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HOMOMORPHISMS IN PROPER LIE CQ^*-ALGEBRAS

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Abstract. Using the Hyers-Ulam-Rassias stability method of functional equations, we investigate homomorphisms in proper CQ^*-algebras and proper Lie CQ^*-algebras, and derivations on proper CQ^*-algebras and proper Lie CQ^*-algebras associated with the following functional equation

$$\frac{1}{k}f(kx+ky+kz)=f(x)+f(y)+f(z)$$

for a fixed positive integer k.

1. Introduction and preliminaries

Ulam [46] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if $f:G\to G'$ satisfies $\rho(f(xy),f(x)f(y))<\delta$ for all $x,y\in G$, then a homomorphism $h:G\to G'$ exists with $\rho(f(x),h(x))<\epsilon$ for all $x\in G$?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated.

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Hyers [18] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

Th.M. Rassias [34] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

THEOREM 1.1. (Th.M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where θ and p are positive real numbers with $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

Th.M. Rassias [35] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [14] following the same approach as in Th.M. Rassias [34], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [14], as well as by Th.M. Rassias and Šemrl [40] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$. The counterexamples of Gajda [14], as well as of Th.M. Rassias and Šemrl [40] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. P. Găvruta [15], who among others studied the Hyers-Ulam stability of functional equations. The inequality (1.1) that was introduced for the first time by

Th.M. Rassias [34] provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as *Hyers-Ulam-Rassias stability* of functional equations (cf. the books of P. Czerwik [10, 11], D.H. Hyers, G. Isac and Th.M. Rassias [19]).

Beginning around the year 1980, the topic of approximate homomorphisms and their stability theory in the field of functional equations and inequalities was taken up by several mathematicians (cf. D.H. Hyers and Th.M. Rassias [20], Th.M. Rassias [38] and the references therein).

J.M. Rassias [32] following the spirit of the innovative approach of Th.M. Rassias [34] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p+\|y\|^p$ by $\|x\|^p\cdot\|y\|^q$ for $p, q \in \mathbb{R}$ with $p+q \neq 1$ (see also [33] for a number of other new results).

Theorem 1.2. [31, 32, 33] *Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p \in \mathbb{R} - \{1\}$ such that f satisfies inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \cdot ||x||^{\frac{p}{2}} \cdot ||y||^{\frac{p}{2}}$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^p - 2|}||x||^p$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

Several mathematicians have contributed works on these subjects (see [22]–[28], [36]–[39], [42]).

In a series of papers [1]–[13] and [43]–[45], many authors have considered a special class of quasi $*$-algebras, called *proper CQ^*-algebras*, which arise as completions of C^*-algebras. They can be introduced in the following way:

Let A be a Banach module over the C^*-algebra A_0 with involution $*$ and C^*-norm $\|\cdot\|_0$ such that $A_0 \subset A$. We say that (A, A_0) is a *proper CQ^*-algebra* if

(i) A_0 is dense in A with respect to its norm $\|\cdot\|$;

(ii) an involution $*$, which extends the involution of A_0, is defined in A with the property $(xy)^* = y^*x^*$ for all $x, y \in A$ whenever the multiplication is defined. ;

(iii) $\|y\|_0 = \sup_{x\in A, \|x\|\leq 1} \|xy\|$ for all $y \in A_0$.

Definition 1.3. Let (A, A_0) and (B, B_0) be proper CQ^*-algebras.

(i) A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *proper CQ^*-algebra homomorphism* if $H(z) \in B_0$ and $H(zx) = H(z)H(x)$ for all $z \in A_0$ and all $x \in A$.

(ii) A $\mathbb{C}$-linear mapping $\delta : A_0 \to A$ is called a *derivation* if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A_0$ (see [3]).

A C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] := \frac{xy-yx}{2}$ on $\mathcal{C}$, is called a *Lie C^*-algebra.* (see [22], [24], [30]).

Definition 1.4. A proper CQ^*-algebra (A, A_0), endowed with the Lie product $[z, x] := \frac{zx-xz}{2}$ for all $z \in A_0$ and all $x \in A$, is called a *proper Lie CQ^*-algebra*.

Definition 1.5. Let (A, A_0) and (B, B_0) be proper Lie CQ^*-algebras.

(i) A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *proper Lie CQ^*-algebra homomorphism* if $H(z) \in B_0$ and $H([z, x]) = [H(z), H(x)]$ for all $z \in A_0$ and all $x \in A$.

(ii) A $\mathbb{C}$-linear mapping $\delta : A_0 \to A$ is called a *Lie derivation* if $\delta([x, y]) = [x, \delta(y)] + [\delta(x), y]$ for all $x, y \in A_0$.

In [16], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\| \tag{1.2}$$

then f satisfies the Jordan-von Neumann functional inequality

$$2f(x) + 2f(y) = f(x + y) + f(x - y).$$

See also [41]. Fechner [12] and Gilányi [17] proved the generalized Hyers–Ulam stability of the functional inequality (1.2). Park, Cho and Han [29] proved the generalized Hyers–Ulam stability of functional inequalities associated with Jordan-von Neumann type additive functional equations.

This paper is organized as follows: In Section 2, we investigate homomorphisms in proper CQ^*-algebras associated with the functional equation

$$\frac{1}{k}f(kx + ky + kz) = f(x) + f(y) + f(z). \tag{1.3}$$

In Section 3, we investigate derivations on proper CQ^*-algebras associated with the functional equation (1.3).

In Section 4, we investigate homomorphisms in proper Lie CQ^*-algebras associated with the functional equation (1.3).

In Section 5, we investigate derivations on proper Lie CQ^*-algebras associated with the functional equation (1.3).

2. Homomorphisms in proper CQ^*-algebras

Throughout this section, assume that (A, A_0) is a proper CQ^*-algebra with C^*-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$, and that (B, B_0) is a proper CQ^*-algebra with C^*-norm $\|\cdot\|_{B_0}$ and norm $\|\cdot\|_B$.

PROPOSITION 2.1. *Let X and Y be normed spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let $f : X \to Y$ be a mapping such that*

$$\|f(x) + f(y) + f(z)\|_Y \leq \|\frac{1}{k}f(kx + ky + kz)\|_Y \tag{2.1}$$

for all $x, y, z \in X$. Then f is Cauchy additive, i.e., $f(x+y) = f(x)+f(y)$.

Proof. Letting $x = y = z = 0$ in (2.1), we get

$$\|3f(0)\|_Y \leq \|\frac{1}{k}f(0)\|_Y.$$

So $f(0) = 0$.

Letting $z = 0$ and $y = -x$ in (2.1), we get

$$\|f(x) + f(-x)\|_Y \leq \|\frac{1}{k}f(0)\|_Y = 0$$

for all $x \in X$. Hence $f(-x) = -f(x)$ for all $x \in X$.

Letting $z = -x - y$ in (2.1), we get

$$\begin{aligned} \|f(x) + f(y) - f(x + y)\|_Y &= \|f(x) + f(y) + f(-x - y)\|_Y \\ &\leq \|\frac{1}{k}f(0)\|_Y = 0 \end{aligned}$$

for all $x, y \in X$. Thus

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$, as desired. □

We investigate homomorphisms in proper CQ^*-algebras associated with the functional equation (1.3).

Theorem 2.2. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying $f(w) \in B_0$ for all $w \in A_0$ such that*

$$\|\mu f(x) + f(y) + f(z)\|_B \leq \|\frac{1}{k} f(k\mu x + ky + kz)\|_B, \tag{2.2}$$

$$\|f(wx) - f(w)f(x)\|_B \leq \theta(\|w\|_A^{2r} + \|x\|_A^{2r}) \tag{2.3}$$

for $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, all $w \in A_0$ and all $x, y, z \in A$. Then the mapping $f : A \to B$ is a proper CQ^*-algebra homomorphism.

Proof. Let $\mu = 1$ in (2.2). By Proposition 2.1, the mapping $f : A \to B$ is Cauchy additive.

Letting $z = 0$ and $y = -\mu x$ in (2.2), we get

$$\mu f(x) - f(\mu x) = \mu f(x) + f(-\mu x) = 0$$

for all $x \in A$. So $f(\mu x) = \mu f(x)$ for all $x \in A$. By the same reasoning as in the proof of Theorem 2.1 of [23], the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (2.3),

$$\begin{aligned} \|f(wx) - f(w)f(x)\|_B &= \lim_{n\to\infty} \frac{1}{4^n} \|f(4^n wx) - f(2^n w)f(2^n x)\|_B \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{4^n} \theta(\|w\|_A^{2r} + \|x\|_A^{2r}) = 0 \end{aligned}$$

for all $w \in A_0$ and all $x \in A$. So

$$f(wx) = f(w)f(x)$$

for all $w \in A_0$ and all $x \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f(wx) = f(w)f(x)$$

for all $w \in A_0$ and all $x \in A$.

Since $f(w) \in B_0$ for all $w \in A_0$, the mapping $f : A \to B$ is a proper CQ^*-algebra homomorphism, as desired. □

Theorem 2.3. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying (2.2) and $f(w) \in B_0$ for all $w \in A_0$ such that*

$$\|f(wx) - f(w)f(x)\|_B \leq \theta \cdot \|w\|_A^r \cdot \|x\|_A^r \tag{2.4}$$

for all $w \in A_0$ and all $x \in A$. Then the mapping $f : A \to B$ is a proper CQ^*-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (2.4),

$$\begin{aligned} \|f(wx) - f(w)f(x)\|_B &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n wx) - f(2^n w)f(2^n x)\|_B \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{4^n}\theta \cdot \|w\|_A^r \cdot \|x\|_A^r = 0 \end{aligned}$$

for all $w \in A_0$ and all $x \in A$. So

$$f(wx) = f(w)f(x)$$

for all $w \in A_0$ and all $x \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f(wx) = f(w)f(x)$$

for all $w \in A_0$ and all $x \in A$.

The rest of the proof is similar to the proof of Theorem 2.2. □

3. Derivations on proper CQ^*-algebras

Throughout this section, assume that (A, A_0) is a proper CQ^*-algebra with C^*-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$.

We investigate derivations on proper CQ^*-algebras associated with the functional equation (1.3).

THEOREM 3.1. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to A$ a mapping such that*

$$\|f(\mu x) + f(y) + f(z)\|_A \leq \|\frac{1}{k}f(k\mu x + ky + kz)\|_A, \tag{3.1}$$

$$\|f(w_0w_1) - f(w_0)w_1 - w_0f(w_1)\|_A \leq \theta(\|w_0\|_A^{2r} + \|w_1\|_A^{2r}) \tag{3.2}$$

for $\mu \in \mathbb{T}^1$, all $w_0, w_1 \in A_0$ and all $x, y, z \in A$. Then the mapping $f : A \to A$ is a derivation on A.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (3.2),

$$\begin{aligned}\|f(w_0w_1) &- f(w_0)w_1 - w_0f(w_1)\|_A \\ &= \lim_{n\to\infty}\frac{1}{4^n}\|f(4^nw_0w_1) - f(2^nw_0)\cdot 2^nw_1 - 2^nw_0f(2^nw_1)\|_A \\ &\leq \lim_{n\to\infty}\frac{4^{nr}}{4^n}\theta(\|w_0\|_A^{2r} + \|w_1\|_A^{2r}) = 0\end{aligned}$$

for all $w_0, w_1 \in A_0$. So

$$f(w_0w_1) = f(w_0)w_1 + w_0f(w_1)$$

for all $w_0, w_1 \in A_0$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to A$ satisfies

$$f(w_0w_1) = f(w_0)w_1 + w_0f(w_1)$$

for all $w_0, w_1 \in A_0$.

Therefore, the mapping $f : A \to A$ is a derivation on A, as desired. □

THEOREM 3.2. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to A$ a mapping satisfying* (3.1) *such that*

$$\|f(w_0w_1) - f(w_0)w_1 - w_0f(w_1)\|_A \leq \theta\cdot\|w_0\|_A^r\cdot\|w_1\|_A^r \tag{3.3}$$

for all $w_0, w_1 \in A_0$. Then the mapping $f : A \to A$ is a derivation on A.

Proof. The proof is similar to the proofs of Theorems 2.2 and 3.1. □

4. Homomorphisms in proper Lie CQ^*-algebras

Throughout this section, assume that (A, A_0) is a proper Lie CQ^*-algebra with C^*-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$, and that (B, B_0) is a proper Lie CQ^*-algebra with C^*-norm $\|\cdot\|_{B_0}$ and norm $\|\cdot\|_B$.

We investigate homomorphisms in proper Lie CQ^*-algebras associated with the functional equation (1.3).

THEOREM 4.1. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying* (2.2) *and $f(w) \in B_0$ for all $w \in A_0$ such that*

$$\|f([w,x]) - [f(w), f(x)]\|_B \leq \theta(\|w\|_A^{2r} + \|x\|_A^{2r}) \tag{4.1}$$

for all $w \in A_0$ and all $x \in A$. Then the mapping $f : A \to B$ is a proper Lie CQ^*-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (4.1),

$$\begin{aligned} \|f([w,x]) - [f(w), f(x)]\|_B &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n[w,x]) - [f(2^n w), f(2^n x)]\|_B \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{4^n}\theta(\|w\|_A^{2r} + \|x\|_A^{2r}) = 0 \end{aligned}$$

for all $w \in A_0$ and all $x \in A$. So

$$f([w,x]) = [f(w), f(x)]$$

for all $w \in A_0$ and all $x \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f([w,x]) = [f(w), f(x)]$$

for all $w \in A_0$ and all $x \in A$.

Therefore, the mapping $f : A \to B$ is a proper Lie CQ^*-algebra homomorphism. □

THEOREM 4.2. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying (2.2) and $f(w) \in B_0$ for all $w \in A_0$ such that*

$$\|f([w,x]) - [f(w), f(x)]\|_B \leq \theta \cdot \|w\|_A^r \cdot \|x\|_A^r \tag{4.2}$$

for all $w \in A_0$ and all $x \in A$. Then the mapping $f : A \to B$ is a proper Lie CQ^*-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (4.2),

$$\begin{aligned} \|f([w,x]) - [f(w), f(x)]\|_B &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n[w,x]) - [f(2^n w), f(2^n x)]\|_B \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{4^n}\theta \cdot \|w\|_A^r \cdot \|x\|_A^r = 0 \end{aligned}$$

for all $w \in A_0$ and all $x \in A$. So

$$f([w,x]) = [f(w), f(x)]$$

for all $w \in A_0$ and all $x \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f([w, x]) = [f(w), f(x)]$$

for all $w \in A_0$ and all $x \in A$.

Therefore, the mapping $f : A \to B$ is a proper Lie CQ^*-algebra homomorphism. □

REMARK 4.3. If the Lie products $[\cdot, \cdot]$ in the statements of Theorems 4.1 and 4.2 are replaced by the Jordan products $\cdot \circ \cdot$, then one obtains proper Jordan CQ^*-algebra homomorphisms instead of proper Lie CQ^*-algebra homomorphisms.

5. Derivations on proper Lie CQ^*-algebras

Throughout this section, assume that (A, A_0) is a proper Lie CQ^*-algebra with C^*-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$.

We investigate derivations on proper Lie CQ^*-algebras associated with the functional equation (1.3).

THEOREM 5.1. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to A$ a mapping satisfying* (3.1) *such that*

$$\begin{aligned} \|f([w_0, w_1]) - [f(w_0), w_1] &- [w_0, f(w_1)]\|_A \\ &\leq \theta(\|w_0\|_A^{2r} + \|w_1\|_A^{2r}) \end{aligned} \tag{5.1}$$

for all $w_0, w_1 \in A_0$. Then the mapping $f : A \to A$ is a Lie derivation on A.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (5.1),

$$\begin{aligned} &\|f([w_0, w_1]) - [f(w_0), w_1] - [w_0, f(w_1)]\|_A \\ &= \lim_{n\to\infty} \frac{1}{4^n} \|f(4^n[w_0, w_1]) - [f(2^n w_0), 2^n w_1] - [2^n w_0, f(2^n w_1)]\|_A \\ &\leq \lim_{n\to\infty} \frac{4^{nr}}{4^n} \theta(\|w_0\|_A^{2r} + \|w_1\|_A^{2r}) = 0 \end{aligned}$$

for all $w_0, w_1 \in A_0$. So

$$f([w_0, w_1]) = [f(w_0), w_1] + [w_0, f(w_1)]$$

for all $w_0, w_1 \in A_0$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to A$ satisfies

$$f([w_0, w_1]) = [f(w_0), w_1] + [w_0, f(w_1)]$$

for all $w_0, w_1 \in A_0$.

Therefore, the mapping $f : A \to A$ is a Lie derivation on A, as desired. □

THEOREM 5.2. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to A$ a mapping satisfying* (3.1) *such that*

$$\|f([w_0, w_1]) - [f(w_0), w_1] - [w_0, f(w_1)]\|_A \leq \theta \cdot \|w_0\|_A^r \cdot \|w_1\|_A^r \tag{5.2}$$

for all $w_0, w_1 \in A_0$. Then the mapping $f : A \to A$ is a Lie derivation on A.

Proof. The proof is similar to the proofs of Theorems 2.2 and 5.1. □

REMARK 5.3. If the Lie products $[\cdot, \cdot]$ in the statements of Theorems 5.1 and 5.2 are replaced by the Jordan products $\cdot \circ \cdot$, then one obtains Jordan derivations instead of Lie derivations.

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