theorem 2.2.

Corollary 2.3. *Let $T \in \mathcal{A}^*$. If $(T - \lambda)x = 0$, then $(T - \lambda)^*x = 0$*

Proof. Let $\mathcal{M} = \text{span}\{x\}$, $T = \begin{pmatrix} \lambda & A \\ 0 & B \end{pmatrix}$ on $\mathcal{M} \oplus \mathcal{M}^\perp$, and P the orthogonal projection of $\mathscr{H}$ onto $\mathcal{M}$. Then $T|_{\mathcal{M}} = \lambda$ and $T|_{\mathcal{M}}$ is an normal operator. By Theorem 2.2, $\mathcal{M}$ reduces T, and so $A = 0$. □

P. R. Halmos([4], Problem 161) proved that a partial isometry is subnormal if and only if it is hyponormal. The following result is analogous to this one.

Corollary 2.4. *A partial isometry T is quasinormal if and only if $T \in \mathcal{A}^*$.*

Proof. Let $T \in \mathcal{A}^*$ be a partial isometry. Then it is suffices to show that T is quasinormal. First, we claim that $R(T)$, the range of T, is contained in $N(T)^\perp$, the initial space of T. It follows from Theorem 2.2 and Corollary 2.3 that $N(T)$ and $N(T)^\perp$ are reducing subspaces of T. Since T is a partial isometry, T is of the form $U \oplus 0$, where U is an isometry. So simple calcuations show that $T(T^*T) = (T^*T)T$, i.e., T is quasinormal. □

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