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ON STEPANOV WEIGHTED PSEUDO ALMOST AUTOMORPHIC SOLUTIONS OF NEURAL NETWORKS

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Abstract. In this paper we investigate some sufficient conditions to guarantee the existence and uniqueness of Stepanov-like weighted pseudo almost periodic solutions of cellular neural networks on Clifford algebra for non-automomous cellular neural networks with multi-proportional delays. Our analysis is based on the differential inequality techniques and the Banach contraction mapping principle.

1. Introduction

In the past decades, the dynamics of various neural networks have been extensively studied. Many kinds of neural networks such as Hopfield neural networks and cellular neural networks etc., have received much more attention from many fields ([2], [12], [15]). They are a good tool for the approximation of dynamical systems, and so their successful application requires an understanding of their long term behavior with dynamical properties, in specially, their existence, uniqueness and stability.

The mathematical theory that enables machine learning of artificial intelligence is Kolmog- orov Arnold theorem [9], which is the starting point of neural network models. A sufficiently large function space can be constructed by choosing a suitable activation function and repeating only this function and arithmetic operation.

It is known that, as a generalization of real-valued neural networks, the research of complex-valued and quaternion-valued neural networks have been investigated in several kinds of neural networks have attracted more and more attention due to have more advantages than real-valued neural networks in many aspects [5]. However they are sometimes inapplicable for some for some engineering problems for instance such as neural computing, computer and robot vision, image and signal processing. For this reason, researchers attempted recently a more general and complicated neural networks, which is Clifford-valued neural networks [2], [12], [13]. Clifford-valued neural networks are a kind of neural networks whose state variables, connection weights and external inputs are Clifford numbers. They are generalizations of real-valued, complex-valued and quaternion-valued neural networks. However, because the multiplication of quaternion numbers does not satisfy the commutative law. In order

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to avoid the non-commutativity of the quaternion multiplication, researchers decomposed given system into real-valued systems [15].
Recently, there have been investigate interesting results on the problem of the existence and stability various type of almost periodic solution for the following:

$$\begin{aligned} x_i'(t) &= -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) \cdots\cdots (1) \\ &+ \sum_{j=1}^{n} b_{ij}(t)g_j(x_j(q_{ij}t)) + I_i(t),\ t \geq 0. \end{aligned}$$

Some authors argue that the first term in each of the right side of (1) corresponds to stabilizing negative feedback of the system which acts instantaneously without time delay; these terms are variously known as "forgetting" or a leakage terms ([8], [13]). The model which has time-varing leakage delays is more general than the previous ones(model). Therefore, some authors focused on the existence and stability of equibrium and periodic solutions for neural networks model involving leakage delay. And it is known from the stabilizing negative feedback terms will have a tendency to destabilize a system.

Motivated by the aforementioned works, to illustrate our abstract result, we investigate some sufficient conditions to guarantee the existence and uniqueness of Stepanov-like weighted pseudo almost periodic solutions of cellular neural networks on Clifford algebra for non-automomous cellular neural networks with multi-proportional and time-varying leakage delays as follow:

$$\begin{aligned} x_i'(t) &= -c_i(t)x_i(t-\eta_i(i)) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) \cdots\cdots (2) \\ &+ \sum_{j=1}^{n} b_{ij}(t)g_j(x_j(q_{ij}t)) + I_i(t),\ t \geq 0. \end{aligned}$$

The initial conditions associated with system (1) are of the form

$$x_i(s) = \varphi_i(s),\ s \in [-\tau_i, 0], \varphi_i \in C([-\tau_i, 0], \mathcal{A})\ , i \in I = \{1, 2, \cdots n\},$$

n is the number of units in a neural network, $x_i(t) \in \mathcal{A}$, which is known as Clifford number, corresponds to the state vector of the i-th unit at time t, $c_i(t) > 0$ represents the rate with which the i-th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $q_{ij}, i, j \in I$ are proportional delay factors and satisfy $0 < q_{ij} \leq 1$, and $q_{ij}t = t-(1-q_{ij})t$, in which $\rho_{ij}(t) = (1-q_{ij})t$ is the transmission delay function and $(1-q_{ij})t \to \infty$ as $q_{ij} \neq 1, t \to \infty$:φ_i denotesthe initial value of x_i at $s \in [\tau_i, 0]$, $\tau_i = \min_{1 \leq j \leq n}\{q_{ij}\}$. $a_{ij}(t), b_{ij}(t) \in \mathcal{A}$ are first-order and second-order connection weights of the natural network, $\eta_i(t) > 0$ and $\rho_{ij}(t), \sigma_{ij}(t), \nu_{ij}(t) > 0$ correspond to the leakage and transmission delays, respectively, $I_i(t) \in \mathcal{A}$ denotes the external inputs at time t, and $f_j, g_j : \mathcal{A} \to \mathcal{A}$ is the activation function for signal transmission of the i-th neuron.

2. Preliminaries and notations

The Clifford algebra was establishment by the British mathematician William K. Clifford in 1878 which is a generalization of the plural, quaternion, and Glassman algebra.

To begin with, we introduce the definition and properties of Clifford algebra which is well known. We shall refer to [2], [13], [15] and references therein.

Clifford algebra over $\mathbb{R}^n$ is defined as $\mathcal{A} = \{\sum_{A\subset\{1,2,3,\cdots,m\}} a^A e_A, a^A \in \mathbb{R}\}$ where $e_A = e_{h_1}e_{h_2}\cdots e_{h_\mu}$, with $A = \{h_1, h_2, \cdots, h_\mu\}$, $1 \le h_1 < h_2 < h_2 < \cdots < h_\mu \le m$.

Moreover, $e_\emptyset = e_0 = 1$ and $e_{\{h\}} = e_h$, $h = 1, 2, \cdots, m$ are called Clifford generators which satisfy the Hamilton's multiplication rules; the relations $e_i^2 = -1$ and $e_ie_j + e_je_i = 0$, $i = j$, $i, j = 1, 2, 3, \cdots, m$. For simplicity, when one element is the product of multiple Clifford generators, we will write its subscripts together.For example, $e_1e_2 = e_{12}$ and $e_3e_7e_2e_5 = e_{3725}$. We define $\Delta = \{\emptyset, 1, 2, \cdots A, \cdots, 12\cdots m\}$ then it is easy to see that $\mathcal{A} = \{\sum_A a^A e_A, a^A \in \mathbb{R}\}$, where $\sum_A$ is a brief form of $\sum_{A\in\triangle}$ and $dim_{\mathbb{R}}\mathcal{A} = \sum_{k=o}^m \binom{m}{0} = 2^m$.
For any Clifford number $x = \sum_A x^A e_A \in \mathcal{A}$, the involution of x is defined as $\bar{x} = \sum_A a^A \bar{e}_A$ where $\bar{e}_A = (-1)^{\frac{\sigma[A](\sigma[A]+1)}{2}}$ and

$$\sigma[A] = \begin{cases} 0, \text{ if } A = \emptyset \\ \mu, \text{ if } A = h_1h_2\cdots h_\mu. \end{cases}$$

From the definition, it is directly deduced that $e_A\bar{e}_A = \bar{e}_A e_A = 1$. Moreover, for Clifford-valued function $x = \sum_A x^A e_A$ where $x^A : \mathbb{R} \to \mathbb{R}$, $A \in \mathcal{A}$, and its derivative is given by $\frac{dx(t)}{dt} = \sum_A \frac{dx^A}{dt} dt e_A$. Since $e_B\bar{e}_A = (-1)^{\frac{\sigma[A](\sigma[A]+1)}{2}} e_Be_A$, we can write $e_B\bar{e}_A = e_c$ or $e_B\bar{e}_A = -e_c$, where e_c is a basis of Clifford algebra $\mathcal{A}$. For example, $e_{h_1h_2}\bar{e}_{h_2h_3} = -e_{h_1h_2}e_{h_2h_3} = -e_{h_1}e_{h_2}e_{h_3} = -e_{h_1}(-1)e_{h_3}e_{h_1}e_{h_3} = e_{h_1h_3}$. Hence it is possible to find a unique corresponding basis e_c for the given $e_B\bar{e}_A$.
Define

$$\sigma[B\cdot\bar{A}] = \begin{cases} 0, \text{ if } e_B\bar{e}_A = e_c \\ \mu, \text{ if } e_B\bar{e}_A = -e_c \end{cases}$$

and then $e_B\bar{e}_A = (-1)^{\sigma[B\cdot\bar{A}]}e_c$.

In addition, for any $g \in \mathcal{A}$, we can find g^c a unique satisfying $g^{B\cdot\bar{A}} = (-1)^{\sigma[B\cdot\bar{A}]}g^c$ for $e_B\bar{e}_A = (-1)^{\sigma[B\cdot\bar{A}]}e_c$. Hence $g^{B\cdot\bar{A}}e_B\bar{e}_A = g^{B\cdot\bar{A}}(-1)^{\sigma[B\cdot\bar{A}]}\, e_C = (-1)^{\sigma[B\cdot\bar{A}]}g^C b(-1)^{\sigma[B\cdot\bar{A}]}e^C$ $= g^C e_C$ and $g = \sum_C g^C e_C \in \mathcal{A}$.

Remark 1. Clifford-valued system (1) includes real-valued systems and complex-valued systems as its special cases. In fact system (1), when m, the number of the generators of A, equals $m = 0$, $m = 1$ and $m = 2$, system (1) degenerates into real-valued, complex-valued, and quternion-valued systems as its special cases, respectively [8].
Next, let $(X, ||\cdot||)$ be a Banach space and $BC(\mathbb{R}, X)$ be the set of all bounded continuous functions from $\mathbb{R}$ to X. For a given $T > 0$ and each $\rho(weights) \in U$, set $\mu(T, \rho)= \int_{-T}^{T} \rho(t)dt$.

In order to facilitate our discussion, we introduce the following notations:

$$\begin{aligned} \mathbb{U} &= \{\rho : \mathbb{R} \to (0,\infty) : \text{ locally integrable on } \mathbb{R} \text{ with } \rho > 0\ (a.e.)\}, \\ \mathbb{U}_\infty &= \{\rho \in U : \lim_{T\to\infty} \mu(T,\rho) = \infty\}, \\ BC(\mathbb{R},\mathbb{R}^n) &= \{f : \mathbb{R} \to \mathbb{R}^n, \text{ the bounded continuous functions}\}. \end{aligned}$$

Note that $(BC(\mathbb{R},\mathbb{R}^n), ||\cdot||_\infty)$ is a Banach space where $||\cdot||_\infty$ denote the sup norm

$$||f||_\infty := \max_{1\le i\le n} \sup_{t\in\mathbb{R}} |f_i(t)|.$$

Conveniently, we introduce some notations. We will use $x = (x_1, \cdots, x_n)^T$ to denote a column vector, in which the symbol $(\cdot^T)$ denotes the transpose of a vector. We let $|x|$ denote the absolute-value vector given by $|x| = (|x_1|, |x_2|, \cdots, |x_n|)^T$ and define $||x|| = \max_{1\le i\le n} |x_i|$. And we put $\varphi = \{\varphi_j(t)\} = (\varphi_1(t), \varphi_2(t), \cdots \varphi_n(t))^T$.
Lastly, we review some definitions and lemmas well known from our references ([2], [3], [4], [10], [11], [14], [1], [12], [13], [15]) and references therein.

Definition 2.1. A function $f \in BC(\mathbb{R},\mathcal{A})$ is called *almost automorphic*, if for every sequence of real numbers $(s_n^{'})_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t) = \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n) = f(t)$$

for each $t \in \mathbb{R}$. Denote by $AA(\mathbb{R},\mathcal{A})$ the collection of such functions.

Lemma 2.2. *If $\alpha \in \mathbb{R}$, $f, g \in AA(\mathbb{R},\mathcal{A})$, then $\alpha f, f+g \in AA(\mathbb{R},\mathcal{A})$.*

Lemma 2.3. *If $x \in C(\mathbb{R},\mathcal{A})$ satisfy the Lipschitz condition and $\varphi \in AA(\mathbb{R},\mathcal{A})$, then $f(\varphi(\cdot)) \in AA(\mathbb{R},\mathcal{A})$.*

Lemma 2.4. *If $f \in AA(\mathbb{R},\mathcal{A}), \eta \in (\mathbb{R},\mathbb{R})$, then $f(\cdot,\eta(\cdot)) \in AA(\mathbb{R},\mathcal{A})$.*

Definition 2.5. The Bochner transform $f^b(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$ of a function $f : \mathbb{R} \to X$ is defined by $f^b(t,s) := f(t+s)$.

Definition 2.6. Let $p \in [1,\infty)$. The space $BS^p(\mathbb{R},\mathcal{A})$ of all Stepanov bounded functions, with the exponent p consists of all measurable functions $f : \mathbb{R} \to \mathcal{A}$ such that $f^b \in L^p(\mathbb{R}; L^p((0,1),\mathcal{A}))$. This is a Banach space with the norm

$$||f||_{S^p} := \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} ||f(\tau)||^p d\tau \Big)^{\frac{1}{p}}.$$

We define the Stepanov weighted ergodic space, for $f \in BC(\mathbb{R},\mathcal{A})$,

$$\begin{aligned} & PAA_0(L^p([0,1],\mathcal{A}),\rho) \\ = & \Big\{ f : \lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \Big(\int_t^{t+1} ||f(s)||^p ds \Big)^{\frac{1}{p}} \rho(t) dt = 0 \Big\}. \end{aligned}$$

Definition 2.7. Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. A function $f \in BS^p(\mathbb{R},\mathcal{A})$ is said to be a Stepanov-weighted pseudo almost automorphic(S^p-weighted pseudo almost automorphic) if it can be expressed as $f = h + \varphi$, where $h^b \in AA(\mathbb{R},(L^p((0,1),\mathcal{A}))$, $\varphi^b \in PAA_0(\mathbb{R}, L^p((0,1),\mathcal{A}),\rho)$.

The collection of such functions will be denoted by $S^pWPAA(\mathbb{R},\mathcal{A},\rho,p)$ which is a closed subspace of $BC(\mathbb{R},L^p([0,1],\mathcal{A}))$ relatively to the norm $||\cdot||_{S^p}$, and therefore is a Banach space [1].

Definition 2.8. A function $f=\sum_{i=1}^n f^Ae_A:\mathbb{R}\to\mathcal{A}$ is said to be Stepanov weighted pseudo almost automorphic, if $f^A\in S^pWPAA(\mathbb{R},\mathbb{R},\rho,p)$ for all $A\in\Delta$.

Note that $M[a_i]=\lim_{T\to+\infty}\frac{1}{T}\int_t^{t+T}a_i(s)ds>0$, using the theory of exponential dichotomy in [7], one can easily get the following lemma.

Lemma 2.9. *For $i=1,2,3,\cdots,n$, and $a_i\in BC(\mathbb{R},\mathbb{R}^n)$ with $\inf_{t\in\mathbb{R}}a_i(t)>0$. If $f\in BC(\mathbb{R},\mathbb{R}^n)$, then the linear system*

$$x^{'}(t)=A(t)x(t)+f(t)$$

has a unique bounded solution

$$x(t)=\int_{-\infty}^{t}e^{\int_s^t A(u)du}f(s)ds,$$

where $A(t)$=diag$(-a_1(t),-a_2(t),\cdots,-a_n(t))$.

3. Existence results for Stepanov weighted pseudo almost periodic solution

To overcome the difficulty for the non-commutativity of multiplication of Clifford numbers, firstly we transform the Clifford-valued system (1) into the real-valued system which is easily to handle. This can be established using by $e_A\bar{e}_A=\bar{e}_Ae_A=1$ and $\bar{e}_Ae_A=e_B$.
For any $g\in\mathcal{A}$, we can find g^c a unique satisfying $g^{B\cdot\bar{A}}=(-1)^{\sigma[B\cdot\bar{A}]}g^c$ for $e_B\bar{e}_A=(-1)^{\sigma[B\cdot\bar{A}]}e_c$. So $g^{B\cdot\bar{A}}e_B\bar{e}_A=g^{B\bar{A}}(-1)^{\sigma[B\cdot\bar{A}]}e_C=(-1)^{\sigma[B\cdot\bar{A}]}g^Cb(-1)^{\sigma[B\cdot\bar{A}]}e^C=g^Ce_C$ and $g=\sum_C g^Ce_C\in\mathcal{A}$. By decomposing (1) into $x=\sum_A x^Ae_A$, we obtain that

$$\begin{aligned} x_i'^A(t) &= -c_i(t)x_i^A(t-\eta_i(t))+\sum_{j=1}^n\sum_B a_{ij}^{A\cdot\bar{B}}(t)f_j^B(x_j(t)),\\ &+ \sum_{j=1}^n\sum_{B\in\Lambda}b_{ij}^{A\cdot\bar{B}}(t)g_j^B(x_j(q_{ij}t))+I_i^A(t),\cdots\cdots(3),\\ &\quad x_i^A(s)=\varphi_i^A(s),\ s\in[-\tau_i,0],\ i\in I,\end{aligned}$$

where

$$x_i(t)=\sum_A x_i^A(t)e_A,\ I_i(t)=\sum_A I_i^A(t)e_A,$$

$$\begin{aligned} &a_{ij}(t)=\sum_A a_{ij}^C(t)e_C,\ a_{ij}^{A\cdot\bar{B}}(t)=(-1)^{n[A\cdot\bar{B}]}a_{ij}^C(t),\\ &b_{ij}(t)=\sum_A b_{ij}^C(t)e_C,\ b_{ij}^{A\cdot\bar{B}}(t)=(-1)^{n[A\cdot\bar{B}]}b_{ij}^C(t),\\ &g_j(x_j(q_{ij}t))=\sum_{B\in\lambda}g_j^B(\varphi_j^{C_1}(q_{ij}t),\varphi_j^{C_2}(q_{ij}t),\cdots,x_j^{C_2m}(q_{ij}t))e_B.\end{aligned}$$

Remark 2. It is clear that if $x = (x_1^0, x_1^1, \cdots, x_1^{1\cdot 2\cdots m}, x_2^0, x_2^1, \cdots, x_2^{1\cdot 2\cdots m}, \cdots, x_n^0, x_n^1, \cdots, x_n^{1\cdot 2\cdots m})^T : \{x_i^A\}$ is a solution to system (2), then $x = (x_1, x_2, \cdots x_n)^T$ must be a solution to (1), where $x_i = \sum_A x_i^A e_A, A \in \Delta$.

For the sake of convenience to work (3) we established some hypothesis and sufficient criteria, which will be used in this paper, as following:

(H_1) For $i, j \in I$ and $A, B \in \Delta$, $c_i(t) \in S^pWPAA(\mathcal{A}, \rho, p)$, $a_{ij}^{A\cdot\bar{B}}(t)$, $b_{ij}^{A\cdot\bar{B}}(t), I_i(t) \in S^pWPAA(\mathbb{R}, \mathcal{A}, \rho, p)$, $\tau_{ij}(t), \sigma_{ij}(t), \mu_{ij}(t) \in S^pWPAA(\mathbb{R}, \mathcal{A}, \rho, p)$.

(H_2) For any $u, v \in \mathcal{A}$, functions $f_j^B, g_j^B \in BC(\mathcal{A}, \mathbb{R})$,there exist positive constant L_j^f, L_j^g such that

$$||f_j^B(u) - f_j^B(v)|| \leq L_j^f \sum_{C\in\Delta} ||u^C - v^C||,\ f_j^B : \mathbb{R}^{2^m} \to \mathbb{R},$$

$$||g_j^B(u) - g_j^B(v)|| \leq L_j^g \sum_{C\in\Delta} ||u^C - v^C||,\ f_j^B : \mathbb{R}^{2^m} \to \mathbb{R}.$$

Additionally, we suppose that $f_j^B(0) = g_j^B(0) = 0$, where $j \in I, A, B \in \Delta$.

(H_3) Let $\mathcal{D} = \{\varphi : \varphi \in S^pWPAA(\mathbb{R}, \mathcal{A}, \rho, p)\}$, $||x||_{S^p} = \max_{i\in I} \{\max_{A\in\Delta} |x_i^A|_{S^p}\}$ and $\varphi_0 = \{(\varphi_0)_i^A\}$, where

$$|x_i^A|_{S^p} = \sup_{t\in\mathbb{R}}(\int_t^{t+1} |x_i^A(s)|^p\ ds)^{\frac{1}{p}},\quad (\varphi_0)_i^A(t) = \int_{-\infty}^t e^{-\int_s^t a_i(u)du} I_i^A(s)ds,$$

respectively. It is clear that $\mathcal{D}$ is a Banach space.

(H_4) For $i \in I$, there is a function $\tilde{a}_i \in BC(\mathbb{R}, (0, +\infty))$ and a constant $k_i > 0$ satisfying the following inequality:

$$e^{-\int_s^t a_i(u)du} \leq k_i e^{-\int_s^t \tilde{a}_i(u)du},\ \text{for all}\ \ t, s, k_i \in \mathbb{R},\ t - s > 0,$$

and $\varphi^* = \sup_{t\in\mathbb{R}} |\varphi(t)|$ for $f \in BC(\mathbb{R}, \mathbb{R})$.

(H_5) Put

$$\Big(\frac{2k_i}{\tilde{a_{i*}}q}\Big)^{\frac{1}{q}} \Big(\frac{2k_i}{\tilde{a_{i*}}p}\Big)^{\frac{1}{p}} ||I||_{s^p} = k,\ \rho < 1, \frac{\rho k}{1-\rho} < 1$$

where

$$\Big(\frac{k_i}{p\tilde{a_{i*}}}\Big)^{\frac{1}{p}} \Big[\Big(c_i^*\eta_i^*\Big)^{\frac{1}{p}} + 2^m \max_{A\in\Delta} \sum_{j=1}^n \Big(\sum_B a_{ij}^{A\cdot\bar{B}}(s) L_j^f + \sum_B b_{ij}^{A\cdot\bar{B}}(s) L_j^g\Big)\Big] = \rho.$$

Using similar ideas as in [4], [13]. one can easily show the following result.

Lemma 3.1. *Suppose that assumptions $(H_1) \sim (H_4)$ hold. Define the nonlinear operator Γ as follows: for each $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n) \in S^pWPAA(\mathcal{A}, \rho, p)$,$\Gamma\varphi(t) := x_\varphi(t)$, where*

$$x_\varphi(t) = \Big(\int_{-\infty}^t e^{-\int_s^t a_u du} F_1(s)ds, \int_{-\infty}^t e^{-\int_s^t a_u du} F_2(s)ds, \cdots, \int_{-\infty}^t e^{-\int_s^t a_u du} F_n(s)ds\Big)^T$$

and

$$\begin{aligned} F_i(s) &= -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) \\ &+ \sum_{j=1}^{n} b_{ij}(t)g_j(x_j(q_{ij}t)) + I_i(t),\ i = 1,2,3\ldots,n. \end{aligned}$$

Then Γ *maps* $S^pWPAA(\mathbb{R},\mathcal{A},\rho,p)$ *into itself.*

Proof. For all $1 \leq i \leq n$, the function F_i is Stepanov weighted pseudo almost automorphic, by using Lemma 2.2 $\sim$ Lemma 2.4 and the composition theorem of pseudo almost automorphic function [11]. Consequently, F_i can be expressed as

$$F_i = F_i^1 + F_i^2$$

where $F_i^1 \in AA(\mathbb{R},\mathcal{A})$ and $F_i^2 \in PAA_0(L^p([0,1],\mathcal{A}),\rho)$. So

$$\begin{aligned} &(\Gamma_i\varphi)(t) \\ =\ & \int_{-\infty}^{t} e^{-\int_s^t a_i(u)du}F_i^1(s)ds + \int_{-\infty}^{t} e^{-\int_s^t a_i(u)du}F_i^2(s)ds \\ =\ & (\Gamma_iF_i^1)(t) + (\Gamma_iF_i^2)(t). \end{aligned}$$

Let $(s_n^{'})_{n\in\mathbb{N}}$ be a sequence of real numbers; since $a_i \in AA(\mathbb{R},\mathcal{A})$ and $\Gamma_i \in AA(\mathbb{R},\mathcal{A})$, we can extract a subsequence $(s_n)_{n\in\mathbb{N}}$ of $(s_n^{'})_{n\in\mathbb{N}}$ such that, for each $t \in \mathbb{R}$,

$$\lim_{n\to\infty} a_i(t+s_n) = \overline{a_i},\ \lim_{n\to\infty} \overline{a_i}(t-s_n) = a_i,\ i = 1,2,\cdots,n$$

and

$$\lim_{n\to\infty} \Gamma_i(t+s_n) = \overline{\Gamma_i},\ \lim_{n\to\infty} \overline{\Gamma_i}(t-s_n) = \Gamma_p,\ i = 1,2,\cdots,n.$$

Set

$$(\overline{T_i\Gamma_i})(t) = \int_{-\infty}^{t} e^{-\int_s^t \bar{a}_i(u)du}\bar{\Gamma}_i(s)ds,\ i = 1,2,\cdots,n,$$

and we have

$$\begin{aligned} &||T_i\Gamma_i(t+s_n) - \overline{T_i\Gamma_i}(t)||_{\mathcal{A}} \\ =\ & \int_{-\infty}^{t+s_n} e^{-\int_s^{t+s_n} a_i(u)du}F_i^1(s)ds + \int_{-\infty}^{t} e^{-\int_s^t a_i(u)du}F_i^2(s)ds \\ =\ & \int_{-\infty}^{t+s_n} e^{-\int_{s-s_n}^t a_i(\rho+s_n)d\rho}F_i^1(s)ds + \int_{-\infty}^{t} e^{-\int_s^t a_i(u)du}F_i^2(s)ds \\ =\ & \int_{-\infty}^{t} e^{-\int_s^t a_i(\rho+s_n)d\rho}F_i^1(s)ds + \int_{-\infty}^{t} e^{-\int_s^t a_i(\rho+s_n)d\rho}F_i^2(s)ds \\ &+ \int_{-\infty}^{t} e^{-\int_s^t a_i(\rho+s_n)d\rho}F_i^1(s)ds + \int_{-\infty}^{t} e^{-\int_s^t a_i(u)du}F_i^2(s)ds \\ =\ & \int_{-\infty}^{t} e^{-\int_s^t a_i(u+s_n)du}(F_i^1(s+s_n) - F_i^1))ds + \int_{-\infty}^{t} e^{-\int_s^t a_i(u+s_n)du} - e^{\int_s^t a_1(u)du})F_i^2(s)ds. \end{aligned}$$

By the Lebesgue dominated convergence theorem, we obtain that

$$\lim_{n\to+\infty}(T_i\Gamma_i)(t+s_n)=\overline{(T_i\Gamma_i)}(t),\ \text{for each } t\in\mathbb{R},\ p=1,2,\cdots,n.$$

Similarly, one can prove that

$$\lim_{n\to+\infty}\overline{(T_i\Gamma_i)}(t-s_n)=(T_i\Gamma_i)(t),\ \text{for each } t\in\mathbb{R},\ i=1,2,\cdots,n,$$

which implies that $\Gamma_i F_i^1\in AA(R,\mathcal{A})$.

Next we show that $\Gamma_i F_i^2\in AA(R,\mathcal{A})$, by the Hölder inequality and Fubinis's theorem
$(p^{-1}+q^{-1}=1)$, we obtain that

$$\begin{aligned}
&||\Gamma_i F_i^2||\\
=&\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\Big[\sup_{t\in\mathbb{R}}\int_t^{t+1}|\int_{-\infty}^{z}e^{-\int_s^t a_i(u)du}F_i^2(s)ds|^p dz\Big]^{\frac{1}{p}}\rho(t)dt\\
=&\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\Big[\sup_{t\in\mathbb{R}}\int_t^{t+1}|\int_{-\infty}^{z}k_ie^{-\int_s^t \tilde{a_i(u)}du}F_i^2(s)ds|^p dz\Big]^{\frac{1}{p}}\rho(t)dt\\
\le&\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\Big[\sup_{t\in\mathbb{R}}\int_t^{t+1}|\int_{-\infty}^{z}k_ie^{-(z-s)\tilde{a_{i*}}}F_i^2(s)ds|^p dz\Big]^{\frac{1}{p}}\rho(t)dt\\
\le&\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\Big[\sup_{t\in\mathbb{R}}\int_t^{t+1}\Big(\frac{2}{\tilde{a_{i*}}}\Big)^{\frac{p}{q}}|\int_{-\infty}^{0}k_ie^{\frac{\tilde{a_{i*}}ps}{2}}F_i^2(z+s)|^p dsdz\Big]^{\frac{1}{p}}\rho(t)dt\\
\le&\lim_{T\to\infty}\frac{1}{\mu(T,\rho)}(2T)^{\frac{1}{q}}\int_{-T}^{T}\Big[\sup_{t\in\mathbb{R}}\int_t^{t+1}\Big(\frac{2}{\tilde{a_{i*}}}\Big)^{\frac{p}{q}}|\int_{-\infty}^{0}k_ie^{\frac{\tilde{a_{i*}}ps}{2}}F_i^2(z+s)|^p\rho(t)dsdzdt\Big]^{\frac{1}{p}}\\
\le&\lim_{T\to\infty}(\frac{1}{\mu(T,\rho)})^{\frac{1}{p}}(2T)^{\frac{1}{q}}\Big(\frac{2}{\tilde{a_{i*}}}\Big)^{\frac{1}{q}}\int_{-\infty}^{0}k_ie^{\frac{\tilde{a_{i*}}ps}{2}}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\Big[\sup_{t\in\mathbb{R}}\int_t^{t+1}|F_i^2(z+s)|^p\rho(t)dzdtds\Big]^{\frac{1}{p}}\\
\le&\lim_{T\to\infty}(\frac{1}{\mu(T,\rho)})^{\frac{1}{p}}(2T)^{\frac{1}{q}}\Big(\frac{2}{\tilde{a_{i*}}}\Big)^{\frac{1}{q}}\int_{-\infty}^{0}k_ie^{\frac{\tilde{a_{i*}}ps}{2}}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\Big[\sup_{t\in\mathbb{R}}\int_t^{t+1}|F_i^2(z+s)|^q dz\Big)^{\frac{p}{q}}\rho(t)\\
&dtds\Big]^{\frac{1}{p}}\Big(\frac{2}{\tilde{a_{i*}}}\Big)^{\frac{1}{q}}\int_{-\infty}^{0}k_ie^{\frac{\tilde{a_{i*}}ps}{2}}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\Big[\sup_{t\in\mathbb{R}}\int_t^{t+1}|F_i^2(z+s)|^q dz\Big)^{\frac{1}{q}}\rho(t)dtds\Big]^{\frac{1}{p}}\\
&\cdot\int_t^{t+1}|F_i^2(z+s)|^q dz\Big)^{\frac{p-1}{q}}\rho(t)dtds\Big]^{\frac{1}{p}}\le\Big(\frac{2k_i}{\tilde{a_{i*}}}\Big)^{\frac{1}{q}}||F_i^2||_\infty^{\frac{p-1}{p}}\int_{-\infty}^{0}e^{\frac{\tilde{a_{i*}}ps}{2}}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\Big[\sup_{t\in\mathbb{R}}\int_t^{t+1}\\
&\cdot|F_i^2(z+s)|^q dz\Big)^{\frac{1}{q}}\rho(t)dtds\Big]^{\frac{1}{p}}.
\end{aligned}$$

Since $F_i^2\in PAA_0$, $\Gamma_i F_i^2\in AA(R,\mathcal{A})$,
Γ maps into $S^pWPAA(\mathbb{R},\mathcal{A},\rho,p)$ itself in the region $\mathcal{D}$. □

By applying the similar mathematical analysis techniques in [5], we derive some new sufficient conditions ensuring the existence, uniqueness and of weighted pseudo almost periodic solutions of system (3).

THEOREM 3.2. *Assume that* (H_1-H_5) *hold, then system* (3) *has a unique* S^p *weighted pseudo almost automorphic solution in the region* $\mathcal{D}^*=\Big\{\varphi|\varphi\in\mathcal{D}:||\varphi-\varphi_0||_{S^p}\le\frac{\rho k}{1-\rho}\Big\}$, *where*

$$\varphi_0(t) = \Big(\int_{-\infty}^{t} e^{-\int_s^t a_u du} I_1(s)ds, \int_{-\infty}^{t} e^{-\int_s^t a_u du} I_2(s)ds, \cdots, \int_{-\infty}^{t} e^{-\int_s^t a_u du} I_n(s)ds \Big)^T.$$

Proof. Firstly, it is easy to see that the region $\mathcal{D}$ is a closed convex subset of $S^pWPAA(\mathbb{R},$
$\mathcal{A}, \rho, p)$. Using the Holder inequality and Fubinis's theorem we obtain

$$\Big| \int_{-\infty}^{0} e^{-\int_s^t a_i(u)du} I(s)ds \Big| \leq \Big(\frac{2k_i}{\tilde{a_{i*}} q} \Big)^{\frac{1}{q}} \Big[\int_{-\infty}^{0} e^{\frac{\tilde{a_{i*}} sp}{2}} |I(s+t)|ds \Big]^{\frac{1}{p}},$$

$$\begin{aligned} ||\varphi_0(t)||_{S^p} &= \sup_{t\in\mathbb{R}} \Big[\int_t^{t+1} | \int_{-\infty}^{0} e^{\int_s^t a_i(u)du} I(s)|^p ds dt \Big]^{\frac{1}{p}} \\ &\leq \Big(\frac{2k_i}{\tilde{a_{i*}} q} \Big)^{\frac{1}{q}} \sup_{t\in\mathbb{R}} \Big[\int_{-\infty}^{0} e^{\frac{\tilde{a_{i*}} sp}{2}} \int_t^{t+1} |I(s+t)|^p dt ds \Big]^{\frac{1}{p}} \\ &\leq \Big(\frac{2k_i}{\tilde{a_{i*}} q} \Big)^{\frac{1}{q}} \sup_{t\in\mathbb{R}} \Big[\int_{-\infty}^{0} e^{\frac{\tilde{a_{i*}} sp}{2}} \int_t^{t+1} |I(t)|^p dt ds \Big]^{\frac{1}{p}} \\ &\leq \Big(\frac{2k_i}{\tilde{a_{i*}} q} \Big)^{\frac{1}{q}} \Big(\frac{2k_i}{\tilde{a_{i*}} p} \Big)^{\frac{1}{p}} ||I||_{s^p} \\ &= k. \end{aligned}$$

Therefore, for all $\varphi \in \mathcal{D}^*$, by applying the estimate just obtained technique, we can easily obtain:

$$||\varphi||_{S^P} \leq ||\varphi - \varphi_0||_{S^p} + ||\varphi_0||_{S^p} \leq \frac{\rho k}{1-\rho} + k$$

and

$$\begin{aligned} & ||(\Gamma_\varphi)_i^A(t) - (\varphi_0)_i^A||_{S^p} \\ = & \sup_{t\in R} \Big\{ \Big[\int_t^{t+1} \Big| \int_{-\infty}^{w} e^{-\int_s^w a_i(u)du} \big(c_i(t) \int_{t-\eta_i(t)}^{t} \varphi_i^{'}(s)ds + \sum_{j=1}^{n} \sum_{B\in\Delta} a_{ij}^{A\cdot\bar{B}} \\ & (s) \cdot f_j^B(\varphi_j(s)) + \sum_{j=1}^{n} \sum_{B\in\Delta} b_{ij}^{A\cdot\bar{B}}(s) g_j^B(\varphi_j(q_{ij}t)) \big] ds \Big|^p dt \Big\}^{\frac{1}{p}} \\ \leq & \sup_{t\in\mathbb{R}} \Big\{ \Big[\int_t^{t+1} \Big| \int_{-\infty}^{w} e^{-\int_s^w a_i(u)du} \big(c_i(t) \int_{t-\eta_i(t)}^{t} \varphi_i^{'}(s) \big) ds \Big|^p dw \Big]^{\frac{1}{p}} \\ & + \Big[\int_t^{t+1} \Big| \int_{-\infty}^{w} e^{-\int_s^w a_i(u)du} \sum_{j=1}^{n} \sum_{B\in\Delta} a_{ij}^{A\cdot\bar{B}}(s) f_j^B \big(\varphi_j(s)\big) ds \Big|^p dw \Big]^{\frac{1}{p}} \\ & + \Big[\int_t^{t+1} \Big| \int_{-\infty}^{w} e^{-\int_s^w a_i(u)du} \sum_{j=1}^{n} \sum_{B\in\Delta} b_{ij}^{A\cdot\bar{B}}(s) g_j^B(\varphi_j(q_{ij}t)) \big] ds|^p dw]^{\frac{1}{p}} dt \} \end{aligned}$$

$$\begin{aligned}
\leq \;& \int_t^{t+1}\int_0^{\infty} k_i e^{-p\tilde{a_{i*}}} c_i^* \eta_i^* ||\varphi||_{S^p} + \sum_{j=1}^{n}\Big\{\Big[\sum_{B\in\Delta} a_{ij}^{A\cdot\bar{B}}(s) L_j^f \sum_{C\in\Delta}\big(\int_t^{t+1}|\int_0^{\infty} k_i e^{-p\tilde{a_{i*}}} \\
& \cdot|\varphi_j^C(w-\sigma)|^p d\sigma dw\big)^{\frac{1}{p}}\Big] + \sum_{j=1}^{n}\Big\{\Big[\sum_{B\in\Delta} b_{ij}^{A\cdot\bar{B}}(s) L_j^g \sum_{C\in\Delta}\big(\int_t^{t+1}|\int_0^{\infty} k_i e^{-p\tilde{a_{i*}}} \\
& \cdot|\varphi_j^C(1-\tau(w-\sigma))\big|^p d\sigma dw\big)^{\frac{1}{p}} \leq \Big(\frac{k_i}{p\tilde{a_{i*}}}\Big)^{\frac{1}{p}}\big(c_i^*\eta_i^* + 2^m \sum_{j=1}^{n}\big(\sum_{B\in\Delta} b_{ij}^{A\cdot\bar{B}}(s) L_j^g \\
& + \sum_{B\in\Delta} b_{ij}^{A\cdot\bar{B}}(s) L_j^g\big)||\varphi||_{S^p} \leq \frac{\rho k}{1-\rho},
\end{aligned}$$

which implies that $\Gamma\varphi \in \mathcal{D}^*$, therefore the mapping Γ is a self mapping from $\mathcal{D}^*$ to $\mathcal{D}^*$.
We see that the equation (3) has a unique weighted pseudo almost periodic solution as following

$$\begin{aligned}
(x^{\varphi})_i^A(t) \;=\;& \int_{-\infty}^{t} e^{-\int_s^t a_i(u)du}\Big[c_i(t)\int_{s-\eta_p(s)}^{s} x_i(u)du + \sum_{j=1}^{n}\sum_{B\in\Delta} a_{ij}^{A\cdot\bar{B}}(s) \\
& \cdot f_j^B(\varphi_j(t)) + \sum_{j=1}^{n}\sum_{B\in\Delta} b_{ij}^{A\cdot\bar{B}}(s)\cdot f_j^B(\varphi_j(q_{ij}t)) + I_i^A(s)\Big]ds.
\end{aligned}$$

Define a mapping $\Gamma : \mathcal{D}^* \to \mathcal{D}^*$ by given (Φ_φ) By using the Minkowski's inequality, we have

$$\begin{aligned}
& ||(\Gamma\varphi)_i^A - (\Gamma\psi)_i^A||_{S^p} \\
=\;& \sup_{t\in\mathbb{R}}\Big\{\int_t^{t+1}\Big|\int_{-\infty}^{t} e^{-\int_s^t a_i(u)du}\Big[c_i(t)\int_{t-\eta_i(t)}^{t}(\varphi_i'(s)-\psi_i'(s))ds \\
& + \sum_{j=1}^{n}\sum_{B\in\Delta} a_{ij}^{A\cdot\bar{B}}(s)\cdot g_j^B(\varphi_j(t)-\psi_j) + \sum_{j=1}^{n}\sum_{B\in\Delta} b_{ij}^{A\cdot\bar{B}}(s) g_j^B(\varphi_j(q_{ij}t)-\psi_j(q_{ij}t))\Big]ds\Big|^p dt\Big\}^{\frac{1}{p}} \\
\leq\;& \sup_{t\in\mathbb{R}}\int_t^{t+1}\Big|\int_{-\infty}^{t} k_i e^{-\int_s^t \tilde{a}_i(u)du}\Big[(c_i(t)\int_{t-\eta_i(t)}^{t}(\varphi_i'(s)-\psi_i'(s))ds\Big|^p dt\Big\}^{\frac{1}{p}} \\
& + \int_t^{t+1}\Big|\int_{-\infty}^{t} k_i e^{-\int_s^t \tilde{a}_i(u)du}\sum_{j=1}^{n}\sum_{B\in\Delta} a_{ij}^{A\cdot\bar{B}}(s)\cdot f_j^B(\varphi_j(t)-\psi_j(t))ds\Big|^p dt\Big\}^{\frac{1}{p}} \\
& + \int_t^{t+1}\Big|\int_{-\infty}^{t} k_i e^{-\int_s^t \tilde{a}_i(u)du}\sum_{j=1}^{n}\sum_{B\in\Delta} b_{ij}^{A\cdot\bar{B}}(s)\cdot g_j^B(\varphi_j(q_{ij}t)-\psi_j(q_{ij}t))ds\Big|^p dt\Big\}^{\frac{1}{p}} \\
\leq\;& \Big(\frac{k_i}{p\tilde{a_{i*}}}\Big)^{\frac{1}{p}} 2^m \max_{A\in\Delta}\Big[(c_i^*\eta_i^*)^{\frac{1}{p}} + \sum_{B\in\Delta}\big(\sum_{j=1}^{n} a_{ij}^{A\cdot\bar{B}}(s) L_j^f + \sum_{j=1}^{n} b_{ij}^{A\cdot\bar{B}}(s) L_j^g\big)\Big]||\varphi-\psi||_{S^p}, \\
\leq\;& \rho||\varphi-\psi||_{S^p}
\end{aligned}$$

where

$$\rho = \Big(\frac{k_i}{p\tilde{a_{i*}}}\Big)^{\frac{1}{p}} 2^m \max_{A\in\Delta}\Big[(c_i^*\eta_i^*)^{\frac{1}{p}} + \sum_{B\in\Delta}\big(\sum_{j=1}^n a_{ij}^{A\cdot\bar{B}}(s)L_j^f + \sum_{j=1}^n b_{ij}^{A\cdot\bar{B}}(s)L_j^g\big)\Big].$$

Since $\rho < 1$, it implies that $\Gamma : \mathcal{D}^* \to \mathcal{D}^*$ is a contraction mapping. By Contraction mapping principle of the $\mathcal{D}^*$, we obtain that the mapping Γ has a unique fixed point $z \in \mathcal{D}^*$ such that Γ which means that the equation (2) has a unique weighted pseudo almost periodic solution. The proof of the theorem is completed. □

4. Examples

In this section we consider a simple application of our abstracts results we give an modified example [13], [15] for $n = 2$ as the following neural networks with time-varing leakage delays:

$$\begin{aligned} x_i'(t) &= -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^2 a_{ij}(t)f_j(x_j(t)) \cdots\cdots (4) \\ &+ \sum_{j=1}^2 b_{ij}(t)g_j(x_j(t-\rho_{ij}(t))) + I_i(t),\ t \geq 0, \end{aligned}$$

where

$x_i(t) = x_i^0(t)e_0 + x_i^1(t)e_1 + x_i^2(t)e_2 + x_i^{12}(t)e_{12} \in \mathcal{A}$,

$f_j(x_j) = \frac{1}{20}e_0 sinx_j^0 + \frac{1}{21}e_1 sinx_j^1 + +\frac{1}{25}e_2 sinx_j^2 + \frac{1}{27}e_{12} sinx_j^{12},\ j = 1, 2,$

$g_j(x_j) = \frac{1}{13}e_0 sinx_j^0 + \frac{1}{15}e_1 sinx_j^1 + +\frac{1}{17}e_2 sinx_j^2 + \frac{1}{18}e_{12} sinx_j^{12},\ j = 1, 2,$

$$\begin{pmatrix} I_1(t) \\ I_2(t) \end{pmatrix} = \begin{pmatrix} (\frac{1}{2}sin2\sqrt{5}t + \frac{2}{1+t^2})e_0 + \frac{1}{20}sin\sqrt{3}t & \frac{1}{12}e_2 cos2\sqrt{3}t + \frac{1}{15}e_{12}sin\sqrt{6}t \\ (\frac{1}{2}sin2\sqrt{3}t + \frac{3}{1+t^2})e_0 + \frac{1}{12}sin\sqrt{6}t & \frac{1}{10}e_2 cos\sqrt{7}t + \frac{1}{20}e_{12}sin\sqrt{3}t \end{pmatrix},$$

$$\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} = \begin{pmatrix} 1 + 0.1sin\sqrt{2}t \\ 1.2 + 0.2cos\sqrt{3}t \end{pmatrix}, \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = \begin{pmatrix} 0.15 + 0.02sin\sqrt{2}t \\ 0.16 + 0.012cos\sqrt{3}t \end{pmatrix},$$

$$\begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix} = \begin{pmatrix} 0.1e_0 sin\sqrt{6}t + 0.2e_1 sin\sqrt{6}t & 0.13e_0 + 0.1e_{12}sin\sqrt{7}t \\ 0.1e_0 + 0.1e_1 cos\sqrt{5}t + 0.2e_{12}cos\sqrt{2}t & 0.11e_0 + 0.2e_2 sin\sqrt{3}t \end{pmatrix},$$

$$\begin{pmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{pmatrix} = \begin{pmatrix} 0.16e_0 sin\sqrt{3}t + 0.12e_1 sin\sqrt{3}t & 0.12e_0 + 0.1e_{12}sin\sqrt{7}t \\ 0.15e_0 + 0.13e_1 cos\sqrt{5}t + 0.12e_{12}cos\sqrt{4}t & 0.12e_0 + 0.12e_2 sin\sqrt{3}t \end{pmatrix}.$$

By detailed calculation, we get:

$\rho = \Big(\frac{k_i}{p\tilde{a_{i*}}}\Big)^{\frac{1}{p}} 2^m \max_{A\in\Delta}\Big[(c_i^*\eta_i^*)^{\frac{1}{p}} + \sum_{B\in\Delta}\big(\sum_{j=1}^n a_{ij}^{A\cdot\bar{B}}(s)L_j^f + \sum_{j=1}^n b_{ij}^{A\cdot\bar{B}}(s)L_j^g\big)\Big] < 1.$

Henceforth, we can show easily that all the conditions in our main Theorem 3.2 are satisfied, which means the existence unique Stepanov weighted pseudo almost automorphic solution of (4).

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