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# STUDY OF BRÜCK CONJECTURE AND UNIQUENESS OF RATIONAL FUNCTION AND DIFFERENTIAL POLYNOMIAL OF A MEROMORPHIC FUNCTION

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Abstract. Let $f$ be a non-constant meromorphic function in the open complex plane $\mathbb{C}$. In this paper we prove under certain essential conditions that $R(f)$ and $P[f]$, rational function and differential polynomial of $f$ respectively, share a small function of $f$ and obtain a conclusion related to *Brück* conjecture. We give some examples in support to our result.

## 1. Introduction and Main Result

Let $\mathbb{C}$ denote the open complex plane and let $f$ be a non-constant meromorphic function defined on $\mathbb{C}$. We assume that the reader is familiar with the standard definitions and notations used in the Nevanlinna value distribution theory, such as $T(r,f), m(r,f), N(r,f)$ (see [8, 15, 16]). By $S(r,f)$ we denote any quantity satisfying $S(r,f) = \circ(T(r,f))$ as $r \to \infty$ possibly outside an exceptional set of finite linear measure. A meromorphic function $a$ is called a small function with respect to $f$ if either $a \equiv \infty$ or $T(r,a) = S(r,f)$. We denote by $S(f)$ the collection of all small functions with respect to $f$. Clearly $\mathbb{C} \cup \{\infty\} \subset S(f)$ and $S(f)$ is a field over the set of complex numbers. For $a \in \mathbb{C} \cup \{\infty\}$ the quantities

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)}$$

and

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

are respectively called the deficiency and ramification index of $a$ for the function $f$.

Throughout this paper, we use the symbol,

$$\mathfrak{X}_m = \begin{cases} 0, & \text{if } m = 0; \\ 1, & \text{if } m \geq 1. \end{cases}$$

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For any two non-constant meromorphic functions $f$ and $g$, and $a \in S(f) \cap S(g)$ we say that $f$ and $g$ share $a$ IM (CM) provided that $f-a$ and $g-a$ have the same zeros ignoring (counting) multiplicities. If $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM (CM), we say that $f$ and $g$ share $\infty$ IM (CM) respectively.
The hyper order $\rho_2(f)$ of a non-constant meromorphic function $f$ is defined by

$$\rho_2(f) = \limsup_{r \to \infty} \frac{\log \log T(r,f)}{\log r}.$$

In connection to find the relation between an entire function with its derivatives when they share one value CM, in 1996 the following famous conjecture was proposed by *Brück*.

**Conjecture**: Let $f$ be a non-constant entire function such that the hyper order $\rho_2(f)$ of $f$ is not a positive integer or infinite. If $f$ and $f^{(1)}$ share a value $a$ CM, then $\frac{f^{(1)}-a}{f-a} = c$, where $c$ is a non-zero constant.

Many authors including Zhang and Yang [17], Chen and Zhang [4], Lahiri [11], Chakraborty [6], Banerjee and Chakraborty [2,3], Li and Yang [12], Yang and Liu [13] and others also worked on this conjecture and its extensions. Subsequently, similar considerations have been made with respect to the higher order derivatives and more general expressions as well.

In the mean time a new notion of scalings between CM and IM known as weighted sharing is introduced in [9, 10] in the uniqueness literature. Below we are giving the definition.

Definition 1.1. [9, 10]. Let $l$ be a nonnegative integer or infinity and $a \in S(f)$. We denote by $E_l(a,f)$ the set of all zeros of $f-a$, where a zero of multiplicity $m$ is counted $m$ times if $m \leq l$ and $l+1$ times if $m > l$. If $E_l(a,f) = E_l(a,g)$, we say that $f,\ g$ share the function $a$ with weight $l$. We write $f$ and $g$ share $(a,l)$ to mean that $f$ and $g$ share the function $a$ with weight $l$. Since $E_l(a,f) = E_l(a,g)$ implies that $E_s(a,f) = E_s(a,g)$ for any integer $s\,(0 \leq s < l)$, if $f,\ g$ share $(a,l)$, then $f,\ g$ share $(a,s)$. Moreover, we note that $f$ and $g$ share the function $a$ IM or CM if and only if $f$ and $g$ share $(a,0)$ or $(a,\infty)$ respectively.

Definition 1.2. Let $p$ be a positive integer. Let $f$ be a meromorphic function and $a \in S(f)$.

(i) $\overline{N}_{p)}(r,a;f)$ denotes the counting function of those $a$-points of $f$ whose multiplicities are not greater than $p$, where each $a$-point is counted only once.

(ii) $\overline{N}_{(p}(r,a;f)$ denotes the counting function of those $a$-points of $f$ whose multiplicities are not less than $p$, where each $a$-point is counted only once.

(iii) $N_p(r,a;f)$ denotes the counting function of those $a$-points of $f$, where an $a$-point of $f$ with multiplicity $m$ is counted $m$ times if $m \leq p$ and $p$ times if $m > p$.

We denote $\delta_p(a, f)$ by the quantity

$$\delta_p(a, f) = 1 - \limsup_{r\to\infty} \frac{N_p(r, a; f)}{T(r, f)}.$$

Clearly $0 \leq \delta(a, f) \leq \delta_p(a, f) \leq \delta_{p-1}(a, f) \leq .... \leq \delta_2(a, f) \leq \delta_1(a, f) = \Theta(a, f)$.

Definition 1.3. Suppose $f$ and $g$ share $a$ IM and let $z_0$ be a zero of $f - a$ of multiplicity $p$ and a zero of $g - a$ of multiplicity $q$.

(i) By $\overline{N}_L(r, a; f)$ we denotes the reduced counting function of those a-points of $f$ and $g$ where $p > q \geq 1$; $\overline{N}_L(r, a; g)$ is defined similarly.

(ii) By $N_E^{1)}(r, a; f)$ the counting function of those a-points of $f$ and $g$ where $p = q = 1$ and

(iii) by $\overline{N}_E^{(2}(r, a; f)$ the counting function of those a-points of $f$ and $g$ where $p = q \geq 2$, where each such zero is counted only once.

Definition 1.4. Let $n_{0j}, n_{1j}, n_{2j}, ..., n_{kj}$ be non-negative integers. The expression

$$M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}}(f^{(2)})^{n_{2j}}...(f^{(k)})^{n_{kj}}$$

is called a differential monomial generated by $f$ of degree $d(M_j) = \sum\limits_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum\limits_{i=0}^{k}(i + 1)n_{ij}$. Let $a_j \in S(f)$ and $a_j \not\equiv 0 (j = 1, 2, ..., t)$. The sum $P[f] = \sum\limits_{j=1}^{t} a_j M_j[f]$ is called a differential polynomial generated by $f$ of degree $\overline{d}(P) = \max\{d(M_j) : 1 \leq j \leq t\}$ and weight $\Gamma = \max\{\Gamma_{M_j} : 1 \leq j \leq t\}$. The numbers $\underline{d}(P) = \min\{d(M_j) : 1 \leq j \leq t\}$ and $k$ (the highest order of the derivative of $f$ in $P[f]$) are called respectively the lower degree and the order of $P[f]$. $P[f]$ is said to be homogeneous differential polynomial of degree $d$ if $\overline{d}_P = \underline{d}_P = d$.

We denote by $R(f)$ as defined in Lemma 2.1 and so we mean $\lambda = \max\{m, n\}$, $p_i$ $(1 \leq i \leq u)$ and $q_j$ $(1 \leq j \leq v)$ are positive integers. Let $P_n(f) = \sum_{k=0}^{n} a_k f^k = a_n \prod_{i=1}^{u}(f - d_i)^{p_i}$, $1 \leq u \leq n$ and $P_m(f) = \sum_{j=0}^{m} b_j f^j = b_m \prod_{j=1}^{v}(f - c_j)^{q_j}$, $1 \leq v \leq m$, where $d_i$ $(1 \leq i \leq u)$, $c_j$ $(1 \leq j \leq v)$ are complex constant and $u$, $v$ are two positive integers. Let $c_0 \neq c_j$ $(j = 1, 2...v)$ be a complex constant. We now define

$$v^* = \begin{cases} \mathfrak{X}_m, & \text{if } m = 0; \\ v\mathfrak{X}_m, & \text{if } m \geq 1. \end{cases}$$

Also for any positive integer $r \leq 3$, $\mu_r^i = \min\{p_i, r\}$ and $\mu_r^{i*} = (r + 1) - \mu_r^i$, for all $i = 1, 2, ...u$.

In 2016 with the notion of weighted sharing of small functions Li, Yang and Liu [13] obtained the following result for homogeneous differential polynomial .

Theorem 1.5. *Let $f$ be a non-constant meromorphic function and $a$ $(\not\equiv 0, \infty) \in S(f)$. Suppose $P[f]$ be a non-constant homogeneous differential polynomial of degree*

*$d$, weight $\Gamma$ and order $k$ satisfying $\Gamma > (k+1)d-2$. If $f-a$ and $P[f]-a$ share $(0,l)$ with one of the following conditions:*
*(i) $l \geq 2$ and*

$$3\Theta(\infty,f)+d\delta_{2+\Gamma-d}(0,f^d)+\delta_2(0,f)+\delta(a,f)>4,$$

*(ii) $l=1$ and*

$$\frac{7+\Gamma-d}{2}\Theta(\infty,f)+d\delta_{2+\Gamma-d}(0,f^d)+\frac{d}{2}\delta_{1+\Gamma-d}(0,f^d)+\delta_2(0,f)+\delta(a,f)>\frac{9+\Gamma}{2},$$

*(iii) $l=0$ and*

$$(6+2\Gamma-2d)\Theta(\infty,f)+d\delta_{2+\Gamma-d}(0,f^d)+d\delta_{1+\Gamma-d}(0,f^d)+\delta_2(0,f)+\Theta(0,f)+\delta(a,f)>8+2\Gamma,$$

*then $\frac{P[f]-a}{f-a}=C$, where $C$ is a non-zero constant.*

In [6] B. Chakraborty improved Theorem 1.5 by replacing the homogeneous differential polynomial to arbitrary differential polynomial and proved the following theorem.

THEOREM 1.6. *Let $f$ be a non-constant meromorphic function and $a\ (\not\equiv 0,\infty) \in S(f)$. Suppose $P[f]$ be a non-constant differential polynomial of degree $\overline{d}(P)$, weight $\Gamma$ and order $k$ satisfying $\Gamma > (k+1)\underline{d}(P)-2$. If $f-a$ and $P[f]-a$ share $(0,l)$ with one of the following conditions:*
*(i) $l \geq 2$ and*

$$3\Theta(\infty,f)+\underline{d}(P)\delta_{2+\Gamma-\underline{d}(P)}(0,f^d)+\delta_2(0,f)+\delta(a,f)>4,$$

*(ii) $l=1$, $2\underline{d}(P)>\overline{d}(P)$ and*

$$\frac{7+\Gamma-\underline{d}(P)}{2}\Theta(\infty,f)+\underline{d}(P)\delta_{2+\Gamma-\underline{d}(P)}(0,f^{\underline{d}(P)})+\frac{\underline{d}(P)}{2}\delta_{1+\Gamma-\underline{d}(P)}(0,f^d)+\delta_2(0,f)+\delta(a,f)$$
$$>\frac{9+\Gamma}{2}+\overline{d}(P)-\underline{d}(P),$$

*(iii) $l=0$, $5\underline{d}(P)>4\overline{d}(P)$ and*

$$(6+2\Gamma-2\underline{d}(P))\Theta(\infty,f)+\underline{d}(P)\delta_{2+\Gamma-\underline{d}(P)}(0,f^{\underline{d}(P)})+2\underline{d}(P)\delta_{1+\Gamma-\underline{d}(P)}(0,f^{\underline{d}(P)})+$$
$$\delta_2(0,f)+\Theta(0,f)+\delta(a,f)>8+2\Gamma+4(\overline{d}(P)-\underline{d}(P)),$$

*then $\frac{P[f]-a}{f-a}=C$, where $C$ is a non-zero constant.*

Very recently B. Chakraborty [7] improved Theorem 1.6 and proved the following.

THEOREM 1.7. *[7] Let $k(\geq 1)$, $n(\geq 1)$ be integers and $f$ be a non-constant meromorphic function. Let $P[f]$ be a homogeneous differential polynomial of degree $d$, weight $\Gamma$ and order $k$ satisfying $\Gamma > (k+1)d-2$. Also $a\ (\not\equiv 0,\infty)$ be a small function with respect to $f$. Suppose $f^n-a$ and $P[f]-a$ share $(0,l)$. If $l \geq 2$ and*

$$(\Gamma-d+3)\Theta(\infty,f)+d\delta_{2+\Gamma-d}(0,f)+\mu_2\delta_{\mu_2^*}(0,f)>\Gamma+\mu_2+3-n,$$

*or, $l=1$ and*

$$\left(\Gamma-d+\frac{7}{2}\right)\Theta(\infty,f)+d\delta_{2+\Gamma-d}(0,f)+\frac{1}{2}\Theta(0,f)+\mu_2\delta_{\mu_2^*}(0,f)$$
$$>\Gamma+\mu_2+4-n,$$

*or,* $l=0$ *and*

$$(6+2\Gamma-2d)\,\Theta(\infty,f)+d\delta_{2+\Gamma-d}(0,f)+d\delta_{1+\Gamma-d}(0,f)$$
$$+2\Theta(0,f)+\mu_2\delta_{\mu_2^*}(0,f)>2\Gamma+\mu_2+8-n,$$

*then* $f^n=P[f]$.

In the same paper the following question was asked:

QUESTION 1.8. *Is it possible to extend Theorem 1.7 up to an arbitrary differential polynomial instead of homogeneous differential polynomial* ?

Regarding the above mentioned Theorems (1.5-1.7) it is quite natural to raise the following question:

QUESTION 1.9. *What will happen if we replace* $f$ *or* $f^n$ *by a rational function* $R(f)$ *of* $f$ ?

In this paper we answer the above two questions and prove the following theorem which is the main result of the paper.

THEOREM 1.10. *Let* $f$ *be a non-constant meromorphic function and* $a$ $(\not\equiv 0,\infty)\in S(f)$. *Suppose* $P[f]$ *be a non-constant differential polynomial of degree* $\overline{d}(P)$, *weight* $\Gamma$ *and order* $k$ *satisfying* $\Gamma>(k+1)\underline{d}(P)-2$. *If* $R(f)$ *and* $P[f]$ *share* $(a,l)$ *with one of the following conditions:*
*(i)* $l\geq 2$ *and*

$$3\Theta(\infty,f)+\sum_{i=1}^{u}\mu_2^i\delta_{\mu_2^{i*}}(d_i,f)+\underline{d}(P)\delta_{2+\Gamma-\underline{d}(P)}(0,f^{\underline{d}(P)})+$$
$$\sum_{j=0}^{v^*}\mathfrak{X}_j\left\{\Theta(c_j,f)+\Theta_{(2}(c_j,f)\right\}$$
$$>3+\sum_{i=1}^{u}\mu_2^i+2v^*, \tag{1}$$

*(ii)* $l=1$ *and*

$$\frac{7+\Gamma-\underline{d}(P)}{2}\Theta(\infty,f)+\sum_{i=1}^{u}\mu_2^i\delta_{\mu_2^{i*}}(d_i,f)+\underline{d}(P)\delta_{2+\Gamma-\underline{d}(P)}(0,f^{\underline{d}(P)})+$$
$$\frac{\underline{d}(P)}{2}\delta_{1+\Gamma-\underline{d}(P)}(0,f^{\underline{d}(P)})+\sum_{j=0}^{v^*}\mathfrak{X}_j\left\{\Theta(c_j,f)+\Theta_{(2}(c_j,f)\right\}$$
$$>\frac{7+\Gamma}{2}+\sum_{i=1}^{u}\mu_2^i+\overline{d}(P)-\underline{d}(P)+2v^*, \tag{2}$$

*(iii) $l = 0$ and*

$$(6+2\Gamma-2\underline{d}(P))\Theta(\infty,f)+\sum_{i=1}^{u}\mu_2^i\delta_{\mu_2^{i*}}(d_i,f)+\underline{d}(P)\delta_{2+\Gamma-\underline{d}(P)}(0,f^{\underline{d}(P)})$$

$$+2\underline{d}(P)\delta_{1+\Gamma-\underline{d}(P)}(0,f^{\underline{d}(P)})+2\sum_{j=0}^{v^*}\mathfrak{X}_j\Theta(c_j,f)+\sum_{j=0}^{v^*}\mathfrak{X}_j\Theta_{(2}(c_j,f)+\sum_{i=1}^{u}\Theta(d_i,f)$$

$$> 6+2\Gamma+\sum_{i=1}^{u}\mu_2^i+u+4(\overline{d}(P)-\underline{d}(P))+3v^*, \tag{3}$$

*then $\frac{P[f]-a}{R(f)-a}=C$, where $C$ is a non-zero constant.*

The following example shows that the condition $a \not\equiv 0$ is necessary in Theorem 1.10.

Example 1.11. Let $P[f] = -ff^{(1)}$ and $R(f) = \frac{f^n}{f^2-1}$ with $n \geq 14$, where $f = \frac{e^z}{e^z-1}$. Then $P[f]$ and $R(f)$ share $(0,\infty)$ all the conditions (1)-(3) in Theorem 1.10 are satisfied but $\frac{P[f]}{R(f)} \neq C$, for a non-zero constant $C$.

The following examples show that the conditions (1)-(3) in Theorem 1.10 cannot be removed.

Example 1.12. Let $f = e^z$, $P[f] = f^2f^{(1)}+3ff^{(1)}+3f$ and $R(f) = \frac{(f+1)^3-(af^2+b)}{af^2+b}$, where $a, b \in \mathbb{C}$ with $a \neq 0$, $b \neq 1$. Then $P[f]+1 = (e^z+1)^3$ and $R(f)+1 = (e^z+1)^3$ share $(0,\infty)$ but none of the conditions (1)-(3) in Theorem 1.10 is satisfied, hence $\frac{P[f]+1}{R(f)+1} = af^2+b \neq C$, for a non-zero constant $C$.

Example 1.13. Let $P[f] = f^2+2f^{(1)}$ and $R(f) = f^3+2f^2+f-1$, where $f = e^z$. Then $P[f]+1 = (e^z+1)^2$ and $R(f)+1 = e^z(e^z+1)^2$ share $(0,\infty)$ but none of the conditions (1)-(3) in Theorem 1.10 is satisfied, hence $\frac{P[f]+1}{R(f)+1} = \frac{1}{f} \neq C$, for a non-zero constant $C$.

## 2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let $F$ and $G$ be two non-constant meromorphic functions. We shall define $H$ by the following function.

$$H = \left(\frac{F^{(2)}}{F^{(1)}}-2\frac{F^{(1)}}{F-1}\right)-\left(\frac{G^{(2)}}{G^{(1)}}-2\frac{G^{(1)}}{G-1}\right).$$

Lemma 2.1. [14] *Let $f$ be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}$$

*be an irreducible rational function in $f$ with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then*

$$T(r,R(f)) = \lambda T(r,f)+S(r,f),$$

*where $\lambda = \max\{n,m\}$.*

Lemma 2.2. [1] *Let $f$ be a meromorphic function and P[f] be a differential polynomial. Then*

$$m\left(r, \frac{P[f]}{f^{\underline{d}(P)}}\right) \leq (\overline{d}(P) - \underline{d}(P))m(r, \frac{1}{f}) + S(r, f).$$

$$N\left(r, \infty; \frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\overline{d}(P) - \underline{d}(P))N(r, 0; f) + Q\left[\overline{N}(r, \infty; f) + \overline{N}(r, 0; f)\right] + S(r, f).$$

Lemma 2.3. [6] *Let $j$ and $p$ be two positive integers satisfying $j \geq p+1$ and $\Gamma > (k+1)\underline{d}(P) - (p+1)$. Then for differential polynomial $P[f]$*

$$N_p(r, 0; P[f]) \leq T(r, P) - \underline{d}(P)T(r, f) + N_{p+\Gamma-d}(r, 0; f^{\underline{d}(P)}) + S(r, f).$$

Lemma 2.4. [6] *Let $j$ and $p$ be two positive integers satisfying $j \geq p+1$ and $\Gamma > (k+1)\underline{d}(P) - (p+1)$. Then for differential polynomial $P[f]$*

$$N_p(r, 0; P[f]) \leq N_{p+\Gamma-d}(r, 0; f^{\underline{d}(P)}) + (\Gamma - \underline{d}(P))\overline{N}(r, \infty; f) + (\overline{d}(P) - \underline{d}(P))\{m(r, \frac{1}{f}) + T(r, f)\} + S(r, f).$$

Lemma 2.5. [6] *If $F$ and $G$ be two non-constant meromorphic functions sharing $(1, l)$, then*

$$\overline{N}_L(r, 1; F) + 2\overline{N}_L(r, 1; G) + \overline{N}_E^{(2}(r, 1; F) + \overline{N}(r, 1; G) \leq N(r, 1; F) + S(r, F) + S(r, G), \text{ when } l = 1,$$

$$2\overline{N}_L(r, 1; F) + 2\overline{N}_L(r, 1; G) + \overline{N}_E^{(2}(r, 1; F) + \overline{N}(r, 1; G) \leq N(r, 1; F) + S(r, F) + S(r, G), \text{ when } l \geq 2.$$

Lemma 2.6. [2] *If $F$ and $G$ be non-constant meromorphic functions sharing $(1, l)$, then*

$$\overline{N}_L(r, 1; F) \leq \frac{1}{2}\overline{N}(r, \infty; F) + \frac{1}{2}\overline{N}(r, 0; F) + S(r, F) \text{ when } l \geq 1$$

$$\overline{N}_L(r, 1; F) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + S(r, F) \text{ when } l = 0.$$

Lemma 2.7. [5] *If $F$ and $G$ be non-constant meromorphic functions sharing $(1, 0)$, then*

$$\begin{aligned}\overline{N}(r, 1; F) + \overline{N}(r, 1; G) &\leq \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}_{(2}(r, 0; G) + \overline{N}_{(2}(r, 0; F) \\ &+ 2\overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + N(r, 1; G) + \overline{N}_0(r, 0; F^{(1)}) \\ &+ \overline{N}_0(r, 0; G^{(1)}) + S(r, F) + S(r, G).\end{aligned}$$

Lemma 2.8. [2] *Let $F$ and $G$ be non-constant meromorphic functions sharing $(1, l)$ and $H \not\equiv 0$. Then*

$$\begin{aligned}N(r, \infty, H) &\leq \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}_{(2}(r, 0; G) + \overline{N}_{(2}(r, 0; F) + \overline{N}_L(r, 1; F) \\ &+ \overline{N}_L(r, 1; G) + \overline{N}_0(r, 0; F^{(1)}) + \overline{N}_0(r, 0; G^{(1)}) + S(r, F) + S(r, G).\end{aligned}$$

Lemma 2.9. [2] *If $F$ and $G$ be non-constant meromorphic functions sharing $(1, l)$ and $H \not\equiv 0$. Then*

$$\begin{aligned}\overline{N}(r, 1; F) + \overline{N}(r, 1; G) &\leq \overline{N}(r, \infty; H) + \overline{N}_E^{(2}(r, 1; F) + \overline{N}_L(r, 1; F) \\ &+ \overline{N}_L(r, 1; G) + \overline{N}(r, 1; G) + S(r, F) + S(r, G).\end{aligned}$$

## 3. Proof of Main Theorem

Proof of Theorem 1.10.

*Proof.* Let

$$F=\frac{R(f)}{a},\ G=\frac{P[f]}{a}.$$

Since $R(f)$ and $P[g]$ share $(a,l)$, it follows that $F$, $G$ share $(1,l)$ except at the zeros and poles of $a$.

Now we consider the following two cases:
Case 1: $H\not\equiv 0$.
Assume that $l\geq 1$. By Second Fundamental Theorem, Lemmas 2.8, 2.9, we get

$$\begin{aligned}
T(r,F)+T(r,G) &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F)+\overline{N}(r,0;G)\\
&+ \overline{N}(r,\infty;H)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\\
&+ \overline{N}(r,1;G)-\overline{N}_0(r,0;F^{(1)})-\overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G)\\
&\leq 3\overline{N}(r,\infty;f)+\sum_{j=0}^{v^*}\mathfrak{X}_j\left\{\overline{N}(r,c_j;f)+\overline{N}(r,c_j;f|\geq 2)\right\}+N_2(r,0;F)\\
&+ N_2(r,0;G)+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)\\
&+ \overline{N}(r,1;G)+S(r,F)+S(r,G). \qquad (4)
\end{aligned}$$

Subcase 1.1: Let $l\geq 2$. Using Lemmas 2.3, 2.5 in (4) we get

$$\begin{gathered}
T(r,F)+T(r,G)\leq 3\overline{N}(r,\infty;f)+\textstyle\sum_{j=0}^{v^*}\mathfrak{X}_j\left\{\overline{N}(r,c_j;f)+\overline{N}(r,c_j;f|\geq 2)\right\}\\
+N_2(r,0;F)+N_2(r,0;G)+N(r,1;F)+S(r,f)\leq 3\overline{N}(r,\infty;f)\\
+\textstyle\sum_{i=1}^{u}\mu_2^iN_{\mu_2^{i*}}(r,d_i;f)+T(r,P)-\underline{d}(P)T(r,f)+N_{2+\Gamma-\underline{d}(P)}(r,0;f^{\underline{d}(P)})\\
+T(r,F)-m(r,1;F)+\textstyle\sum_{j=0}^{v^*}\mathfrak{X}_j\left\{\overline{N}(r,c_j;f)+\overline{N}(r,c_j;f|\geq 2)\right\}+S(r,f)
\end{gathered}$$

$$\begin{aligned}
\Rightarrow \underline{d}(P)T(r,f) &\leq 3\overline{N}(r,\infty;f)+N_{2+\Gamma-\underline{d}(P)}(r,0;f^{\underline{d}(P)})+\sum_{i=1}^{u}\mu_2^iN_{\mu_2^{i*}}(r,d_i;f)\\
&+ \sum_{j=0}^{v^*}\mathfrak{X}_j\left\{\overline{N}(r,c_j;f)+\overline{N}(r,c_j;f|\geq 2)\right\}+S(r,f).
\end{aligned}$$

$$\begin{aligned}
&3\Theta(\infty,f)+\sum_{i=1}^{u}\mu_2^i\delta_{\mu_2^{i*}}(d_i,f)+\underline{d}(P)\delta_{2+\Gamma-\underline{d}(P)}(0,f^{\underline{d}(P)})+\sum_{j=0}^{v^*}\mathfrak{X}_j\left\{\Theta(c_j,f)+\Theta_{(2}(c_j,f)\right\}\\
&\qquad\leq 3+\sum_{i=1}^{u}\mu_2^i+\sum_{j=0}^{v^*}2\mathfrak{X}_j=3+\sum_{i=1}^{u}\mu_2^i+2v^*
\end{aligned}$$

which contradicts (1).

Subcase 1.2 : Let $l = 1$. From (4) and using Lemmas 2.3, 2.4, 2.5 and 2.6 we get

$$
\begin{aligned}
& T(r,F)+T(r,G) \\
\leq\ & 3\overline{N}(r,\infty;f)+\sum_{j=0}^{v^*}\mathfrak{X}_j\left\{\overline{N}(r,c_j;f)+\overline{N}(r,c_j;f|\geq 2)\right\}+N_2(r,0;F) \\
+\ & N_2(r,0;G)+N(r,1;F)+\overline{N}_L(r,1;G)+S(r,f). \\
\leq\ & 3\overline{N}(r,\infty;f)+\sum_{j=0}^{v^*}\mathfrak{X}_j\left\{\overline{N}(r,c_j;f)+\overline{N}(r,c_j;f|\geq 2)\right\}+N_2(r,0;F) \\
+\ & N_2(r,0;G)+N(r,1;F)+\frac{1}{2}\overline{N}(r,\infty;G)+\frac{1}{2}\overline{N}(r,0;G)+S(r,f). \\
\leq\ & 3\overline{N}(r,\infty;f)+\sum_{i=1}^{u}\mu_2^i N_{\mu_2^{i*}}(r,d_i;f)+T(r,P)-\underline{d}(P)T(r,f)+N_{2+\Gamma-\underline{d}(P)}(r,0;f^{\underline{d}(P)}) \\
+\ & T(r,F)-m(r,1;F)+\sum_{j=0}^{v^*}\mathfrak{X}_j\left\{\overline{N}(r,c_j;f)+\overline{N}(r,c_j;f|\geq 2)\right\} \\
+\ & \frac{1}{2}(\Gamma-\underline{d}(P))\overline{N}(r,\infty;f)+\frac{1}{2}N_{1+\Gamma-\underline{d}(P)}(r,0;f^{\underline{d}(P)}) \\
+\ & \frac{\overline{d}(P)-\underline{d}(P)}{2}\{m(r,0;f)+T(r,f)\}+\frac{1}{2}\overline{N}(r,\infty;f)+S(r,f)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Rightarrow\ & \underline{d}(P)T(r,f) \\
\leq\ & \frac{7+\Gamma-\underline{d}(P)}{2}\overline{N}(r,\infty;f)+N_{2+\Gamma-\underline{d}(P)}(r,0;f^{\underline{d}(P)})+\sum_{i=1}^{u}\mu_2^i N_{\mu_2^{i*}}(r,d_i;f) \\
+\ & \frac{1}{2}N_{1+\Gamma-\underline{d}(P)}(r,0;f^{\underline{d}(P)})+(\overline{d}(P)-\underline{d}(P))T(r,f) \\
+\ & \sum_{j=0}^{v^*}\mathfrak{X}_j\left\{\overline{N}(r,c_j;f)+\overline{N}(r,c_j;f|\geq 2)\right\}+S(r,f).
\end{aligned}
$$

$$
\begin{aligned}
& \frac{7+\Gamma-\underline{d}(P)}{2}\Theta(\infty,f)+\sum_{i=1}^{u}\mu_2^i\delta_{\mu_2^{i*}}(d_i,f)+\underline{d}(P)\delta_{2+\Gamma-\underline{d}(P)}(0,f^{\underline{d}(P)})+\frac{\underline{d}(P)}{2}\delta_{1+\Gamma-\underline{d}(P)}(0,f^{\underline{d}(P)}) \\
& \quad +\sum_{j=0}^{v^*}\mathfrak{X}_j\left\{\Theta(c_j,f)+\Theta_{(2}(c_j,f)\right\}\leq\frac{7+\Gamma}{2}+\sum_{i=1}^{u}\mu_2^i+\overline{d}(P)-\underline{d}(P)+\sum_{j=0}^{v^*}2\mathfrak{X}_j \\
& \quad =\frac{7+\Gamma}{2}+\sum_{i=1}^{u}\mu_2^i+\overline{d}(P)-\underline{d}(P)+2v^*
\end{aligned}
$$

5 which contradicts assumption (2).

Subcase 1.3: Let us assume $l=0$. By Second Fundamental Theorem, Lemmas 2.3, 2.4, 2.6, 2.7, 2.8 and 2.9 we get

$$\begin{aligned}
&T(r,F)+T(r,G)\\
\leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F)+\overline{N}(r,0;G)\\
+\ & \overline{N}(r,1;F)+\overline{N}(r,1;G)-\overline{N}_0(r,0;F^{(1)})-\overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G)\\
\leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F)+\overline{N}(r,0;G)+N(r,\infty;H)\\
+\ & \overline{N}_E^{(2}(r,1;G)+\overline{N}_L(r,1;G)+\overline{N}_L(r,1;F)+\overline{N}(r,1;G)\\
-\ & \overline{N}_0(r,0;F^{(1)})-\overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G)\\
\leq\ & 2\overline{N}(r,\infty;f)+\overline{N}(r,\infty;G)+\sum_{j=0}^{v^*}\mathfrak{X}_j\left\{\overline{N}(r,c_j;f)+\overline{N}(r,c_j;f|\geq 2)\right\}\\
+\ & N_2(r,0;F)+N_2(r,0;G)+\overline{N}_E^{(2}(r,1;G)+2\overline{N}_L(r,1;F)\\
+\ & 2\overline{N}_L(r,1;G)+\overline{N}(r,1;G)+S(r,f)\\
\leq\ & 3\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;G)+\sum_{j=0}^{v^*}\mathfrak{X}_j\left\{2\overline{N}(r,c_j;f)+\overline{N}(r,c_j;f|\geq 2)\right\}\\
+\ & N_2(r,0;F)+N_2(r,0;G)+2\overline{N}(r,0;G)+\overline{N}(r,0;F)+N(r,1;F)+S(r,f).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Rightarrow \underline{d}(P)T(r,f)\\
&\leq\ (6+2\Gamma-2\underline{d}(P))\overline{N}(r,\infty;f)+N_{2+\Gamma-\underline{d}(P)}(r,0;f^{\underline{d}(P)})+\sum_{i=1}^{u}\mu_2^i N_{\mu_2^{i*}}(r,d_i;f)\\
&+\ 2N_{1+\Gamma-\underline{d}(P)}(r,0;f^{\underline{d}(P)})+4\left(\overline{d}(P)-\underline{d}(P)\right)T(r,f)+\sum_{i=1}^{u}\overline{N}(r,d_i;f)\\
&+\ 2\sum_{j=0}^{v^*}\mathfrak{X}_j\overline{N}(r,c_j;f)+\sum_{j=0}^{v^*}\mathfrak{X}_j\overline{N}(r,c_j;f|\geq 2)+S(r,f).
\end{aligned}$$

$$\begin{aligned}
(6+2\Gamma-2\underline{d}(P))\Theta(\infty,f)+\sum_{i=1}^{u}\mu_2^i\delta_{\mu_2^{i*}}(d_i,f)+\underline{d}(P)\delta_{2+\Gamma-\underline{d}(P)}(0,f^{\underline{d}(P)})\\
+2\underline{d}(P)\delta_{1+\Gamma-\underline{d}(P)}(0,f^{\underline{d}(P)})+2\sum_{j=0}^{v^*}\mathfrak{X}_j\Theta(c_j,f)+\sum_{j=0}^{v^*}\mathfrak{X}_j\Theta_{(2}(c_j,f)\\
+\sum_{i=1}^{u}\Theta(d_i,f)\leq 6+2\Gamma+\sum_{i=1}^{u}(\mu_2^i+1)+4(\overline{d}(P)-\underline{d}(P))+\sum_{j=0}^{v^*}3\mathfrak{X}_j\\
=6+2\Gamma+\sum_{i=1}^{u}\mu_2^i+u+4(\overline{d}(P)-\underline{d}(P))+3v^*
\end{aligned}$$

which contradicts (3).

Case 2: $H \equiv 0$. That is

$$\left(\frac{G^{(2)}}{G^{(1)}} - 2\frac{G^{(1)}}{G-1}\right) = \left(\frac{F^{(2)}}{F^{(1)}} - 2\frac{F^{(1)}}{F-1}\right).$$

Integrating twice we get

$$\frac{1}{G-1} = \frac{A}{F-1} + B,$$

where $A\ (\neq 0)$ and $B$ are constant.
Thus

$$F = \frac{(B-A)G + (A-B-1)}{BG-(B+1)}. \tag{5}$$

Next we consider following three subcases:
Subcase 2.1: $B \neq 0, -1$. Then from (5) we have

$$\overline{N}(r, \frac{B+1}{B}; G) = \overline{N}(r, \infty; F).$$

By Nevanlinna second fundamental theorem and Lemma 2.3 we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\frac{B+1}{B};G) + S(r,G) \\ &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + S(r,G) \\ &\leq 2\overline{N}(r,\infty;f) + \sum_{j=0}^{v^*}\mathfrak{X}_j\overline{N}(r,c_j;f) + T(r,G) - \underline{d}(P)T(r,f) \\ &+ N_{1+\Gamma-\underline{d}(P)}(r,0;f^{\underline{d}(P)}) + S(r,f) \end{aligned}$$

i.e.,

$$\underline{d}T(r,f) \leq 2\overline{N}(r,\infty;f) + \sum_{j=0}^{v^*}\mathfrak{X}_j\overline{N}(r,c_j;f) + N_{1+\Gamma-\underline{d}(P)}(r,0;f^{\underline{d}(P)}) + S(r,f),$$

$$\Rightarrow 2\Theta(\infty,f) + \sum_{j=0}^{v^*}\mathfrak{X}_j\Theta(c_j,f) + \underline{d}(P)\delta_{1+\Gamma-\underline{d}(P)}(0,f^{\underline{d}(P)}) \leq 2 + \sum_{j=0}^{v^*}\mathfrak{X}_j = 2 + v^*,$$

which contradicts (1)-(3).
Subcase 2.2: $B = -1$. Then

$$F = \frac{(1+A)G - A}{G}.$$

If $A + 1 \neq 0$,

$$\overline{N}(r, \frac{A}{A+1}; G) = \overline{N}(r, 0; F).$$

Again by Nevanlinna second fundamental theorem and Lemma 2.3 we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\frac{A}{A+1};G) + S(r,G) \\ &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,0;F) + S(r,G) \\ &\leq \overline{N}(r,\infty;f) + T(r,G) - \underline{d}T(r,f) + N_{1+\Gamma-\underline{d}(P)}(r,0;f^{\underline{d}(P)}) \\ &+ \sum_{i=1}^{u}\overline{N}(r,d_i;f) + S(r,f) \end{aligned}$$

i.e.,

$$\underline{d}T(r,f) \leq \overline{N}(r,\infty;f) + N_{1+\Gamma-\underline{d}(P)}(r,0;f^{\underline{d}(P)}) + \sum_{i=1}^{u} \overline{N}(r,d_i;f) + S(r,f)$$

$$\Rightarrow \Theta(\infty,f) + \underline{d}(P)\delta_{1+\Gamma-\underline{d}(P)}(0,f^{\underline{d}(P)}) + \sum_{i=1}^{u} \Theta(d_i,f) \leq 1 + \sum_{i=1}^{u} 1 = 1+u,$$

which again contradicts (1)-(3).
If $A+1=0$ then

$$FG = 1$$

$$\Rightarrow R(f).P[f] \equiv a^2. \tag{6}$$

Thus from (6) we have $N(r,0;f) = S(r,f)$ and $N(r,\infty;f) = S(r,f)$. Now by 1st Fundamental Theorem and Lemma 2.2 we have

$$\begin{aligned}
(\lambda + \overline{d}(P))T(r,f) &= T\left(r, \frac{a^2}{R(f)f^{\overline{d}(P)}}\right) + S(r,f) \\
&\leq T\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) + S(r,f) \\
&= N\left(r,\infty; \frac{P[f]}{f^{\overline{d}(P)}}\right) + m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) + S(r,f) \\
&\leq (\overline{d}(P) - \underline{d}(P))m(r, \frac{1}{f}) + S(r,f) \\
&\leq (\overline{d}(P) - \underline{d}(P))(T(r,f) - N(r,0;f)) + S(r,f),
\end{aligned}$$

$\Rightarrow (\lambda + \underline{d}(P))T(r,f) \leq S(r,f)$, which is impossible .
Subcase 2.3: $B = 0$. Then $\frac{G-1}{F-1} = \frac{1}{A} \Rightarrow \frac{P[f]-a}{R(f)-a} = C$, where $C = \frac{1}{A}$ is a non-zero constant.

□

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