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ESTIMATES FOR ANALYTIC FUNCTIONS ASSOCIATED WITH SCHWARZ LEMMA ON THE BOUNDARY

Ayşan Kaynakkan and Bülent Nafi Örnek

Abstract. In this paper, we will introduce the class of analytic functions called $\mathcal{R}(\alpha,\lambda)$ and explore the different 5properties of the functions belonging to this class.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z) = z + \sum\limits_{p=2}^{\infty} c_p z^p$ that are analytic in U. Also, let $\mathcal{R}(\alpha,\lambda)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ satisfying

$$\sum_{p=2}^{\infty} p\,|c_p| \leq \frac{\beta(1-\alpha)}{1-\beta}\cos\lambda, \tag{1.1}$$

where $0 \leq \alpha < 1$, $0 < \beta \leq \frac{1}{2}$, $|\lambda| < \frac{\pi}{2}$.

In this paper, we study some of the properties of the classes $\mathcal{R}(\alpha,\lambda)$ and assign coefficient bounds for functions belonging to these classes. Namely, the modulus of the second coefficient c_2 in the expansion of $f(z) = z + c_2 z^2 + \ldots$ belonging to the given class will be estimated from above. To find an upper bound for the coefficients of such functions, we need to give the following lemma.

Lemma 1.1 (Schwarz lemma). *Suppose that p is analytic in the unit disc, $|p(z)| < 1$ for $|z| < 1$ and $p(0) = 0$. Then*

$$i-)\ |p(z)| \leq |z|$$

$$ii-)\ |p'(0)| \leq 1$$

with equality in either of the above if and only if $p(z) = ze^{i\theta}$, θ real ([5], p.329).

Let $f \in \mathcal{R}(\alpha,\lambda)$ and consider the following function

$$\Theta(z) = \frac{f'(z)-1}{(2\beta-1)(f'(z)-1) + 2\beta(1-\alpha)\cos\lambda e^{-i\lambda}}. \tag{1.2}$$

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It is an analytic function in U and $\Theta(0)=0$. Now, let us show that $|\Theta(z)|<1$ in U. Now let us check the difference of the modules of the numerator and denominator of the function $\Theta(z)$ given in (1.2). Therefore, we take

$$\begin{aligned} & |f'(z)-1| - \left|(2\beta-1)(f'(z)-1)+2\beta(1-\alpha)\cos\lambda e^{-i\lambda}\right| \\ = & \left|\sum_{p=2}^{\infty} pc_p z^{p-1}\right| - \left|(2\beta-1)\sum_{p=2}^{\infty} pc_p z^{p-1} + 2\beta(1-\alpha)\cos\lambda e^{-i\lambda}\right| \\ \leq & \sum_{p=2}^{\infty} 2(1-\beta)\, p\, |c_p|\, |z|^{p-1} - 2\beta(1-\alpha)\cos\lambda. \end{aligned}$$

If we pass to limit in the last expression $|z|\to 1^-$, we take

$$\begin{aligned} & |f'(z)-1| - \left|(2\beta-1)(f'(z)-1)+2\beta(1-\alpha)\cos\lambda e^{-i\lambda}\right| \\ < & \sum_{p=2}^{\infty} 2(1-\beta)\, p\, |c_p| - 2\beta(1-\alpha)\cos\lambda. \end{aligned}$$

Since $\sum\limits_{p=2}^{\infty} p\,|c_p| \leq \frac{\beta(1-\alpha)}{1-\beta}\cos\lambda$, we obtain

$$|f'(z)-1| - \left|(2\beta-1)(f'(z)-1)+2\beta(1-\alpha)\cos\lambda e^{-i\lambda}\right| < 0$$

and

$$\left|\frac{f'(z)-1}{(2\beta-1)(f'(z)-1)+2\beta(1-\alpha)\cos\lambda e^{-i\lambda}}\right| < 1.$$

Therefore, if we substitute the Taylor expansion of $f(z)$ in the function $\Theta(z)$, we obtain

$$\Theta(z) = \frac{2c_2 z + 3c_3 z^2 + \ldots}{(2\beta-1)(2c_2 z + 3c_3 z^2 + \ldots) + 2\beta(1-\alpha)\cos\lambda e^{-i\lambda}}$$

$$\frac{\Theta(z)}{z} = \frac{2c_2 + 3c_3 z + \ldots}{(2\beta-1)(2c_2 z + 3c_3 z^2 + \ldots) + 2\beta(1-\alpha)\cos\lambda e^{-i\lambda}}.$$

Thus, from the Schwarz lemma, we obtain

$$|\Theta'(0)| = \left|\frac{2c_2}{2\beta(1-\alpha)\cos\lambda e^{-i\lambda}}\right| \leq 1$$

and

$$|c_2| \leq \beta(1-\alpha)\cos\lambda.$$

We thus obtain the following lemma.

Lemma 1.2. *If $f\in\mathcal{R}(\alpha,\lambda)$, then we have the inequality*

$$|f''(0)| \leq 2\beta(1-\alpha)\cos\lambda.$$

Now let us consider the following function by taking into account of the critical points, which are different from zero, of the function $f(z)-z$,

$$w(z) = \frac{\Theta(z)}{\prod\limits_{i=1}^{n} \frac{z-a_i}{1-\overline{a_i}z}}.$$

Since $w(z)$ function satisfies the conditions of the Schwarz lemma, we obtain

$$\begin{aligned} w(z) &= \frac{f'(z)-1}{(2\beta-1)\left(f'(z)-1\right)+2\beta\left(1-\alpha\right)\cos\lambda e^{-i\lambda}}\frac{1}{\prod\limits_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}} \\ &= \frac{2c_2z+3c_3z^2+...}{(2\beta-1)\left(2c_2z+3c_3z^2+...\right)+2\beta\left(1-\alpha\right)\cos\lambda e^{-i\lambda}}\frac{1}{\prod\limits_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}}, \end{aligned}$$

$$\frac{w(z)}{z} = \frac{2c_2+3c_3z+...}{(2\beta-1)\left(2c_2z+3c_3z^2+...\right)+2\beta\left(1-\alpha\right)\cos\lambda e^{-i\lambda}}\frac{1}{\prod\limits_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}},$$

$$|w'(0)| = \frac{|c_2|}{\beta\left(1-\alpha\right)\cos\lambda\prod\limits_{i=1}^{n}|a_i|} \leq 1$$

and

$$|c_2| \leq \beta\left(1-\alpha\right)\cos\lambda\prod_{i=1}^{n}|a_i|.$$

As a result, we get the following lemma.

Lemma 1.3. *Let $f \in \mathcal{R}(\alpha,\lambda)$ and $a_1, a_2, ..., a_n$ be critical points of the function $f(z)-z$ in D that are different from zero. Then we have the inequality*

$$|f''(0)| \leq 2\beta\left(1-\alpha\right)\cos\lambda\prod_{i=1}^{n}|a_i|.$$

This lemma shows that if the critical points of the $f(z)-z$ function are included, a stronger upper bound for the coefficient c_2 is obtained. Also, these two lemmas are results for analytic functions inside the unit disk. To examine the behavior of the derivative of this function at the boundary of the unit disk, the following lemma is needed [10, 15].

Lemma 1.4. *Let $p(z)$ be an analytic function in U, $f(0)=0$ and $|f(z)|<1$ for $z \in U$. If $p(z)$ extends continuously to some boundary point $1 \in \partial U = \{z : |z| = 1\}$, and if $|p(1)| = 1$ and $p'(1)$ exists, then*

$$|p'(1)| \geq \frac{2}{1+|p'(0)|} \tag{1.3}$$

and

$$|p'(1)| \geq 1. \tag{1.4}$$

Moreover, the equality in (1.3) holds if and only if

$$p(z) = z\frac{z-a}{1-az}$$

for some $a \in (-1, 0]$. Also, the equality in (1.4) holds if and only if $p(z) = ze^{i\theta}$.

Inequality (1.5) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1–4, 6–13].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [14]).

LEMMA 1.5 (Julia-Wolff lemma). *Let p be an analytic function in U, $p(0) = 0$ and $p(U) \subset U$. If, in addition, the function p has an angular limit $p(1)$ at $1 \in \partial U$, $|p(1)| = 1$, then the angular derivative $p'(1)$ exists and $1 \leq |p'(1)| \leq \infty$.*

COROLLARY 1.6. *The analytic function p has a finite angular derivative $p'(1)$ if and only if p' has the finite angular limit $p'(1)$ at $1 \in \partial U$.*

2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma for $\mathcal{R}(\alpha, \lambda)$ class. Also, in a class of analytic functions on the unit disc, assuming the existence of angular limit on the boundary point, the estimations below of the modulus of angular derivative have been obtained.

THEOREM 2.1. *Let $f \in \mathcal{R}(\alpha, \lambda)$. Assume that, for $1 \in \partial U$, f has an angular limit $f(1)$ at the point 1, $f'(1) = 1 + \frac{\beta(1-\alpha)}{1-\beta} \cos \lambda e^{-i\lambda}$. Then we have the inequality*

$$|f''(1)| \geq \frac{\beta(1-\alpha)\cos\lambda}{2(1-\beta)^2}. \tag{2.1}$$

Proof. Consider the function

$$\Theta(z) = \frac{f'(z) - 1}{(2\beta - 1)(f'(z) - 1) + 2\beta(1-\alpha)\cos\lambda e^{-i\lambda}}.$$

Also, since $f'(1) = 1 + \frac{\beta(1-\alpha)}{1-\beta} \cos \lambda e^{-i\lambda}$, we have

$$\Theta(1) = \frac{\frac{\beta(1-\alpha)}{1-\beta}\cos\lambda e^{-i\lambda}}{(2\beta - 1)\left(\frac{\beta(1-\alpha)}{1-\beta}\cos\lambda e^{-i\lambda}\right) + 2\beta(1-\alpha)\cos\lambda e^{-i\lambda}} = 1.$$

Therefore, from (1.4), we obtain

$$\begin{aligned} 1 \leq |\Theta'(1)| &= \frac{2\beta(1-\alpha)\cos\lambda\,|f''(1)|}{|(2\beta-1)(f'(1)-1) + 2\beta(1-\alpha)\cos\lambda e^{-i\lambda}|^2} \\ &= \frac{2\beta(1-\alpha)\cos\lambda\,|f''(1)|}{\left|(2\beta - 1)\left(\frac{\beta(1-\alpha)}{1-\beta}\cos\lambda e^{-i\lambda}\right) + 2\beta(1-\alpha)\cos\lambda e^{-i\lambda}\right|^2} \\ &= \frac{2(1-\beta)^2}{\beta(1-\alpha)\cos\lambda}|f''(1)| \end{aligned}$$

and

$$|f''(1)| \geq \frac{\beta(1-\alpha)\cos\lambda}{2(1-\beta)^2}.$$

□

The inequality (2.1) can be strengthened from below by taking into account, $c_2 = \frac{f''(0)}{2}$, the first coefficient of the expansion of the function $f(z) = z + c_2 z^2 + c_3 z^3 + ...$

THEOREM 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$|f''(1)| \geq \frac{1}{(1-\beta)^2} \frac{\beta^2 (1-\alpha)^2 \cos^2 \lambda}{\beta (1-\alpha) \cos \lambda + |c_2|}. \tag{2.2}$$

Proof. Let $w(z)$ function be the same as (1.2). So, from (1.3), we obtain

$$\frac{2}{1+|\Theta'(0)|} \leq |\Theta'(1)| = \frac{2(1-\beta)^2}{\beta(1-\alpha)\cos\lambda} |f''(1)|.$$

Since

$$|\Theta'(0)| = \left| \frac{2c_2}{2\beta(1-\alpha)\cos\lambda e^{-i\lambda}} \right| = \frac{|c_2|}{\beta(1-\alpha)\cos\lambda},$$

we take

$$\frac{2}{1+\frac{|c_2|}{\beta(1-\alpha)\cos\lambda}} \leq \frac{2(1-\beta)^2}{\beta(1-\alpha)\cos\lambda} |f''(1)|$$

and

$$|f''(1)| \geq \frac{1}{(1-\beta)^2} \frac{\beta^2 (1-\alpha)^2 \cos^2 \lambda}{\beta (1-\alpha) \cos \lambda + |c_2|}.$$

□

The inequality (2.2) can be strengthened as below by taking into account $c_3 = \frac{f'''(0)}{3!}$ which is the coefficient in the expansion of the function $f(z) = z + c_2 z^2 + c_3 z^3 + ...$

THEOREM 2.3. *Let $f \in \mathcal{R}(\alpha, \lambda)$. Assume that, for $1 \in \partial U$, f has an angular limit $f(1)$ at the point 1, $f'(1) = 1 + \frac{\beta(1-\alpha)}{1-\beta} \cos\lambda e^{-i\lambda}$. Then we have the inequality*

$$\begin{aligned} |f''(1)| \geq & \frac{\beta(1-\alpha)\cos\lambda}{2(1-\beta)^2} \Bigg(1+ \\ & \frac{4(\beta(1-\alpha)\cos\lambda - |c_2|)^2}{2\left((\beta(1-\alpha)\cos\lambda)^2 - |c_2|^2\right) + |3\beta(1-\alpha)\cos\lambda e^{-i\lambda} c_3 - 2c_2^2(2\beta-1)|}\Bigg). \end{aligned} \tag{2.3}$$

Proof. Let $\Theta(z)$ be the same as in the proof of Theorem 2.1 and $b(z) = z$. By the maximum principle, for each $z \in U$, we have the inequality $|\Theta(z)| \leq |b(z)|$. Therefore, we take

$$\begin{aligned} \vartheta(z) &= \frac{\Theta(z)}{b(z)} = \frac{1}{z} \left(\frac{f'(z)-1}{(2\beta-1)(f'(z)-1) + 2\beta(1-\alpha)\cos\lambda e^{-i\lambda}} \right) \\ &= \frac{1}{z} \frac{2c_2 z + 3c_3 z^2 + ...}{(2\beta-1)(2c_2 z + 3c_3 z^2 + ...) + 2\beta(1-\alpha)\cos\lambda e^{-i\lambda}} \\ &= \frac{2c_2 + 3c_3 z + ...}{(2\beta-1)(2c_2 z + 3c_3 z^2 + ...) + 2\beta(1-\alpha)\cos\lambda e^{-i\lambda}} \end{aligned}$$

is an analytic function in U and $|\vartheta(z)| \leq 1$ for $z \in U$. In particular, we have

$$|\vartheta(0)| = \frac{|c_2|}{\beta(1-\alpha)\cos\lambda} \leq 1 \tag{2.4}$$

and

$$|\vartheta'(0)| = \frac{\left|3\beta\left(1-\alpha\right)\cos\lambda e^{-i\lambda}c_3 - 2c_2^2\left(2\beta-1\right)\right|}{2\beta^2\left(1-\alpha\right)^2\cos^2\lambda}.$$

The auxiliary function

$$d(z) = \frac{\vartheta(z)-\vartheta(0)}{1-\overline{\vartheta(0)}\vartheta(z)}$$

is analytic in U, $d(0)=0$, $|d(z)|<1$ for $|z|<1$ and $|d(1)|=1$ for $1\in\partial U$. From (1.3), we obtain

$$\begin{aligned}\frac{2}{1+|d'(0)|} &\leq |d'(1)| = \frac{1-|\vartheta(0)|^2}{\left|1-\overline{\vartheta(0)}\vartheta(1)\right|^2}|\vartheta'(1)| \\ &\leq \frac{1+|\vartheta(0)|}{1-|\vartheta(0)|}\left\{|\Theta'(1)|-|b'(1)|\right\} \\ &= \frac{|\beta\left(1-\alpha\right)\cos\lambda|+|c_2|}{\beta\left(1-\alpha\right)\cos\lambda-|c_2|}\left(\frac{2\left(1-\beta\right)^2}{\beta\left(1-\alpha\right)\cos\lambda}|f''(1)|-1\right).\end{aligned}$$

Since

$$d'(z) = \frac{1-|\vartheta(0)|^2}{\left(1-\overline{\vartheta(0)}\vartheta(z)\right)^2}\vartheta'(z)$$

and

$$\begin{aligned}|d'(0)| &= \frac{|\vartheta'(0)|}{1-|\vartheta(0)|^2} = \frac{\frac{\left|3\beta(1-\alpha)\cos\lambda e^{-i\lambda}c_3-2c_2^2(2\beta-1)\right|}{2\beta^2(1-\alpha)^2\cos^2\lambda}}{1-\left(\frac{|c_2|}{\beta(1-\alpha)\cos\lambda}\right)^2} \\ &= \frac{\left|3\beta\left(1-\alpha\right)\cos\lambda e^{-i\lambda}c_3-2c_2^2\left(2\beta-1\right)\right|}{2\left(\left(\beta\left(1-\alpha\right)\cos\lambda\right)^2-|c_2|^2\right)},\end{aligned}$$

we obtain

$$\frac{2}{1+\frac{\left|3\beta(1-\alpha)\cos\lambda e^{-i\lambda}c_3-2c_2^2(2\beta-1)\right|}{2\left((\beta(1-\alpha)\cos\lambda)^2-|c_2|^2\right)}} \leq \frac{|\beta(1-\alpha)\cos\lambda|+|c_2|}{\beta(1-\alpha)\cos\lambda-|c_2|}\left(\frac{2(1-\beta)^2}{\beta(1-\alpha)\cos\lambda}|f''(1)|-1\right),$$

$$\frac{4(\beta(1-\alpha)\cos\lambda-|c_2|)^2}{2\left((\beta(1-\alpha)\cos\lambda)^2-|c_2|^2\right)+\left|3\beta(1-\alpha)\cos\lambda e^{-i\lambda}c_3-2c_2^2(2\beta-1)\right|} \leq \frac{2(1-\beta)^2}{\beta(1-\alpha)\cos\lambda}|f''(1)|-1$$

and

$$|f''(1)| \geq \frac{\beta(1-\alpha)\cos\lambda}{2(1-\beta)^2}\left(1+\frac{4(\beta(1-\alpha)\cos\lambda-|c_2|)^2}{2\left((\beta(1-\alpha)\cos\lambda)^2-|c_2|^2\right)+\left|3\beta(1-\alpha)\cos\lambda e^{-i\lambda}c_3-2c_2^2(2\beta-1)\right|}\right).$$ □

If $f(z)-z$ have critical points different from $z=0$, taking into account these critical points, the inequality (2.3) can be strengthened in another way. This is given by the following Theorem.

THEOREM 2.4. *Let $f\in\mathcal{R}(\alpha,\lambda)$ and $a_1,a_2,...,a_n$ be critical points of the function $f(z)-z$ in D that are different from zero. Assume that, for $1\in\partial U$, f has an angular limit $f(1)$ at the point 1, $f'(1)=1+\frac{\beta(1-\alpha)}{1-\beta}\cos\lambda e^{-i\lambda}$. Then we have the inequality*

$$|f''(1)| \geq \frac{\beta(1-\alpha)\cos\lambda}{2(1-\beta)^2}\left(1+\sum_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2}\right. \tag{2.5}$$

$$+\frac{4\left(\beta(1-\alpha)\cos\lambda\prod\limits_{i=1}^{n}|a_i|-|c_2|\right)^2}{2\left(\left(\beta(1-\alpha)\cos\lambda\prod\limits_{i=1}^{n}|a_i|\right)^2-|c_2|^2\right)+\prod\limits_{i=1}^{n}|a_i|\left|3\beta(1-\alpha)\cos\lambda e^{-i\lambda}c_3-2(2\beta-1)c_2^2+\beta(1-\alpha)\cos\lambda e^{-i\lambda}c_2\sum\limits_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}\Bigg).$$

Proof. Let $\Theta(z)$ be as in (1.2) and $a_1, a_2, ..., a_n$ be critical points of the function $f(z)-z$ in U that are different from zero. Also, consider the function

$$B(z)=z\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}.$$

By the maximum principle for each $z\in U$, we have

$$|\Theta(z)|\leq|B(z)|.$$

Consider the function

$$\begin{aligned} l(z) &= \frac{\Theta(z)}{B(z)}=\left(\frac{f'(z)-1}{(2\beta-1)\left(f'(z)-1\right)+2\beta\left(1-\alpha\right)\cos\lambda e^{-i\lambda}}\right)\frac{1}{z\prod\limits_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}}\\ &= \frac{2c_2z+3c_3z^2+...}{(2\beta-1)\left(2c_2z+3c_3z^2+...\right)+2\beta\left(1-\alpha\right)\cos\lambda e^{-i\lambda}}\frac{1}{z\prod\limits_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}}\\ &= \frac{2c_2+3c_3z+...}{(2\beta-1)\left(2c_2z+3c_3z^2+...\right)+2\beta\left(1-\alpha\right)\cos\lambda e^{-i\lambda}}\frac{1}{\prod\limits_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}}. \end{aligned}$$

$l(z)$ is analytic in U and $|l(z)|<1$ for $|z|<1$. In particular, we have

$$|l(0)|=\frac{|c_2|}{\beta\left(1-\alpha\right)\cos\lambda\prod\limits_{i=1}^{n}|a_i|}$$

and

$$|l'(0)|=\frac{\left|3\beta\left(1-\alpha\right)\cos\lambda e^{-i\lambda}c_3-2\left(2\beta-1\right)c_2^2+\beta\left(1-\alpha\right)\cos\lambda e^{-i\lambda}c_2\sum\limits_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{2\left(\beta\left(1-\alpha\right)\cos\lambda\right)^2\prod\limits_{i=1}^{n}|a_i|}.$$

The auxiliary function

$$g(z)=\frac{l(z)-l(0)}{1-\overline{l(0)}l(z)}$$

is analytic in U, $|g(z)|<1$ for $|z|<1$ and $g(0)=0$. For $1\in\partial U$ and $f'(1)=1+\frac{\beta(1-\alpha)}{1-\beta}\cos\lambda e^{-i\lambda}$, we take $|g(1)|=1$.

From (1.3), we obtain

$$\begin{aligned} \frac{2}{1+|g'(0)|} &\leq |g'(1)|=\frac{1-|l(0)|^2}{\left|1-\overline{l(0)}l(1)\right|}|l'(1)|\\ &\leq \frac{1+|l(0)|}{1-|l(0)|}\left(|\Theta'(1)|-|B'(1)|\right). \end{aligned}$$

It can be seen that

$$|g'(0)| = \frac{|l'(0)|}{1-|l(0)|^2}$$

and

$$|g'(0)| = \frac{\frac{\left|3\beta(1-\alpha)\cos\lambda e^{-i\lambda}c_3 - 2(2\beta-1)c_2^2 + \beta(1-\alpha)\cos\lambda e^{-i\lambda}c_2 \sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{2(\beta(1-\alpha)\cos\lambda)^2 \prod_{i=1}^{n}|a_i|}}{1-\left(\frac{|c_2|}{\beta(1-\alpha)\cos\lambda \prod_{i=1}^{n}|a_i|}\right)^2}$$

$$= \prod_{i=1}^{n}|a_i| \frac{\left|3\beta(1-\alpha)\cos\lambda e^{-i\lambda}c_3 - 2(2\beta-1)c_2^2 + \beta(1-\alpha)\cos\lambda e^{-i\lambda}c_2 \sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{2\left(\left(\beta(1-\alpha)\cos\lambda \prod_{i=1}^{n}|a_i|\right)^2 - |c_2|^2\right)}$$

Also,we have

$$|B'(1)| = 1 + \sum_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2},\ 1 \in \partial U.$$

Therefore, we obtain

$$\frac{2}{1+\prod_{i=1}^{n}|a_i| \frac{\left|3\beta(1-\alpha)\cos\lambda e^{-i\lambda}c_3 - 2(2\beta-1)c_2^2 + \beta(1-\alpha)\cos\lambda e^{-i\lambda}c_2 \sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{2\left(\left(\beta(1-\alpha)\cos\lambda \prod_{i=1}^{n}|a_i|\right)^2 - |c_2|^2\right)}}$$

$$\leq \frac{\beta(1-\alpha)\cos\lambda \prod_{i=1}^{n}|a_i| + |c_2|}{\beta(1-\alpha)\cos\lambda \prod_{i=1}^{n}|a_i| - |c_2|}\left(\frac{2(1-\beta)^2}{\beta(1-\alpha)\cos\lambda}|f''(1)| - 1 - \sum_{i=1}^{n}\frac{1-|s_i|^2}{|1-s_i|^2}\right),$$

$$\frac{4\left(\beta(1-\alpha)\cos\lambda \prod_{i=1}^{n}|a_i| - |c_2|\right)^2}{2\left(\left(\beta(1-\alpha)\cos\lambda \prod_{i=1}^{n}|a_i|\right)^2 - |c_2|^2\right) + \prod_{i=1}^{n}|a_i| \left|3\beta(1-\alpha)\cos\lambda e^{-i\lambda}c_3 - 2(2\beta-1)c_2^2 + \beta(1-\alpha)\cos\lambda e^{-i\lambda}c_2 \sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}$$

$$\leq \frac{2(1-\beta)^2}{\beta(1-\alpha)\cos\lambda}|f''(1)| - 1 - \sum_{i=1}^{n}\frac{1-|s_i|^2}{|1-s_i|^2}$$

and so, we get inequality (2.5). □

If $f(z)-z$ has no critical points different from $z=0$ in Theorem 2.3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

THEOREM 2.5. *Let $f \in \mathcal{R}(\alpha,\lambda)$, $f(z)-z$ has no critical points in U except $z=0$ and $c_2 > 0$. Assume that, for $1 \in \partial U$, f has an angular limit $f(1)$ at the point 1, $f'(1) = 1 + \frac{\beta(1-\alpha)}{1-\beta}\cos\lambda e^{-i\lambda}$. Then we have the inequality*

$$|f''(1)| \geq \frac{\beta(1-\alpha)\cos\lambda}{2(1-\beta)^2}\Bigg(1- \tag{2.6}$$

$$\left.\frac{4\beta(1-\alpha)\cos\lambda c_2 \ln^2\left(\frac{c_2}{\beta(1-\alpha)\cos\lambda}\right)}{\frac{2c_2}{\beta(1-\alpha)\cos\lambda}\ln\left(\frac{c_2}{\beta(1-\alpha)\cos\lambda}\right) - \left|3\beta(1-\alpha)\cos\lambda e^{-i\lambda}c_3 - 2c_2^2(2\beta-1)\right|}\right).$$

Proof. Let $c_2 > 0$ in the expression of the function $f(z)$. Having in mind the inequality (2.4) and the function $f(z)-z$ has no critical points in U except $z=0$, we denote by $\ln\vartheta(z)$ the analytic branch of the logarithm normed by the condition

$$\ln\vartheta(0) = \ln\left(\frac{c_2}{\beta(1-\alpha)\cos\lambda}\right) < 0.$$

The auxiliary function

$$\phi(z)=\frac{\ln\vartheta(z)-\ln\vartheta(0)}{\ln\vartheta(z)+\ln\vartheta(0)}$$

is analytic in the unit disc U, $|\phi(z)|<1$, $\phi(0)=0$ and $|\phi(1)|=1$ for $1\in\partial U$.

From (1.3), we obtain

$$\begin{aligned}\frac{2}{1+|\phi'(0)|} &\leq |\phi'(1)|=\frac{|2\ln\vartheta(0)|}{|\ln\vartheta(1)+\ln\vartheta(0)|^2}\left|\frac{\vartheta'(1)}{\vartheta(1)}\right|\\ &= \frac{-2\ln\vartheta(0)}{\ln^2\vartheta(0)+\arg^2\vartheta(1)}\left\{|\Theta'(1)|-1\right\}.\end{aligned}$$

Replacing $\arg^2\vartheta(1)$ by zero, then

$$\frac{1}{1-\frac{\frac{\left|3\beta(1-\alpha)\cos\lambda e^{-i\lambda}c_3-2c_2^2(2\beta-1)\right|}{2\beta^2(1-\alpha)^2\cos^2\lambda}}{\frac{2c_2}{\beta(1-\alpha)\cos\lambda}\ln\left(\frac{c_2}{\beta(1-\alpha)\cos\lambda}\right)}}\leq\frac{-1}{\ln\left(\frac{c_2}{\beta(1-\alpha)\cos\lambda}\right)}\left\{\frac{2\,(1-\beta)^2}{\beta\,(1-\alpha)\cos\lambda}\,|f''(1)|-1\right\}$$

and

$$1-\frac{4\beta(1-\alpha)\cos\lambda c_2\ln^2\left(\frac{c_2}{\beta(1-\alpha)\cos\lambda}\right)}{\frac{2c_2}{\beta(1-\alpha)\cos\lambda}\ln\left(\frac{c_2}{\beta(1-\alpha)\cos\lambda}\right)-\left|3\beta(1-\alpha)\cos\lambda e^{-i\lambda}c_3-2c_2^2(2\beta-1)\right|}\leq\frac{2(1-\beta)^2}{\beta(1-\alpha)\cos\lambda}\,|f''(1)|\,.$$

Thus, we obtain the inequality (2.6). □

The following theorem shows the relationship between the coefficients c_2 and c_3 in the Maclaurin expansion of the $f(z)=z+c_2z^2+c_3z^3+...$ function.

THEOREM 2.6. *Let $f\in\mathcal{R}(\alpha,\lambda)$, $f(z)-z$ has no critical points in U except $z=0$ and $c_2>0$. Then we have the inequality*

(2.7)

$$\left|3\beta\,(1-\alpha)\cos\lambda e^{-i\lambda}c_3-2c_2^2\,(2\beta-1)\right|\leq 4\left|\beta\,(1-\alpha)\cos\lambda c_2\ln\left(\frac{c_2}{\beta\,(1-\alpha)\cos\lambda}\right)\right|.$$

Proof. Let $\phi(z)$ be the same as in the proof of Theorem 2.5. Here, $\phi(z)$ is analytic in the unit disc U, $|\phi(z)|<1$, $\phi(0)=0$. Therefore, the function $\phi(z)$ satisfies the assumptions of the Schwarz Lemma. Thus, we obtain

$$\begin{aligned}1 &\geq |\phi'(0)|=\frac{|2\ln\vartheta(0)|}{|\ln\vartheta(0)+\ln\vartheta(0)|^2}\left|\frac{\vartheta'(0)}{\vartheta(0)}\right|\\ &= \frac{-1}{2\ln\vartheta(0)}\left|\frac{\vartheta'(0)}{\vartheta(0)}\right|\\ &= -\frac{\frac{\left|3\beta(1-\alpha)\cos\lambda e^{-i\lambda}c_3-2c_2^2(2\beta-1)\right|}{2\beta^2(1-\alpha)^2\cos^2\lambda}}{\frac{2c_2}{\beta(1-\alpha)\cos\lambda}\ln\left(\frac{c_2}{\beta(1-\alpha)\cos\lambda}\right)}\end{aligned}$$

and

$$\left|3\beta\,(1-\alpha)\cos\lambda e^{-i\lambda}c_3-2c_2^2\,(2\beta-1)\right|\leq 4\left|\beta\,(1-\alpha)\cos\lambda c_2\ln\left(\frac{c_2}{\beta\,(1-\alpha)\cos\lambda}\right)\right|.$$

□

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Ayşan Kaynakkan
Department of Mathematics, Amasya University,
Merkez-Amasya, 05100, Turkey
E-mail: aysankaynakkan@gmail.com

Bülent Nafi Örnek
Department of Computer Engineering, Amasya University,
Merkez-Amasya, 05100, Turkey
E-mail: nafiornek@gmail.com, nafi.ornek@amasya.edu.tr