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A NOTE ON ENSTRÖM-KAKEYA THEOREM FOR QUATERNIONIC POLYNOMIALS

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Abstract. In this paper, we are concerned with the problem of locating the zeros of regular polynomials of a quaternionic variable with quaternionic coefficients. We derive new bounds of Eneström-Kakeya type for the zeros of these polynomials by virtue of a maximum modulus theorem and the structure of the zero sets in the newly developed theory of regular functions and polynomials of a quaternionic variable. Our results generalize some recently proved results about the distribution of zeros of a quaternionic polynomial.

1. Introduction and statement of results

Various experimental observations and investigations when translated into mathematical language lead to mathematical models. The solution of these models could lead to problems of solving algebraic polynomial equations of certain degree. The exact computation of zeros of polynomials of degree at most four made possible by virtue of algorithms having been devised for such polynomials, no such method is available for accomplishing the same task for polynomials of higher degree. The impossibility of achieving this feat, or in other words, the impossibility of solving by radicals the polynomial equations of degree five or greater is an important milestone in the history of mathematics, occasioned by ground breaking discoveries in algebra by N. H. Abel and E. Galois in the first quarter of the nineteenth century. In view of this and significant applications of zero bounds in scientific disciplines such as stability theory, mathematical biology, communication theory and computer engineering, it became interesting to identify the suitable regions in the complex plane which contain the zeros of a given polynomial. A classical result due to Cauchy [3] on the distribution of zeros of a polynomial may be stated as follows:

Theorem 1.1. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then all the zeros of p lie in*

$$|z| < 1 + \max_{1\leq v\leq n-1} \left|\frac{a_v}{a_n}\right|.$$

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Although various results concerning the bounds for zeros of polynomials are available in literature (see [9], [10]), but the remarkable property of the bound in Theorem 1.1 which distinguishes it from other such bounds is its simplicity of computations. However, this simplicity comes at the cost of precision. The following elegant result on the location of zeros of a polynomial with restricted coefficients is known as Eneström-Kakeya theorem (see [4], [9], [10]) which states that:

Theorem 1.2. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *such that* $0 < a_0 \leq a_1 \leq ... \leq a_n$*, then all the zeros of* p *lie in* $|z| \leq 1$*.*

In literature, for example see ([1], [7]- [10]), there exist various extensions and generalizations of Eneström-Kakeya theorem. By removing non-negative restriction over the coefficients of polynomial $p(z)$, Joyal et al. [8] proved the following result:

Theorem 1.3. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *such that* $a_0 \leq a_1 \leq ... \leq a_n$*, then all the zeros of* p *lie in* $|z| \leq \frac{1}{|a_n|}(|a_0| + a_n - a_0)$*.*

In this paper, we will prove some extensions and generalizations of above results for the class of polynomials with quaternionic variable and quaternionic coefficients.

2. Background

Quaternions are the extension of complex numbers to four dimensions, introduced by William Rowan Hamilton in 1843. The set of all quaternions are denoted by $\mathbb{H}$ in honour of Sir Hamilton and are generally represented in the form $q = \alpha + i\beta + j\gamma + k\delta \in \mathbb{H}$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and i, j, k are the fundamental quaternion units, such that $i^2 = j^2 = k^2 = ijk = -1$. Each quaternion q has a conjugate. The conjuate of a quaternion $q = \alpha + i\beta + j\gamma + k\delta$ is denoted by q^* and is defined as $q^* = \alpha - i\beta - j\gamma - k\delta$. Morever, the norm (or length) of a quaternion q is given by

$$||q|| = \sqrt{qq^*} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}.$$

The quaternions are the standard example of a noncommutative division ring and also forms a four dimensional vector space over $\mathbb{R}$ with $\{1, i, j, k\}$ as a basis.
In 2020, Carney et al. [2] proved the following extension of Theorem 1.2 for the quaternionic polynomial $p(q)$. More prescisely they proved the following result:

Theorem 2.1. *If* $p(q) = \sum_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with real coefficients satisfying* $0 < a_0 \leq a_1 \leq ... \leq a_n$*, then all the zeros of* p *lie in* $|q| \leq 1$*.*

As an extension of Theorem 1.3 to quaternionic polynomial $p(q)$, Carney et al. in the same paper proved the following result:

Theorem 2.2. *If* $p(q) = \sum\limits_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with quaternionic coefficients* $a_v = \alpha_v + i\beta_v + j\gamma_v + k\delta_v,\ v = 0, 1, 2, ..., n$*, and satisfying*

$$\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_0,\ \beta_n \geq \beta_{n-1} \geq ... \geq \beta_0,$$

$$\gamma_n \geq \gamma_{n-1} \geq ... \geq \gamma_0,\ \delta_n \geq \delta_{n-1} \geq ... \geq \delta_0,$$

then all the zeros of p lie in

$$|q| \leq \frac{1}{|a_n|}\Big\{(|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n)\Big\}.$$

Recently, Dinesh Tripathi [11] relaxed the hypothesis of Theorem 2.1 and proved the following intresting result which also provides a generalization to Theorem 2.2.

THEOREM 2.3. *If* $p(q) = \sum_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with quaternionic coefficients* $a_v = \alpha_v + i\beta_v + j\gamma_v + k\delta_v,\ v = 0, 1, 2, ..., n$ *and for some* $0 \leq l \leq n$,

$$\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_l,\ \beta_n \geq \beta_{n-1} \geq ... \geq \beta_l,$$
$$\gamma_n \geq \gamma_{n-1} \geq ... \geq \gamma_l,\ \delta_n \geq \delta_{n-1} \geq ... \geq \delta_l,$$

then all the zeros of p lie in

$$|q| \leq \frac{1}{|a_n|}\Big\{|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + (\alpha_n - \alpha_l) + (\beta_n - \beta_l) + (\gamma_n - \gamma_l) + (\delta_n - \delta_l) + M\Big\},$$

where $M = \sum_{v=1}^{l}\Big\{|\alpha_v - \alpha_{v-1}| + |\beta_v - \beta_{v-1}| + |\gamma_v - \gamma_{v-1}| + |\delta_v - \delta_{v-1}|\Big\}$.

3. Main Results

We begin with the following result which gives a generalization of Theorem 2.3 and hence of Theorem 2.2 as well.

THEOREM 3.1. *If* $p(q) = \sum_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with quaternionic coefficients* $a_v = \alpha_v + i\beta_v + j\gamma_v + k\delta_v,\ v = 0, 1, 2, ..., n$ *and for some* $k_1, k_2, k_3, k_4 \geq 1$, $0 \leq l \leq n$, *and* $0 < \rho \leq 1$,

$$k_1\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_{l+1} \geq \rho\alpha_l,\ k_2\beta_n \geq \beta_{n-1} \geq ... \geq \beta_{l+1} \geq \rho\beta_l,$$
$$k_3\gamma_n \geq \gamma_{n-1} \geq ... \geq \gamma_{l+1} \geq \rho\gamma_l,\ k_4\delta_n \geq \delta_{n-1} \geq ... \geq \delta_{l+1} \geq \rho\delta_l,$$

then all the zeros of p lie in

(1)
$$\begin{aligned} |q| \leq \frac{1}{|a_n|}\Big\{&|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + (k_1\alpha_n - \rho\alpha_l) + (k_2\beta_n - \rho\beta_l) + (k_3\gamma_n - \rho\gamma_l) \\ &+ (k_4\delta_n - \rho\delta_l) + (k_1 - 1)|\alpha_n| + (k_2 - 1)|\beta_n| + (k_3 - 1)|\gamma_n| + (k_4 - 1)|\delta_n| \\ &+ (1 - \rho)(|\alpha_l| + |\beta_l| + |\gamma_l| + |\delta_l|) + M\Big\}, \end{aligned}$$

where $M = \sum_{v=1}^{l}\Big\{|\alpha_v - \alpha_{v-1}| + |\beta_v - \beta_{v-1}| + |\gamma_v - \gamma_{v-1}| + |\delta_v - \delta_{v-1}|\Big\}$.

Applying Theorem 3.1 to the polynomial $p(q)$ having real coefficients, i.e., $\beta = \gamma = \delta = 0$, we have the following result:

Corollary 3.2. *If* $p(q)=\sum_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with real coefficients* $a_v,\ v=0,1,2,...,n$ *and for some* $k_1\geq 1$, $0\leq l\leq n$, *and* $0<\rho\leq 1$,

$$k_1 a_n \geq a_{n-1}\geq ...\geq a_{l+1}\geq \rho a_l,$$

then all the zeros of p *lie in*

$$|q|\leq \frac{1}{|a_n|}\Big\{|a_0|+(k_1a_n-\rho a_l)+(k_1-1)|a_n|+(1-\rho)|a_l|+\sum_{v=1}^{l}|a_v-a_{v-1}|\Big\}. \tag{2}$$

If we take $l=0$ in (1) and (2) respectively, we get the following results:

Corollary 3.3. *If* $p(q)=\sum_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with quaternionic coefficients* $a_v=\alpha_v+i\beta_v+j\gamma_v+k\delta_v,\ v=0,1,2,...,n$ *and for some* $k_1,k_2,k_3,k_4\geq 1$, *and* $0<\rho\leq 1$,

$$k_1\alpha_n\geq \alpha_{n-1}\geq ...\geq \alpha_1\geq \rho\alpha_0,\ k_2\beta_n\geq \beta_{n-1}\geq ...\geq \beta_1\geq \rho\beta_0,$$

$$k_3\gamma_n\geq \gamma_{n-1}\geq ...\geq \gamma_1\geq \rho\gamma_0,\ k_4\delta_n\geq \delta_{n-1}\geq ...\geq \delta_1\geq \rho\delta_0,$$

then all the zeros of p *lie in*

(3)

$$\begin{aligned}|q|\leq \frac{1}{|a_n|}\Big\{&|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+(k_1\alpha_n-\rho\alpha_0)+(k_2\beta_n-\rho\beta_0)+(k_3\gamma_n-\rho\gamma_0)\\&+(k_4\delta_n-\rho\delta_0)+(k_1-1)|\alpha_n|+(k_2-1)|\beta_n|+(k_3-1)|\gamma_n|+(k_4-1)|\delta_n|\\&+(1-\rho)(|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|)\Big\}.\end{aligned}$$

Corollary 3.4. *If* $p(q)=\sum_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with real coefficients* $a_v,\ v=0,1,2,...,n$ *and for some* $k_1\geq 1$, *and* $0<\rho\leq 1$,

$$k_1a_n\geq a_{n-1}\geq ...\geq a_1\geq \rho a_0,$$

then all the zeros of p *lie in*

$$|q|\leq \frac{1}{|a_n|}\Big\{|a_0|+(k_1a_n-\rho a_0)+(k_1-1)|a_n|+(1-\rho)|a_0|\Big\}. \tag{4}$$

Remark 3.5. If we take $k_1=k_2=k_3=k_4=\rho=1$ in (3), we get Theorem 2.2. Similarily if we take $k_1=\rho=1$ in (4), we get the following extension of Theorem 1.3 to quaternion polynomials.

Corollary 3.6. *If* $p(q)=\sum_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with real coefficients* $a_v,\ v=0,1,2,...,n$,

$$a_n\geq a_{n-1}\geq ...\geq a_1\geq a_0,$$

then all the zeros of p lie in

$$|q| \leq \frac{1}{|a_n|}\Big\{|a_0| + a_n - a_0\Big\}. \tag{5}$$

If in (5), we assume that $a_0 > 0$, then from Corollary 3.6, we get Theorem 2.1. Next by considering a more general class of polynomials by putting the monotonicity-type condition on the real and imaginary parts of $p(q) = \sum_{v=0}^{n} q^v a_v$ (some are montonic decreasing and some are monotonic increasing) and proved the following generalization of Theorem 2.2.

Theorem 3.7. *If* $p(q) = \sum_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with quaternionic coefficients* $a_v = \alpha_v + i\beta_v + j\gamma_v + k\delta_v,\ v = 0, 1, 2, ..., n$ *and for some* $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 1$,

$$\alpha_n \leq \alpha_{n-1} \leq ... \leq \alpha_{l+1} \leq \lambda_1\alpha_l \geq \alpha_{l-1} \geq ... \geq \alpha_1 \geq \alpha_0,$$

$$\beta_n \leq \beta_{n-1} \leq ... \leq \beta_{l+1} \leq \lambda_2\beta_l \geq \beta_{l-1} \geq ... \geq \beta_1 \geq \beta_0,$$

$$\gamma_n \leq \gamma_{n-1} \leq ... \leq \gamma_{l+1} \leq \lambda_3\gamma_l \geq \gamma_{l-1} \geq ... \geq \gamma_1 \geq \gamma_0,$$

$$\delta_n \leq \delta_{n-1} \leq ... \leq \delta_{l+1} \leq \lambda_4\delta_l \geq \delta_{l-1} \geq ... \geq \delta_1 \geq \delta_0,$$

$0 \leq l \leq n$, *then all the zeros of p lie in*

$$|q| \leq \frac{1}{|a_n|}\Big\{(|\alpha_0| + 2\lambda_1\alpha_l - \alpha_0 - \alpha_n) + (|\beta_0| + 2\lambda_2\beta_l - \beta_0 - \beta_n) + (|\gamma_0| + 2\lambda_3\gamma_l - \gamma_0 - \gamma_n)$$
$$+ (|\delta_0| + 2\lambda_4\delta_l - \delta_0 - \delta_n) + N\Big\},$$

where $N = 2\Big((\lambda_1 - 1)|\alpha_l| + (\lambda_2 - 1)|\beta_l| + (\lambda_3 - 1)|\gamma_l| + (\lambda_4 - 1)|\delta_l|\Big)$.

Applying Theorem 3.7 to the polynomial $p(q)$ having real coefficients, i.e., $\beta = \gamma = \delta = 0$, we have the following result:

Corollary 3.8. *If* $p(q) = \sum_{v=0}^{n} q^v a_v$ *is a quaternionic polynomial of degree* n *with real coefficients* $a_v,\ v = 0, 1, 2, ..., n$ *and for some* $\lambda_1 \geq 1,\ 0 \leq l \leq n$,

$$a_n \leq a_{n-1} \leq ... \leq a_{l+1} \leq \lambda_1 a_l \geq a_{l-1} \geq ... \geq a_1 \geq a_0,$$

then all the zeros of p lie in

$$|q| \leq \frac{1}{|a_n|}\Big\{(|a_0| + 2\lambda_1 a_l - a_0 - a_n) + 2(\lambda_1 - 1)|a_l|\Big\}.$$

Remark 3.9. If we take $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$ and $l = n$ in Theorem 3.7, we easily get Theorem 2.2.

4. Lemmas

For the proofs of our main results, we need the following lemma due to G. Gentili and C. Stoppato [5].

Lemma 4.1. *If* $f(q)=\sum\limits_{v=0}^{\infty}q^v a_v$ *and* $g(q)=\sum\limits_{v=0}^{\infty}q^v b_v$ *be two given quaternionic power series with radii of convergence greater than R. The regular product of* $f(q)$ *and* $g(q)$ *is defined as* $(f\star g)(q)=\sum\limits_{v=0}^{\infty}q^v c_v$*, where* $c_v=\sum\limits_{l=0}^{\infty}a_l b_{v-l}$*. Let* $|q_0|<R$*, then* $(f\star g)(q_0)=0$ *if and only if either* $f(q_0)=0$ *or* $f(q_0)\neq 0$ *implies* $g(f(q_0)^{-1}q_0 f(q_0))=0$*.*

5. Proof of Theorems

Proof of Theorem 3.1. Consider the polynomial

$$f(q)=\sum_{v=1}^{n}q^v(a_v-a_{v-1})+a_0.$$

Let $p(q)\star(1-q)=f(q)-q^{n+1}a_n$, therefore by Lemma 4.1, $p(q)\star(1-q)=0$ if and only if either $p(q)=0$ or $p(q)\neq 0$ implies $p(q)^{-1}qp(q)-1=0$, that is, $p(q)^{-1}qp(q)=1$. If $p(q)\neq 0$, then $q=1$. Therefore, the only zeros of $p(q)\star(1-q)$ are $q=1$ and the zeros of $p(q)$.

For $|q|=1$, we have

$$\begin{aligned}|f(q)|\leq&|a_0|+\sum_{v=1}^{n}|a_v-a_{v-1}|\\ =&|\alpha_0+i\beta_0+j\gamma_0+k\delta_0|\\ &+\sum_{v=1}^{n}\left\{\Big|(\alpha_v-\alpha_{v-1})+i(\beta_v-\beta_{v-1})+j(\gamma_v-\gamma_{v-1})+k(\delta_v-\delta_{v-1})\Big|\right\}\\ \leq&|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|\\ &+\sum_{v=1}^{n}\left\{|\alpha_v-\alpha_{v-1}|+|\beta_v-\beta_{v-1}|+|\gamma_v-\gamma_{v-1}|+|\delta_v-\delta_{v-1}|\right\}\\ =&|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+|\alpha_n-\alpha_{n-1}|+|\alpha_{n-1}-\alpha_{n-2}|+...+|\alpha_{l+1}-\alpha_l|\\ &+|\beta_n-\beta_{n-1}|+|\beta_{n-1}-\beta_{n-2}|+...+|\beta_{l+1}-\beta_l|+|\gamma_n-\gamma_{n-1}|+|\gamma_{n-1}-\gamma_{n-2}|\\ &+...+|\gamma_{l+1}-\gamma_l|+|\delta_n-\delta_{n-1}|+|\delta_{n-1}-\delta_{n-2}|+...+|\delta_{l+1}-\delta_l|+M,\end{aligned}$$

where $M=\sum_{v=1}^{l}\left\{|\alpha_v-\alpha_{v-1}|+|\beta_v-\beta_{v-1}|+|\gamma_v-\gamma_{v-1}|+|\delta_v-\delta_{v-1}|\right\}$,

$$
\begin{aligned}
=&|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+|k_1\alpha_n+\alpha_n-k_1\alpha_n-\alpha_{n-1}|+|\alpha_{n-1}-\alpha_{n-2}|\\
&+...+|\rho\alpha_l+\alpha_{l+1}-\alpha_l-\rho\alpha_l|+|k_2\beta_n+\beta_n-k_2\beta_n-\beta_{n-1}|+|\beta_{n-1}-\beta_{n-2}|\\
&+...+|\rho\beta_l+\beta_{l+1}-\beta_l-\rho\beta_l|+|k_3\gamma_n+\gamma_n-k_3\gamma_n-\gamma_{n-1}|+|\gamma_{n-1}-\gamma_{n-2}|\\
&+...+|\rho\gamma_l+\gamma_{l+1}-\gamma_l-\rho\gamma_l|+|k_4\delta_n+\delta_n-\delta_{n-1}-k_4\delta_n|+|\delta_{n-1}-\delta_{n-2}|\\
&+...+|\rho\delta_l+\delta_{l+1}-\delta_l-\rho\delta_l|+M\\
\le&|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+|k_1\alpha_n-\alpha_{n-1}|+(k_1-1)|\alpha_n|+|\alpha_{n-1}-\alpha_{n-2}|+...+(1-\rho)|\alpha_l|\\
&+|\alpha_{l+1}-\rho\alpha_l|+|k_2\beta_n-\beta_{n-1}|+(k_2-1)|\beta_n|+|\beta_{n-1}-\beta_{n-2}|+...+(1-\rho)|\beta_l|\\
&+|\beta_{l+1}-\rho\beta_l|+|k_3\gamma_n-\gamma_{n-1}|+(k_3-1)|\gamma_n|+|\gamma_{n-1}-\gamma_{n-2}|+...+(1-\rho)|\gamma_l|+|\gamma_{l+1}-\rho\gamma_l|\\
&+|k_4\delta_n-\delta_{n-1}|+(k_4-1)|\delta_n|+|\delta_{n-1}-\delta_{n-2}|+...+(1-\rho)|\delta_l|+|\delta_{l+1}-\rho\delta_l|+M\\
=&|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+(k_1\alpha_n-\rho\alpha_l)+(k_2\beta_n-\rho\beta_l)+(k_3\gamma_n-\rho\gamma_l)+(k_4\delta_n-\rho\delta_l)\\
&+(k_1-1)|\alpha_n|+(k_2-1)|\beta_n|+(k_3-1)|\gamma_n|+(k_4-1)|\delta_n|+(1-\rho)(|\alpha_l|+|\beta_l|+|\gamma_l|+|\delta_l|)+M.
\end{aligned}
$$

Since

$$\max_{|q|=1}\left|q^n\star f\left(\frac{1}{q}\right)\right|=\max_{|q|=1}\left|f\left(\frac{1}{q}\right)\right|=\max_{|q|=1}|f(q)|,$$

therefore, $q^n\star f\left(\frac{1}{q}\right)$ has the same bound on $|q|=1$ as $f(q)$, that is

$$
\begin{aligned}
\left|q^n\star f\left(\frac{1}{q}\right)\right|\le&|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+(k_1\alpha_n-\rho\alpha_l)+(k_2\beta_n-\rho\beta_l)\\
&+(k_3\gamma_n-\rho\gamma_l)+(k_4\delta_n-\rho\delta_l)+(k_1-1)|\alpha_n|+(k_2-1)|\beta_n|+(k_3-1)|\gamma_n|\\
&+(k_4-1)|\delta_n|+(1-\rho)(|\alpha_l|+|\beta_l|+|\gamma_l|+|\delta_l|)+M \text{ for } |q|=1.
\end{aligned}
$$

Applying maximum modulus theorem ([6], Theorem 3.4), it follows that

$$
\begin{aligned}
\left|q^n\star f\left(\frac{1}{q}\right)\right|\le&|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+(k_1\alpha_n-\rho\alpha_l)+(k_2\beta_n-\rho\beta_l)\\
&+(k_3\gamma_n-\rho\gamma_l)+(k_4\delta_n-\rho\delta_l)+(k_1-1)|\alpha_n|+(k_2-1)|\beta_n|+(k_3-1)|\gamma_n|\\
&+(k_4-1)|\delta_n|+(1-\rho)(|\alpha_l|+|\beta_l|+|\gamma_l|+|\delta_l|)+M \text{ for } |q|\le 1.
\end{aligned}
$$

Replacing q by $\frac{1}{q}$, we get for $|q|\ge 1$

$$
\begin{aligned}
|f(q)|\le&\Big\{|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+(k_1\alpha_n-\rho\alpha_l)+(k_2\beta_n-\rho\beta_l)\\
&+(k_3\gamma_n-\rho\gamma_l)+(k_4\delta_n-\rho\delta_l)+(k_1-1)|\alpha_n|+(k_2-1)|\beta_n|\\
&+(k_3-1)|\gamma_n|+(k_4-1)|\delta_n|+(1-\rho)(|\alpha_l|+|\beta_l|+|\gamma_l|+|\delta_l|)+M\Big\}|q|^n. \qquad (6)
\end{aligned}
$$

But $|p(q)\star(1-q)|=|f(q)-q^{n+1}a_n|\ge|a_n||q|^{n+1}-|f(q)|$.

Using (6), we have for $|q| \geq 1$

$$\begin{aligned}
&|p(q) \star (1-q)| \\
&\geq |a_n||q|^{n+1} - \Big\{ |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + (k_1\alpha_n - \rho\alpha_l) + (k_2\beta_n - \rho\beta_l) \\
&\quad + (k_3\gamma_n - \rho\gamma_l) + (k_4\delta_n - \rho\delta_l) + (k_1 - 1)|\alpha_n| + (k_2 - 1)|\beta_n| \\
&\quad + (k_3 - 1)|\gamma_n| + (k_4 - 1)|\delta_n| + (1-\rho)(|\alpha_l| + |\beta_l| + |\gamma_l| + |\delta_l|) + M \Big\} |q|^n.
\end{aligned}$$

This implies that $|p(q) \star (1-q)| > 0$, i.e., $p(q) \star (1-q) \neq 0$ if

$$\begin{aligned}
|q| > \frac{1}{|a_n|} \Big\{ &|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + (k_1\alpha_n - \rho\alpha_l) + (k_2\beta_n - \rho\beta_l) + (k_3\gamma_n - \rho\gamma_l) \\
&+ (k_4\delta_n - \rho\delta_l) + (k_1 - 1)|\alpha_n| + (k_2 - 1)|\beta_n| + (k_3 - 1)|\gamma_n| + (k_4 - 1)|\delta_n| \\
&+ (1-\rho)(|\alpha_l| + |\beta_l| + |\gamma_l| + |\delta_l|) + M \Big\}.
\end{aligned}$$

Note that the only zeros of $p(q) \star (1-q)$ are $q = 1$ and the zeros of $p(q)$. Therefore, $p(q) \neq 0$ for

$$\begin{aligned}
|q| > \frac{1}{|a_n|} \Big\{ &|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + (k_1\alpha_n - \rho\alpha_l) + (k_2\beta_n - \rho\beta_l) + (k_3\gamma_n - \rho\gamma_l) \\
&+ (k_4\delta_n - \rho\delta_l) + (k_1 - 1)|\alpha_n| + (k_2 - 1)|\beta_n| + (k_3 - 1)|\gamma_n| + (k_4 - 1)|\delta_n| \\
&+ (1-\rho)(|\alpha_l| + |\beta_l| + |\gamma_l| + |\delta_l|) + M \Big\}.
\end{aligned}$$

Hence all the zeros of $p(q)$ lie in

$$\begin{aligned}
|q| \leq \frac{1}{|a_n|} \Big\{ &|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + (k_1\alpha_n - \rho\alpha_l) + (k_2\beta_n - \rho\beta_l) + (k_3\gamma_n - \rho\gamma_l) \\
&+ (k_4\delta_n - \rho\delta_l) + (k_1 - 1)|\alpha_n| + (k_2 - 1)|\beta_n| + (k_3 - 1)|\gamma_n| + (k_4 - 1)|\delta_n| \\
&+ (1-\rho)(|\alpha_l| + |\beta_l| + |\gamma_l| + |\delta_l|) + M \Big\}.
\end{aligned}$$

This completes the proof of Theorem 3.1. □

Proof of Theorem 3.7. Consider the polynomial

$$f(q) = \sum_{v=1}^{n} q^v (a_v - a_{v-1}) + a_0.$$

Let $p(q) \star (1-q) = f(q) - q^{n+1}a_n$, therefore by Lemma 4.1, $p(q) \star (1-q) = 0$ if and only if either $p(q) = 0$ or $p(q) \neq 0$ implies $p(q)^{-1}qp(q) - 1 = 0$, that is, $p(q)^{-1}qp(q) = 1$. If $p(q) \neq 0$, then $q = 1$. Therefore the only zeros of $p(q) \star (1-q)$ are $q = 1$ and the zeros of $p(q)$.

For $|q|=1$, we have

$$\begin{aligned}
&|f(q)|\\
\le&|a_0|+\sum_{v=1}^{n}|a_v-a_{v-1}|\\
\le&|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+\sum_{v=1}^{n}\Big\{|\alpha_v-\alpha_{v-1}|+|\beta_v-\beta_{v-1}|+|\gamma_v-\gamma_{v-1}|+|\delta_v-\delta_{v-1}|\Big\}\\
=&|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+|\alpha_1-\alpha_0|+|\alpha_2-\alpha_1|+...+|\alpha_l-\alpha_{l-1}|+|\alpha_{l+1}-\alpha_l|\\
&+...+|\alpha_n-\alpha_{n-1}|+|\beta_1-\beta_0|+|\beta_2-\beta_1|+...+|\beta_l-\beta_{l-1}|+|\beta_{l+1}-\beta_l|\\
&+...+|\beta_n-\beta_{n-1}|+|\gamma_1-\gamma_0|+|\gamma_2-\gamma_1|+...+|\gamma_l-\gamma_{l-1}|+|\gamma_{l+1}-\gamma_l|\\
&+...+|\gamma_n-\gamma_{n-1}|+|\delta_1-\delta_0|+|\delta_2-\delta_1|+...+|\delta_l-\delta_{l-1}|+|\delta_{l+1}-\delta_l|\\
&+...+|\delta_n-\delta_{n-1}|\\
=&|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+|\alpha_1-\alpha_0|+|\alpha_2-\alpha_1|+...+|\lambda_1\alpha_l+\alpha_l-\lambda_1\alpha_l-\alpha_{l-1}|\\
&+|\lambda_1\alpha_l+\alpha_l-\lambda_1\alpha_l-\alpha_{l+1}|+...+|\alpha_n-\alpha_{n-1}|+|\beta_1-\beta_0|+|\beta_2-\beta_1|\\
&+...+|\lambda_2\beta_l+\beta_l-\lambda_2\beta_l-\beta_{l-1}|+|\lambda_2\beta_l+\beta_l-\lambda_2\beta_l-\beta_{l+1}|+...+|\beta_n-\beta_{n-1}|\\
&+|\gamma_1-\gamma_0|+|\gamma_2-\gamma_1|+...+|\lambda_3\gamma_l+\gamma_l-\lambda_3\gamma_l-\gamma_{l-1}|+|\lambda_3\gamma_l+\gamma_l-\lambda_3\gamma_l-\gamma_{l+1}|\\
&+...+|\gamma_n-\gamma_{n-1}|+|\delta_1-\delta_0|+|\delta_2-\delta_1|+...+|\lambda_4\delta_l+\delta_l-\lambda_4\delta_l-\delta_{l-1}|\\
&+|\lambda_4\delta_l+\delta_l-\lambda_4\delta_l-\delta_{l+1}|+...+|\delta_n-\delta_{n-1}|\\
\le&|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+\alpha_1-\alpha_0+\alpha_2-\alpha_1+...+\lambda_1\alpha_l-\alpha_{l-1}+2(\lambda_1-1)|\alpha_l|\\
&+\lambda_1\alpha_l-\alpha_{l+1}+...+\alpha_{n-1}-\alpha_n+\beta_1-\beta_0+\beta_2-\beta_1\\
&+...+\lambda_2\beta_l-\beta_{l-1}+2(\lambda_2-1)|\beta_l|+\lambda_2\beta_l-\beta_{l+1}+...+\beta_{n-1}-\beta_n\\
&+\gamma_1-\gamma_0+\gamma_2-\gamma_1+...+\lambda_3\gamma_l-\gamma_{l-1}+2(\lambda_3-1)|\gamma_l|+\lambda_3\gamma_l-\gamma_{l+1}\\
&+...+\gamma_{n-1}-\gamma_n+\delta_1-\delta_0+\delta_2-\delta_1+...+\lambda_4\delta_l-\delta_{l-1}+2(\lambda_4-1)|\delta_l|\\
&+\lambda_4\delta_l-\delta_{l+1}+...+\delta_{n-1}-\delta_n\\
=&(|\alpha_0|+2\lambda_1\alpha_l-\alpha_0-\alpha_n)+(|\beta_0|+2\lambda_2\beta_l-\beta_0-\beta_n)+(|\gamma_0|+2\lambda_3\gamma_l-\gamma_0-\gamma_n)\\
&+(|\delta_0|+2\lambda_4\delta_l-\delta_0-\delta_n)+N,
\end{aligned}$$

where $N=2\Big((\lambda_1-1)|\alpha_l|+(\lambda_2-1)|\beta_l|+(\lambda_3-1)|\gamma_l|+(\lambda_4-1)|\delta_l|\Big)$.

Since

$$\max_{|q|=1}\left|q^n\star f\left(\frac{1}{q}\right)\right|=\max_{|q|=1}\left|f\left(\frac{1}{q}\right)\right|=\max_{|q|=1}|f(q)|,$$

therefore, $q^n\star f\left(\frac{1}{q}\right)$ has the same bound on $|q|=1$ as $f(q)$, that is

$$\begin{aligned}
\left|q^n\star f\left(\frac{1}{q}\right)\right|\le&(|\alpha_0|+2\lambda_1\alpha_l-\alpha_0-\alpha_n)+(|\beta_0|+2\lambda_2\beta_l-\beta_0-\beta_n)\\
&+(|\gamma_0|+2\lambda_3\gamma_l-\gamma_0-\gamma_n)+(|\delta_0|+2\lambda_4\delta_l-\delta_0-\delta_n)+N \text{ for } |q|=1.
\end{aligned}$$

After few steps as in the proof of Theorem 3.1, we conclude that all the zeros of $p(q)$ lie in

$$|q| \leq \frac{1}{|a_n|}\Big\{(|\alpha_0| + 2\lambda_1\alpha_l - \alpha_0 - \alpha_n) + (|\beta_0| + 2\lambda_2\beta_l - \beta_0 - \beta_n) + (|\gamma_0| + 2\lambda_3\gamma_l - \gamma_0 - \gamma_n) + (|\delta_0| + 2\lambda_4\delta_l - \delta_0 - \delta_n) + N\Big\}.$$

This completes the proof of Theorem 3.7. □

6. Conclusions

Some new results on Eneström-Kakeya theorem for quaternionic polynomials has been established that are benefical in determining the regions containing all the zeros of a polynomial.

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