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FIXED-POINT THEOREMS FOR (ϕ, ψ,β)-GERAGHTY CONTRACTION TYPE MAPPINGS IN PARTIALLY ORDERED FUZZY METRIC SPACES WITH APPLICATIONS

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Abstract. In this paper, we prove some fixed-point theorems in partially ordered fuzzy metric spaces for (ϕ, ψ, β)-Geraghty contraction type mappings which are generalization of mappings with Geraghty contraction type condition. Application of the derived results are shown in proving the existence of unique solution to some boundary value problems.

1. Introduction

Over the last decades, fixed-point theory has been widely extended and worked upon in several aspects by different researchers (refer to [1], [2], [3], [6], [7], [8], [9], [18]). In [13] and [14], authors have formulated existence theorem for first order periodic boundary value problem using the fixed-point theorems in partial ordered metric spaces. In [3], Cho et al. proved some fixed-point theorems in partially ordered fuzzy metric spaces for nonlinear mappings with respect to some contractive type conditions. In [16], Nieto et al. established existence and uniqueness theorem for solution of fuzzy differential equation. In 1973, M. Geraghty [5] refined the Banach contraction principle using an interesting class of test functions in complete metric spaces. In 2012, Gordji et al. [6] generalized the Geraghty's contraction theorem in partially ordered metric spaces. In [12], Gupta et al. introduced (ψ, β)-Geraghty contraction type mappings in partially ordered metric spaces and proved some fixed-point theorems for such mappings with application to integral equations and differential equations with periodic boundary conditions.

In this paper, we extend (ψ, β)-Geraghty contraction type mappings with the help of altering distance function ϕ to partially ordered fuzzy metric spaces. We develop some existence theorems for unique solution to different types of boundary value problems.

Definition 1.1. [13] Let $T : [0, 1] \times [0, 1] \longrightarrow [0, 1]$. Then the mapping T is said to be a triangular norm (t-norm) if

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(i) $T(\mu, 1) = \mu$ for all $\mu \in [0, 1]$,
(ii) $T(\mu, \nu) = T(\nu, \mu)$ for all $\mu, \nu \in [0, 1]$,
(iii) $\mu \geq \nu$, $\sigma \geq \tau \Rightarrow T(\mu, \sigma) \geq T(\nu, \tau)$ for all $\mu, \nu, \sigma, \tau \in [0, 1]$,
(iv) $T(\mu, T(\nu, \sigma)) = T(T(\mu, \nu), \sigma)$ for all $\mu, \nu, \sigma \in [0, 1]$.

Some elementary t-norms are $T_p(\mu, \nu) = \mu.\nu$, $T_m(\mu, \nu) = \min(\mu, \nu)$, $T_L(\mu, \nu) = \max(\mu + \nu - 1, 0)$.

Definition 1.2. [4] For an arbitrary set X, let T be a continuous t-norm and M be a fuzzy set on $X^2 \times (0, \infty)$. The 3-tuple (X, M, T) is called a fuzzy metric space if
(i) $M(a, b, s) > 0$ for all $a, b \in X, s > 0$,
(ii) $M(a, b, s) = 1$ for all $s > 0 \Leftrightarrow a = b$,
(iii) $M(a, b, s) = M(b, a, s)$ for all $a, b \in X, s > 0$,
(iv) $T(M(a, b, s), M(b, c, p)) \leq M(a, c, s + p)$
for all $a, b, c \in X, s, p > 0$,
(v) $M(a, b, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous for all $a, b \in X$.

In this case, M is called a fuzzy metric on X and the 3-tuple, (X, M, T) is called fuzzy metric space.

Example 1.3. [11] For $X = \mathbb{R}$, taking the usual metric $d(x, y) = |x - y|$ and the fuzzy metric $M(x, y, t) = \frac{t}{t+d(x,y)}, x, y \in X$, $t \in (0, \infty)$, we have, (X, M, T) is a fuzzy metric space with respect to the t-norm $T_p(a, b) = a.b$, $a, b \in [0, 1]$.
It is called the fuzzy metric induced by the usual metric d.

Definition 1.4. [19] Let $\phi : [0, 1] \longrightarrow [0, 1]$ be a mapping. If
(i) ϕ is strictly decreasing and left continuous,
(ii) $\phi(\lambda) = 0$ if and only if $\lambda = 1$,
that is, $\lim_{\lambda \longrightarrow 1-} \phi(\lambda) = 0$,
then the function ϕ is called an altering distance function.
$\phi(x) = 1 - x$, $\phi(x) = 1 - \frac{1}{e^{1-x}}$, $x \in [0, 1]$ are some examples of altering distance functions.

Definition 1.5. [10] Let (X, M, T) be a fuzzy metric space and $\{x_n\}$ be a sequence in X. Then the sequence $\{x_n\}$ is a Cauchy sequence if for all $\epsilon \in (0, 1)$, $t > 0$, $\exists\, n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$, $\forall\, n, m \geq n_0$. Also the sequence $\{x_n\}$ converges to x if for all $\epsilon \in (0, 1)$, $t > 0$, $\exists\, n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \epsilon$, $\forall\, n \geq n_0$. Moreover the fuzzy metric space X is complete if and only if every Cauchy sequence converges in X.

In 1973, Geraghty [5] proved a fixed-point theorem, which is known as Geraghty contraction theorem, with the help of the following class of functions:

Definition 1.6. Define $\mathcal{S} = \{\alpha | \alpha : [0, \infty) \longrightarrow [0, 1)\}$ which satisfies the condition $\alpha(t_n) \longrightarrow 1$ implies $t_n \longrightarrow 0$.

Theorem 1.7. (**Geraghty contraction theorem**) [5] *Let $f : X \longrightarrow X$ be a mapping on a complete metric space (X, d). Suppose there exists $\alpha \in \mathcal{S}$ such that for each $x, y \in X$*

$$d(f(x), f(y)) \leq \alpha(d(x, y))d(x, y).$$

Then f has a unique fixed-point $z \in X$.

Example 1.8. Let $X = (0,1)$ with the Euclidean distance d. Then, (X,d) is a bounded and complete metric space. Let $\alpha(t) = \frac{1}{1+t}, t \in [0,\infty)$, then $\alpha \in \mathcal{S}$. Let $f(x) = \frac{x}{2}, x \in X$.
Now for $x = y$, we have,

$$d(f(x), f(y)) = \alpha(d(x,y))d(x,y) = 0.$$

If $x \neq y$, then

$$\begin{aligned} d(f(x), f(y)) &= |\frac{x}{2} - \frac{y}{2}| \\ &= \frac{|x-y|}{2} \end{aligned}$$

and

$$\begin{aligned} \alpha(d(x,y))d(x,y) &= \frac{1}{1+|x-y|}|x-y| \\ &\geq \frac{|x-y|}{2}, \end{aligned}$$

that is,

$$d(f(x), f(y)) \leq \alpha(d(x,y))d(x,y).$$

Thus from the above theorem, f has a unique fixed-point. In fact, here 0 is the unique fixed-point.

In 2010, Altun et al. [1] introduced the following notion of weakly increasing mappings.

Definition 1.9. [1] Let $(X, \preceq)$ be a partially ordered set. Two self mappings f, g on X are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$.

Using the above notion, in 2017, Gupta et al. [12] proved some fixed-point theorems for (ψ, β)-Geraghty contraction type mappings in partially ordered metric space which improve and extend some already established results.

Theorem 1.10. *[12] Let (X,d) be a partially ordered complete metric space. Let f and g be weakly increasing self mappings on X such that*

$$\psi(d(f(x), g(y))) \leq \alpha(d(x,y)\beta(d(x,y)) \quad \text{for all } x \preceq y,$$

*where $\alpha \in \mathcal{S}$, $\psi : [0,\infty) \longrightarrow [0,\infty)$ is continuous and non decreasing such that $\psi(t) = 0$ if and only if $t = 0$ and $\beta : [0,\infty) \longrightarrow [0,\infty)$ is continuous function with condition $\psi(t) > \beta(t)$ for all $t > 0$.
Suppose that for each $x, y \in X$, there exists $z \in X$ which is comparable to x and y. Moreover, if f or g is continuous, then f and g have a unique fixed-point.*

2. Results and discussion

Taking a maximum condition, we prove the following fixed-point theorem for mappings with (ψ, β)-Geraghty [12] contraction type condition extended to partially ordered fuzzy metric spaces with the help of an altering distance function ϕ.

THEOREM 2.1. *Let $(X, M, T, \preceq)$ be a partially ordered complete fuzzy metric space and $x_0 \in X$ be such that $x_0 \preceq f(x_0)$. Let f be a non decreasing continuous self mapping on X such that*

$$\psi(\phi(M(f(x), f(y), t))) \leq \alpha(M_\phi(x, y))\beta(M_\phi(x, y))\psi(M_\phi(x, y)) \tag{1}$$

for all $x \preceq y$, where

$$M_\phi(x, y) = \max\{\phi(M(x, y, t)), \phi(M(x, f(x), t)), \phi(M(y, f(y), t)), \phi(M(f(x), y, t))\},$$

$\alpha, \beta \in \mathcal{S}$, ϕ is an altering distance function and $\psi : [0, 1) \longrightarrow [0, 1)$ is a non decreasing continuous function. Then f has a fixed-point in X.

Proof. We put $x_n = f^n(x_0)$, $n = 1, 2, 3, \ldots.$ Then since $x_0 \preceq f(x_0)$ and f is non decreasing, so by induction we obtain, $x_n \preceq x_{n+1}$, that is, x_n and x_{n+1} are comparable for each $n \in \mathbb{N}$.
By (1),

$$\begin{aligned}\psi(\phi(M(x_{n+1}, x_{n+2}, t))) &= \psi(\phi(M(f(x_n), f(x_{n+1}), t))) \\ &\leq \alpha(M_\phi(x_n, x_{n+1}))\beta(M_\phi(x_n, x_{n+1}))\psi(M_\phi(x_n, x_{n+1})),\end{aligned} \tag{2}$$

where

$$\begin{aligned}M_\phi(x_n, x_{n+1}) &= \max\{\phi(M(x_n, x_{n+1}, t)), \phi(M(x_n, f(x_n), t)), \phi(M(x_{n+1}, f(x_{n+1}), t)), \\ &\qquad \phi(M(f(x_n), x_{n+1}, t))\} \\ &= \max\{\phi(M(x_n, x_{n+1}, t)), \phi(M(x_n, x_{n+1}, t)), \phi(M(x_{n+1}, x_{n+2}, t)), \\ &\qquad \phi(M(x_{n+1}, x_{n+1}, t))\} \\ &= \max\{\phi(M(x_n, x_{n+1}, t)), \phi(M(x_{n+1}, x_{n+2}, t))\}.\end{aligned}$$

Now, if $M_\phi(x_n, x_{n+1}) = \max\{\phi(M(x_n, x_{n+1}, t)), \phi(M(x_{n+1}, x_{n+2}, t))\} = \phi(M(x_{n+1}, x_{n+2}, t))$, and since $\alpha, \beta \in \mathcal{S}$ we have from (2),

$$\psi(\phi(M(x_{n+1}, x_{n+2}, t))) < \psi(\phi(M(x_{n+1}, x_{n+2}, t))),$$

which is impossible.
Thus, $\max\{\phi(M(x_n, x_{n+1}, t)), \phi(M(x_{n+1}, x_{n+2}, t))\} = \phi(M(x_n, x_{n+1}, t))$, and therefore from (2),

$$\psi(\phi(M(x_{n+1}, x_{n+2}, t))) < \psi(\phi(M(x_n, x_{n+1}, t))).$$

Since ψ is non decreasing, it gives

$$\phi(M(x_{n+1}, x_{n+2}, t)) < \phi(M(x_n, x_{n+1}, t)).$$

Thus, $\{\phi(M(x_n, x_{n+1}, t))\}$ is a decreasing sequence which is bounded below. Therefore, it converges to $l \geq 0$ (say).
Now, $M_\phi(x_n, x_{n+1})) \geq \phi(M(x_n, x_{n+1}, t))$
$\Rightarrow \lim_{n\to\infty} M_\phi(x_n, x_{n+1}) \geq l$.
Assume that $l > 0$. Then in (2), taking $n \to \infty$ we have,

$$\begin{aligned}\lim_{n\to\infty} \psi(\phi(M(x_{n+1}, x_{n+2}, t))) &\leq \lim_{n\to\infty} [\alpha(M_\phi(x_n, x_{n+1}))\beta(M_\phi(x_n, x_{n+1})) \\ &\qquad \psi(M_\phi(x_n, x_{n+1}))]\end{aligned}$$

$$\Rightarrow \psi(l)[1 - \lim_{n\to\infty} \alpha(M_\phi(x_n, x_{n+1})) \lim_{n\to\infty} \beta(M_\phi(x_n, x_{n+1}))] \leq 0$$

$$\Rightarrow \lim_{n\to\infty} \alpha(M_\phi(x_n, x_{n+1})) \lim_{n\to\infty} \beta(M_\phi(x_n, x_{n+1})) = 1.$$

Since $\alpha, \beta \in \mathcal{S}$, this implies that $\lim_{n\to\infty} M_\phi(x_n, x_{n+1})) = 0$, that is, $l = 0$, which is a contradiction to our assumption that $l > 0$. Thus,

$$\lim_{n\to\infty} \phi(M(x_n, x_{n+1}, t)) = 0,$$

that is,

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1. \tag{3}$$

Next we show that $\{x_n\}$ is a Cauchy sequence.
To the contrary, suppose there exist $0 < \epsilon < 1, t > 0$ for which we can find two subsequences $\{x_{r(n)}\}$ and $\{x_{s(n)}\}$ of $\{x_n\}$ with $r(n) > s(n) > n, n \in \mathbb{N} \cup \{0\}$ such that

$$M(x_{r(n)}, x_{s(n)}, t) \leq 1 - \epsilon. \tag{4}$$

Further, corresponding to $r(n)$, we can choose $s(n)$ such that it is the smallest integer satisfying (4) with $r(n) > s(n)$. Then

$$M(x_{r(n)-1}, x_{s(n)}, t) > 1 - \epsilon. \tag{5}$$

Now,

$$M(x_{r(n)-1}, x_{s(n)-1}, t) \geq T(M(x_{r(n)-1}, x_{s(n)}, \frac{t}{2}), M(x_{s(n)}, x_{s(n)-1}, \frac{t}{2})), \ n \in \mathbb{N},$$

that is, $M(x_{r(n)-1}, x_{s(n)-1}, t) \geq T(1 - \epsilon, M(x_{s(n)}, x_{s(n)-1}, \frac{t}{2}))$ (From (5))

Taking limit as $n \longrightarrow \infty$ and using (3),

$$\lim_{n\to\infty} M(x_{r(n)-1}, x_{s(n)-1}, t) \geq T(1 - \epsilon, 1) = 1 - \epsilon,$$

that is,

$$M(x_{r(n)-1}, x_{s(n)-1}, t) \geq 1 - \epsilon. \tag{6}$$

Again, from (4),

$$\begin{aligned} 1 - \epsilon &\geq M(x_{r(n)}, x_{s(n)}, 4t), \\ &\geq T(M(x_{r(n)}, x_{r(n)-1}, 2t), T(M(x_{r(n)-1}, x_{s(n)-1}, t), M(x_{s(n)}, x_{s(n)-1}, t))) \end{aligned}$$

So, as $n \to \infty$, $1 - \epsilon \geq T(1, T(\lim_{n\to\infty} M(x_{r(n)-1}, x_{s(n)-1}, t), 1))$ (By (3)),

that is,

$$1 - \epsilon \geq \lim_{n\to\infty} M(x_{r(n)-1}, x_{s(n)-1}, t). \tag{7}$$

From (6) and (7),

$$\lim_{n\to\infty} M(x_{r(n)-1}, x_{s(n)-1}, t) = 1 - \epsilon. \tag{8}$$

Since $r(n)$ and $s(n)$ are comparable, from (4), using (1), we have,

$$\begin{aligned} &\psi(\phi(1 - \epsilon)) \leq \psi(\phi(M(x_{r(n)}, x_{s(n)}, t))) \\ &\quad \leq \alpha(M_\phi(x_{r(n)-1}, x_{s(n)-1}))\beta(M_\phi(x_{r(n)-1}, x_{s(n)-1}))\psi(M_\phi(x_{r(n)-1}, x_{s(n)-1})). \end{aligned}$$

Taking $n \longrightarrow \infty$ and by (8),

$$\psi(\phi(1 - \epsilon)) \leq \lim_{n\to\infty} \alpha(M_\phi(x_n, x_{n+1})) \lim_{n\to\infty} \beta(M_\phi(x_n, x_{n+1}))\psi(\phi(1 - \epsilon))$$

$$\Rightarrow \psi(\phi(1 - \epsilon))[1 - \lim_{n\to\infty} \alpha(M_\phi(x_n, x_{n+1})) \lim_{n\to\infty} \beta(M_\phi(x_n, x_{n+1}))] \leq 0.$$

Since $\alpha, \beta \in \mathcal{S}$, this implies that $\epsilon = 0$, which is a contradiction.
Therefore, $\{x_n\}$ is a Cauchy sequence in X.
Since (X, M, T) is a complete fuzzy metric space, there exists $p \in X$ such that $x_n \to p$.

Next we prove that p is a fixed-point of f.
We have,

$$\begin{aligned}
&M(p, \lim_{n\to\infty} x_n, t) = 1\\
\Rightarrow &M(p, \lim_{n\to\infty} f^n(x_0), t) = 1\\
\Rightarrow &M(p, f(\lim_{n\to\infty} f^{n-1}x_0), t) = 1, \text{ by continuity of } f\\
\Rightarrow &M(p, f(p), t) = 1,\\
&\text{that is, } p = f(p).
\end{aligned}$$

Thus f has a fixed-point. □

Similarly, for the minimum condition, we have the following theorem.

THEOREM 2.2. *Let $(X, M, T, \preceq)$ be a partially ordered complete fuzzy metric space and $x_0 \in X$ be such that $x_0 \preceq f(x_0)$. Let f be a non decreasing continuous self mapping on X such that*

$$\psi(\phi(M(f(x), f(y), t))) \leq \alpha(m_\phi(x, y))\beta(m_\phi(x, y))\psi(m_\phi(x, y)) \tag{9}$$

for all $x \preceq y$, where

$$m_\phi(x, y) = \min\{\phi(M(x, y, t)), \phi(M(x, f(x), t)), \phi(M(y, f(y), t)), \phi(M(x, f(y), 2t))\},$$

$\alpha, \beta \in \mathcal{S}$, ϕ is an altering distance function and $\psi : [0, 1) \longrightarrow [0, 1)$ is a non decreasing continuous function. Then f has a fixed-point.

Proof. As in Theorem 2.1, taking $x_n = f^n(x_0)$, $n = 1, 2, 3, ...$, we can show that x_n and x_{n+1} are comparable for each $n \in \mathbb{N}$.
By (9),

$$\begin{aligned}
\psi(\phi(M(x_{n+1}, x_{n+2}, t))) &= \psi(\phi(M(f(x_n), f(x_{n+1}), t)))\\
&\leq \alpha(m_\phi(x_n, x_{n+1}))\beta(m_\phi(x_n, x_{n+1}))\psi(m_\phi(x_n, x_{n+1})),
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
m_\phi(x_n, x_{n+1}) &= \min\{\phi(M(x_n, x_{n+1}, t)), \phi(M(x_n, f(x_n), t)), \phi(M(x_{n+1}, f(x_{n+1}), t)),\\
&\qquad \phi(M(x_n, f(x_{n+1}), 2t))\}\\
&= \min\{\phi(M(x_n, x_{n+1}, t)), \phi(M(x_n, x_{n+1}, t)), \phi(M(x_{n+1}, x_{n+2}, t)),\\
&\qquad \phi(M(x_n, x_{n+2}, 2t))\}\\
&\leq \min\{\phi(M(x_n, x_{n+1}, t)), \phi(M(x_{n+1}, x_{n+2}, t)),\\
&\qquad \phi\big(T(M(x_n, x_{n+1}, t), M(x_{n+1}, x_{n+2}, t))\big)\}.
\end{aligned}$$

Again,

$$\begin{aligned}
&\phi(u) \leq \phi(T(u, v)) \text{ and } \phi(v) \leq \phi(T(u, v))\\
&(\text{Since, } u = T(u, 1) \geq T(u, v) \text{ and } v = T(v, 1) \geq T(u, v))
\end{aligned}$$

So, $\quad \min\{\phi(u), \phi(v), \phi(T(u, v))\} = \min\{\phi(u), \phi(v)\}$ for all $u, v \in [0, 1]$.

Hence,

$$\begin{aligned} m_\phi(x_n, x_{n+1}) &= \min\{\phi(M(x_n, x_{n+1}, t)), \phi(M(x_{n+1}, x_{n+2}, t)), \\ &\qquad \phi\big(T(M(x_n, x_{n+1}, t), M(x_{n+1}, x_{n+2}, t))\big)\} \\ &\le \min\{\phi(M(x_n, x_{n+1}, t)), \phi(M(x_{n+1}, x_{n+2}, t))\}. \end{aligned}$$

Now, if $\min\{\phi(M(x_n, x_{n+1}, t)), \phi(M(x_{n+1}, x_{n+2}, t))\} = \phi(M(x_{n+1}, x_{n+2}, t))$, and since $\alpha, \beta \in \mathcal{S}$ and ψ is nondecreasing, we have from (10),

$$\psi(\phi(M(x_{n+1}, x_{n+2}, t))) < \psi(\phi(M(x_{n+1}, x_{n+2}, t))),$$

which is a contradiction.
Thus, $\min\{\phi(M(x_n, x_{n+1}, t)), \phi(M(x_{n+1}, x_{n+2}, t))\} = \phi(M(x_n, x_{n+1}, t))$, and therefore from (10),

$$\psi(\phi(M(x_{n+1}, x_{n+2}, t))) < \psi(\phi(M(x_n, x_{n+1}, t))).$$

Since ψ is non decreasing, we have,

$$\phi(M(x_{n+1}, x_{n+2}, t)) < \phi(M(x_n, x_{n+1}, t)).$$

Thus $\{\phi(M(x_n, x_{n+1}, t))\}$ is a decreasing sequence which is bounded below. Now proceeding similarly as in Theorem 2.1, we can show that f has a fixed-point. $\square$

The following theorem uses an additional requirement of a comparable element for the existence of a unique fixed-point.

Theorem 2.3. *Let $(X, M, T, \preceq)$ be a partially ordered complete fuzzy metric space, $x_0 \in X$ be such that $x_0 \preceq f(x_0)$ and f be a non decreasing continuous self mapping on X such that*

$$\psi(\phi(M(f(x), f(y), t))) \le \alpha(\phi(M(x, y, t)))\beta(\phi(M(x, y, t)))\psi(\phi(M(x, y, t))) \quad \text{for all } x \preceq y, \tag{11}$$

where $\alpha, \beta \in \mathcal{S}$, ϕ is an altering distance function and $\psi : [0, 1) \longrightarrow [0, 1)$ is a non decreasing continuous function. Then f has a fixed-point. Moreover, if

$$\textit{for every } x, y \in X, \textit{ there exists } z \in X \textit{ which is comparable to } x \textit{ and } y,$$

then f has a unique fixed-point.

Proof. The existence of fixed-point follows from Theorem 2.1 as a particular case taking $M_\phi(x, y) = \phi(M(x, y, t))$.
Next we establish the uniqueness.
Suppose there is another fixed-point of f, say r. Then there exist $s \in X$ which is comparable to both r and p. Since f is monotonic we have $f^n(s)$ is comparable to $f^n(r) = r$ and $f^n(p) = p$ for $n = 1, 2, 3, \ldots$. Now,

$$\begin{aligned} \psi(\phi(M(p, f^n(s), t))) &= \psi(\phi(M(f^n(p), f^n(s), t))) \\ &\le \alpha(M(f^{n-1}(p), f^{n-1}(s), t)))\beta(\phi(M(f^{n-1}(p), f^{n-1}(s), t))) \\ &\qquad \psi(\phi(M(f^{n-1}(p), f^{n-1}(s), t))) \\ &\le \psi(\phi(M(f^{n-1}(p), f^{n-1}(s), t))) \end{aligned}$$

which implies

$$\phi(M(p, f^n(s), t)) \le \phi(M(f^{n-1}(p), f^{n-1}(s), t)). \tag{12}$$

Thus the sequence $\{\phi(M(p, f^n(s), t))\}$ is a decreasing sequence, that is bounded below and it can be shown to be convergent to 0.

$$\text{Thus} \quad \lim_{n\to\infty} \phi(M(p, f^n(s), t)) = 0,$$
$$\text{that is,} \quad \lim_{n\to\infty} M(p, f^n(s), t) = 1.$$

Similarly, we can show that $\lim_{n\to\infty} M(f^n(s), r, t) = 1$.
Now,

$$\begin{aligned}
&\phi(M(p,r,t)) \leq \phi(T(M(p, f^n(s), \frac{t}{2}), M(f^n(s), r, \frac{t}{2})))\\
\Rightarrow\ &\phi(M(p,r,t)) \leq \phi(1) \quad \text{as } n \longrightarrow \infty\\
\Rightarrow\ &\phi(M(p,r,t)) = 0\\
\Rightarrow\ &M(p,r,t) = 1,\\
\text{that is,}\ \ &p = r.
\end{aligned}$$

Hence f has a unique fixed-point. □

EXAMPLE 2.4. Let $X = [0, 2)$ with the Euclidean distance d. Then, (X, d) is a bounded and complete metric space. Let $M(x, y, t) = 1 - \frac{d(x,y)}{2}, x, y \in X, t > 0$. Then (X, M, T) is a partially ordered complete fuzzy metric space with respect to the t-norm, $T(x, y) = \max(x + y - 1, 0)$ $x, y \in [0, 1]$ and the usual partial ordering $(\leq)$ in $\mathbb{R}$. Also let $\phi(t) = 1 - t$, $t \in [0, 1]$ be the altering distance function.

Next, we consider the function $f : X \longrightarrow X$ given by $f(x) = \frac{x}{2}$. Let $\psi(s) = s^3, \alpha(t) = \frac{1}{1+t^2}$ and $\beta(t) = \frac{1}{1+t^3}$, $s \in [0, 1)$, $t \in [0, \infty)$. Now for $x = y$, the condition (11) is obvious.

Again, for $x \neq y$ we have

$$\begin{aligned}
&\psi(\phi(M(f(x), f(y), t)))\\
=&\psi\left(\frac{|\frac{x}{2} - \frac{y}{2}|}{2}\right)\\
=&(\frac{|\frac{x}{2} - \frac{y}{2}|}{2})^3\\
=&\frac{1}{8}(\frac{|x-y|}{2})^3,
\end{aligned}$$

and

$$\begin{aligned}
&\alpha(\phi(M(x, y, t)))\beta(\phi(M(x, y, t)))\psi(\phi(M(x, y, t)))\\
=&\alpha(\frac{|x-y|}{2})\beta(\frac{|x-y|}{2})\psi(\frac{|x-y|}{2})\\
=&\frac{1}{1+(\frac{|x-y|}{2})^2}\frac{1}{1+(\frac{|x-y|}{2})^3}(\frac{|x-y|}{2})^3\\
=&\frac{1}{1+(\frac{|x-y|}{2})^2}\frac{1}{1+(\frac{|x-y|}{2})^3}(\frac{|x-y|}{2})^3.
\end{aligned}$$

Since the minimum value of $\frac{1}{1+(\frac{|x-y|}{2})^2}\frac{1}{1+(\frac{|x-y|}{2})^3}$ is $\frac{1}{4}$, so

$$\frac{1}{8}(\frac{|x-y|}{2})^3 \leq \frac{1}{1+(\frac{|x-y|}{2})^2}\frac{1}{1+(\frac{|x-y|}{2})^3}(\frac{|x-y|}{2})^3.$$

Hence the condition (11) is satisfied.
Moreover, since all the element of X are comparable, f has a unique fixed-point in X. In fact here 0 is the unique fixed-point of f.

Example 2.5. Let $X = \{a, b, c, d\}$ such that
$d(a, b) = d(a, d) = d(b, c) = d(b, d) = d(c, d) = 4, d(a, c) = 2$
and $d(a, a) = d(b, b) = d(c, c) = d(d, d) = 0$.
Then, (X, d) is a bounded and complete metric space. Let $M(x, y, t) = 1-\frac{d(x,y)}{2}, x, y \in X, t > 0$. Define partial ordering ($\preceq$) in X with the alphabetical order. Then (X, M, T) is a partially ordered complete fuzzy metric space with respect to the t-norm, $T(x, y) = \max(x + y - 1, 0)\ \ x, y \in [0, 1]$. Also let $\phi(t) = 1 - t,\ t \in [0, 1]$ be the altering distance function.

Next, we consider the function $f : X \longrightarrow X$ given by:

$$f(x) = \begin{cases} c, & \text{if } x = b \\ a, & \text{if } x \neq b \end{cases}.$$

Let $\psi(s) = s^3, \alpha(t) = \beta(t) = e^{-\frac{t}{2}}$ and $\alpha(0) = \beta(0) = 0,\ s \in [0, 1),\ t \in [0,\infty)$.
Now for $x = y$, the condition (11) is obvious.

Again, for $x \neq y$,
Case 1: If x, y takes any of the values a, c, d, then again the condition (11) is obvious.
Case 2: If x(or, y) takes value b and y(or, x) takes any value from a, c, d, then we have,

$$\begin{aligned} &\psi(\phi(M(f(x), f(y), t))) \\ =&\psi\left(\frac{d(a, c)}{2}\right) \\ =&1 \end{aligned}$$

and

$$\begin{aligned} &\alpha(\phi(M(x, y, t)))\beta(\phi(M(x, y, t)))\psi(\phi(M(x, y, t))) \\ =&\alpha(\frac{d(x, y)}{2})\beta(\frac{d(x, y)}{2})\psi(\frac{d(x, y)}{2}) \\ =&(e^{-\frac{d(x,y)}{4}})^2(\frac{d(x, y)}{2})^3 \\ =&8(e^{-1})^2 \\ =&1.083 \end{aligned}$$

Hence the condition (11) is satisfied.
Moreover, since all the element of X are comparable, f has a unique fixed-point in X. In fact here a is the unique fixed-point of f.

3. Application to boundary value problems

We consider the following second order boundary value problem [20]:

$$y''(t) = -\lambda^2 y(t) + f(y(t)), \quad t \in I = [0,1],$$
$$y(0) = y(1) = 0, \tag{13}$$

where λ is a constant lying in the interval $(0, \frac{\pi}{2})$,

which is equivalent to the integral equation $u(t) = \int_0^1 G(t,s)f(u(s))ds, \quad 0 < t, s < 1$,
where $G(t,s) = \begin{cases} -\frac{\sin(\lambda s)\ \sin(\lambda(1-t))}{\lambda \sin \lambda}, & t \geq s \\ -\frac{\sin(\lambda t)\ \sin(\lambda(1-s))}{\lambda \sin \lambda}, & t \leq s \end{cases}$
is the Green's function.

Now, we establish a result which gives the condition for existence of unique solution to (13).
Let $C(I,\mathbb{R})$ denote the set of all continuous functions $f : I \longrightarrow \mathbb{R}$ such that for $x, y \in C(I,\mathbb{R})$, $|x(t) - y(t)| < k$ for some $k > 0$, and for all $t \in I$.

Theorem 3.1. *Considering the above problem* (13) *with f continuous, suppose that the following conditions are satisfied:*

(i) *$a, b \in \mathbb{R}$ with $b \geq a$ implies $f(a) \geq f(b)$,*
(ii) *for all $t \in I$ and $a, b \in \mathbb{R}$,*

$$|f(t,a) - f(t,b)| \leq \lambda^2 \frac{k}{3} \xi \left(\frac{|a-b|}{k} \right)$$

where $\xi : [0,\infty) \longrightarrow [0,\infty)$ is defined by $\xi(x) = \frac{x}{\sqrt{(1+x)(1+x^2)}}$, $x \in [0,\infty)$.

Then there exists a unique solution of (13).

Proof. Let $F : C(I,\mathbb{R}) \longrightarrow C(I,\mathbb{R})$ be defined by:

$$(Fu)(t) = \int_0^1 G(t,s)\Big(f(u(s))\Big)ds.$$

If $u \in C(I,\mathbb{R})$ is a fixed-point of F then $u \in C(I,\mathbb{R})$ is a solution of (13). Now we check that hypotheses in Theorem 2.3 are satisfied.

For $x, y \in C(I,\mathbb{R})$, we take $x \succeq y$ if and only if $x(t) \geq y(t)$ for all $t \in I$. Then $X = C(I,\mathbb{R})$ is a partially ordered set.
Also $x, y \in C(I,\mathbb{R})$ implies $|x(t) - y(t)| < k$, for all $t \in I$, that is, $X = C(I,\mathbb{R})$ is a bounded metric space with metric d, where $d(x,y) = \sup_{t \in I} |x(t) - y(t)|, \quad x, y \in C(I,\mathbb{R})$.
Taking $M(x,y,t) = 1 - \frac{d(x,y)}{k}, x, y \in X, t > 0,\ \ (X, M, T)$ is a complete fuzzy metric space with respect to the t-norm, $T(x,y) = \max(x + y - 1, 0) \ \ x, y \in [0,1]$. Also let $\phi(t) = 1 - t,\ t \in [0,1]$ be the altering distance function.
By the hypothesis (i), for $u, v \in X$ with $u \succeq v$, we have,
$f(v(s)) \geq f(u(s))$, for all $s \in I$.
This implies

$$(Fu)(t) = \int_0^1 G(t,s)f(u(s))ds$$

$$\geq \int_0^1 G(t,s)f(v(s))ds = (Fv)(t),$$

that is, the mapping F is non decreasing.
Now, for $u \succeq v$ and $t_1 > 0$,

$$\begin{aligned}
\phi(M(Fu,Fv,t_1)) &= \frac{d(Fu,Fv)}{k}\\
&= \frac{\sup_{t\in I}|(Fu)(t)-(Fv)(t))|}{k}\\
&= \frac{1}{k}\sup_{t\in I}\Big|\int_0^1 G(t,s)f(t,u)ds - \int_0^1 G(t,s)f(t,v)ds\Big|\\
&= \frac{1}{k}\sup_{t\in I}\Big|\int_0^1 G(t,s)(f(t,u)-f(t,v))ds\Big|\\
&\leq \frac{1}{k}\sup_{t\in I}\Big|\int_0^1 \lambda^2 G(t,s)\frac{k}{3}\xi\left(\frac{|u(s)-v(s)|}{k}\right)ds\Big|\\
&= \sup_{t\in I}\Big|\int_0^1 G(t,s)\frac{\lambda^2}{3}\xi\left(\frac{|u(s)-v(s)|}{k}\right)ds\Big|\\
&\leq \sup_{t\in I}\Big|\int_0^1 G(t,s)\frac{\lambda^2}{3}\xi\left(\frac{d(u,v)}{k}\right)ds\Big| \quad \text{(since } \xi \text{ is non-decreasing)}\\
&= \frac{\lambda^2}{3}\xi\left(\frac{d(u,v)}{k}\right)\sup_{t\in I}\Big|\int_0^1 G(t,s)ds\Big|\\
&= \frac{\lambda^2}{3}\xi\left(\frac{d(u,v)}{k}\right)\sup_{t\in I}\Big|\int_0^t -\frac{\sin(\lambda s)\ \sin(\lambda(1-t))}{\lambda\sin\lambda}ds\\
&\qquad + \int_t^1 -\frac{\sin(\lambda t)\ \sin(\lambda(1-s))}{\lambda\sin\lambda}ds\Big|\\
&= \frac{\lambda^2}{3}\xi\left(\frac{d(u,v)}{k}\right)\sup_{t\in I}\Big|\frac{1}{\lambda^2\sin\lambda}[\cos(\lambda t)\sin(\lambda(1-t))\\
&\qquad -\sin(\lambda(1-t))+\sin(\lambda t)-\sin(\lambda t)\cos(\lambda(1-t))]\Big|\\
&= \frac{\lambda^2}{3}\xi\left(\frac{d(u,v)}{k}\right)\sup_{t\in I}\Big|\frac{1}{\lambda^2\sin\lambda}[\sin(\lambda-\lambda t-\lambda t)-\sin(\lambda(1-t))-\sin(\lambda t)]\Big|\\
&= \frac{\lambda^2}{3}\xi\left(\frac{d(u,v)}{k}\right)\sup_{t\in I}\Big|\frac{1}{\lambda^2\sin\lambda}[\sin(\lambda-2\lambda t)-\sin(\lambda(1-t))-\sin(\lambda t)]\Big|\\
&\leq \frac{1}{3}\xi\left(\frac{d(u,v)}{k}\right)3 = \xi\left(\frac{d(u,v)}{k}\right)
\end{aligned}$$

$$\Rightarrow \phi(M(Fu,Fv,t_1)) \leq \frac{\frac{d(u,v)}{k}}{\sqrt{\left(1+\frac{d(u,v)}{k}\right)\left(1+\left(\frac{d(u,v)}{k}\right)^2\right)}}$$

$$\Rightarrow (\phi(M(Fu,Fv,t_1)))^2 \leq \left(\frac{1}{1+\frac{d(u,v)}{k}}\right)\left(\frac{1}{1+\left(\frac{d(u,v)}{k}\right)^2}\right)\left(\frac{d(u,v)}{k}\right)^2$$

$$= \left(\frac{1}{1+\phi(M(u,v,t_1))}\right)\left(\frac{1}{1+(\phi(M(u,v,t_1)))^2}\right)(\phi(M(u,v,t_1)))^2$$
$$\Rightarrow \psi(\phi(M(Fu,Fv,t_1))) = \alpha(\phi(M(u,v,t_1)))\beta(\phi(M(u,v,t_1)))\psi(\phi(M(u,v,t_1)))$$

where $\psi(t) = t^2, \alpha(t) = \frac{1}{1+t}$ and $\beta(t) = \frac{1}{1+t^2}$. Clearly $\alpha, \beta \in \mathcal{S}$ and ψ is a non decreasing continuous function.
Hence, the hypotheses of Theorem 2.3 are satisfied and therefore, F has a unique fixed-point. □

Next, we take another boundary value problem [21]:

$$y''(t) + \frac{1}{t}y(t) = -f(t, y(t)),\ t \in I = [0,1] \qquad (14)$$
$$y'(0) = y(1) = 0,$$

where $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is non negative and continuous function.

The solutions of the above boundary value problem is the fixed-points of the operator F on $C(I, E)$ defined by:

$$(Fu)(t) = \int_0^1 G(t,s)f(s,u(s))ds \qquad (15)$$

where $G(t,s) = \begin{cases} -s(\log t), & 0 \le s < t \le 1 \\ -s(\log s), & 0 \le t \le s \le 1 \end{cases}$
is the Green's function (refer to [21]).

Theorem 3.2. *Consider the problem* (14) *with f continuous and suppose that*

(a) *$a, b \in \mathbb{R}$ with $b \ge a$ implies $f(t,b) \ge f(t,a)$ for all $t \in I$,*
(b) *for all $t \in I$ and $a, b \in \mathbb{R}$,*

$$0 < |f(t,a) - f(t,b)| \le 4k\xi\left(\frac{|a-b|}{k}\right)$$

where $\xi : [0,\infty) \longrightarrow [0,\infty)$ is defined by $\xi(x) = \frac{x}{\sqrt{(1+x)(1+x^2)}}$.

Then there exists a unique solution of (14).

Proof. The unique solution of (14) exists if the fixed-point of (15) exists and is unique.

If $u \in C(I,\mathbb{R})$ is a fixed-point of F then $u \in C(I,\mathbb{R})$ is a solution of (14). Now we check that hypotheses in Theorem 2.3 are satisfied.
We consider X, d, M, T and ϕ as in the Theorem 3.1

By the hypothesis (i), for $u, v \in X$ with $u \succeq v$, we have,
$f(s,u(s)) \ge f(s,v(s))$, for all $s \in I$.
This implies

$$(Fu)(t) = \int_0^T G(t,s)f(s,u(s))ds$$
$$\ge \int_0^T G(t,s)f(s,v(s))ds = (Fv)(t),$$

that is, the mapping F is non decreasing.
Now, for $u \succeq v$ and $t_1 > 0$,

$$\begin{aligned}\phi(M(Fu,Fv,t_1)) =&\frac{d(Fu,Fv)}{k} = \frac{\sup_{t\in I}|(Fu)(t)-(Fv)(t)|}{k}\\ =&\frac{1}{k}\sup_{t\in I}\ |\int_0^1 G(t,s)f(s,u(s))ds - \int_0^1 G(t,s)f(s,v(s))ds|\\ \leq&\frac{1}{k}\sup_{t\in I}|\int_0^1 G(t,s)(f(s,u(s)) - f(s,v(s))ds|\\ \leq&\frac{1}{k}\sup_{t\in I}\Big|\int_0^1 4kG(t,s)\xi\left(\frac{|u(s)-v(s)|}{k}\right)\ ds\Big|.\end{aligned}$$

Since the function $\xi(x)$ is non decreasing for $x \in [0, 1]$, therefore

$$\begin{aligned}\phi(M(Fu,Fv,t_1)) \leq&\sup_{t\in I}\Big|\int_0^1 G(t,s)4\xi\Big(\frac{d(u,v)}{k}\Big)\ ds\Big|\\ =&4\ \xi\Big(\frac{d(u,v)}{k}\Big)\ \sup_{t\in I}\Big|\int_0^1 G(t,s)ds\Big|\\ =&4\ \xi\Big(\frac{d(u,v)}{k}\Big)\ \sup_{t\in I}\Big|\Big(\Big[-\frac{s^2}{2}\log t\Big]_0^t + \Big[\frac{s^2}{4}-\frac{s^2}{2}\log s\Big]_t^1\Big)\Big|\\ =&4\ \xi\Big(\frac{d(u,v)}{k}\Big)\ \sup_{t\in I}\Big|\Big(-\frac{t^2}{2}\log t+\frac{1}{4}-\frac{t^2}{4}+\frac{t^2}{2}\log t\Big)\Big|\\ =&4\ \xi\Big(\frac{d(u,v)}{k}\Big)\ \sup_{t\in I}\Big|\Big(\frac{1}{4}-\frac{t^2}{4}\Big)\Big|\\ =&4\ \xi\Big(\frac{d(u,v)}{k}\Big)\ \frac{1}{4}\\ =&\xi\Big(\frac{d(u,v)}{k}\Big)\end{aligned}$$

$$\Rightarrow \phi(M(Fu,Fv,t_1)) \leq \frac{\frac{d(u,v)}{k}}{\sqrt{\left(1+\frac{d(u,v)}{k}\right)\left(1+\left(\frac{d(u,v)}{k}\right)^2\right)}}$$

$$\begin{aligned}\Rightarrow (\phi(M(Fu,Fv,t_1)))^2 \leq& \left(\frac{1}{1+\frac{d(u,v)}{k}}\right)\left(\frac{1}{1+\left(\frac{d(u,v)}{k}\right)^2}\right)\left(\frac{d(u,v)}{k}\right)^2\\ =&\left(\frac{1}{1+\phi(M(u,v,t_1))}\right)\left(\frac{1}{1+(\phi(M(u,v,t_1)))^2}\right)(\phi(M(u,v,t_1)))^2\\ \Rightarrow \psi(\phi(M(Fu,Fv,t_1))) =&\alpha(\phi(M(u,v,t_1)))\beta(\phi(M(u,v,t_1)))\psi(\phi(M(u,v,t_1)))\end{aligned}$$

where $\psi(t) = t^2, \alpha(t) = \frac{1}{1+t}$ and $\beta(t) = \frac{1}{1+t^2}$. Clearly $\alpha, \beta \in \mathcal{S}$ and ψ is non decreasing continuous function.
Hence, the hypotheses of Theorem 2.3 are satisfied and therefore, F has a unique fixed-point.

4. Conclusion

We have proved some fixed-point theorems in partially ordered fuzzy metric space for some generalized contraction mappings and showed the existence of unique solution to different boundary value problems using these theorems. Similar investigation can be done considering initial value problems.

In [2], Chandok introduced the concept of (α, β)-admissible Geraghty type contractive mappings and proved some fixed-point theorems for such types of mappings in metric spaces. In this context, one may investigate for analogous results for (ϕ, ψ, β)-Geraghty contraction type mappings in partially ordered fuzzy metric spaces.

The study of geometrical aspect of fixed-point theory is also of importance now a days and interesting research work is going on in this area. In [17] authors proved some existence and uniqueness theorems of fixed-circle for some self mappings in metric spaces with geometric interpretation. In [15] authors proved some fixed-circle theorems in an s-metric space for self mappings with different contraction and contractive type conditions. In this context one can investigate fixed-circle theorems for (ϕ, ψ, β)-Geraghty contraction type mappings in partially ordered fuzzy metric spaces.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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