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COEFFICIENT ESTIMATES FOR GENERALIZED LIBERA TYPE BI-CLOSE-TO-CONVEX FUNCTIONS

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Abstract. In a recent paper, Sakar and Güney introduced a new class of bi-close-to-convex functions and determined the estimates for the general Taylor-Maclaurin coefficients of functions therein. The main purpose of this note is to give a generalization of this class. Also we point out the proof by Sakar and Güney is incorrect and present a correct proof.

1. Introduction

Assume that $\mathcal{H}$ is the class of analytic functions in the open unit disc

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions f normalized by

$$f(0) = f'(0) - 1 = 0.$$

Each function $f \in \mathcal{A}$ can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1}$$

We also denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ whose members are univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be *starlike of order* $\beta\,(0 \leq \beta < 1)$ if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \qquad (z \in \mathbb{U}).$$

We denote the class which consists of all functions $f \in \mathcal{A}$ that are starlike of order β by $\mathcal{S}^*(\beta)$. It is well-known that $\mathcal{S}^*(\beta) \subset \mathcal{S}^*(0) = \mathcal{S}^* \subset \mathcal{S}$.

A function $f \in \mathcal{A}$ is said to be *close-to-convex of order* $\alpha\,(0 \leq \alpha < 1)$ if there exists a function $g \in \mathcal{S}^*$ such that the inequality

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

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holds. We denote the class which consists of all functions $f \in \mathcal{A}$ that are close-to-convex of order α by $\mathcal{C}(\alpha)$. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{C}(\alpha) \subset \mathcal{S}$ (see [10]).

Let $0 \leq \alpha, \beta < 1$. A function $f \in \mathcal{A}$ is said to be *close-to-convex of order α and type β* if there exists a function $g \in \mathcal{S}^*(\beta)$ such that the inequality

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

holds. We denote the class which consists of all functions $f \in \mathcal{A}$ that are close-to-convex of order α and type β by $\mathcal{C}(\alpha, \beta)$. This class is introduced by Libera [18].

In particular, when $\beta = 0$ we have $\mathcal{C}(\alpha, 0) = \mathcal{C}(\alpha)$ of close-to-convex functions of order α, and also we get $\mathcal{C}(0,0) = \mathcal{C}$ of close-to-convex functions introduced by Kaplan [17].

Let $0 \leq \alpha < 1$, $\gamma \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$, $0 \leq \lambda \leq 1$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{SC}(\gamma, \lambda, \alpha)$ if it satisfies the condition

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{z\left[(1-\lambda) f(z) + \lambda z f'(z)\right]'}{(1-\lambda) f(z) + \lambda z f'(z)} - 1\right)\right) > \alpha \qquad (z \in \mathbb{U}).$$

This class is introduced by Altıntaş et al. [1]. Clearly, we have the following relationships: $\mathcal{SC}(1,0,\alpha) = \mathcal{S}^*(\alpha)$ and $\mathcal{SC}(1,0,0) = \mathcal{S}^*$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. Indeed, the Koebe one-quarter theorem [10] ensures that the image of $\mathbb{U}$ under every univalent function f contains a disk with radius 1/4. Thus, every function $f \in \mathcal{A}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

The inverse function $F = f^{-1}$ is given by

$$\begin{aligned} F(w) &= f^{-1}(w) \\ &= w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \\ &= w + \sum_{n=2}^{\infty} A_n w^n. \end{aligned} \tag{2}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). For a brief history and interesting examples of functions in the class , see [4,24].

The Faber polynomials introduced by Faber [11] play an important role in various areas of mathematical sciences, especially in geometric function theory. The recent publications like [5,6,14–16,27] applying the Faber polynomial expansions to analytic bi-univalent functions motivated us to apply this technique to classes of analytic bi-univalent functions.

Making use of the Faber polynomial expansion of function $f \in \mathcal{A}$ with the form (1), the coefficients of its inverse map $F = f^{-1}$ may be expressed as follows (see [2,3]):

$$F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n) w^n.$$

In general, for any $p \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, an expansion of K_{n-1}^{p} is given by (see [2])

$$K_{n-1}^{p} = pa_n + \frac{p(p-1)}{2} D_{n-1}^{2} + \frac{p!}{(p-3)!\,3!} D_{n-1}^{3} + \cdots + \frac{p!}{(p-n+1)!\,(n-1)!} D_{n-1}^{n-1},$$

where $D_{n-1}^{p} = D_{n-1}^{p}(a_2, a_3, \ldots, a_n)$. In view of [25], we see that

$$D_{n-1}^{m}(a_2, \ldots, a_n) = \sum \frac{m!}{j_1! \ldots j_{n-1}!} a_2^{j_1} \ldots a_n^{j_{n-1}}$$

and the sum is taken over all non-negative integers $j_1, \ldots, j_{n-1}$ satisfying

$$\begin{cases} j_1 + i_2 + \cdots + j_{n-1} = m, \\ \\ j_1 + 2j_2 + \cdots + (n-1)\, j_{n-1} = n-1. \end{cases}$$

It is clear that $D_{n-1}^{n-1}(a_2, \ldots, a_n) = a_2^{n-1}$.

In particular, the first three terms of K_{n-1}^{-n} are

$$K_1^{-2} = -2a_2, \quad K_2^{-3} = 3\left(2a_2^2 - a_3\right), \quad K_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right).$$

Hamidi and Jahangiri [13] introduced the class of bi-close-to-convex functions of order α as follows: For α $(0 \leq \alpha < 1)$, a function $f \in \mathcal{A}$ is said to be *bi-close-to-convex of order* α if both f and its inverse map $F = f^{-1}$ are close-to-convex of order α in $\mathbb{U}$. We denote the class which consists of all functions $f \in \Sigma$ that are bi-close-to-convex of order α by $\mathcal{C}_{\Sigma}(\alpha)$. In particular, we set $\mathcal{C}_{\Sigma}(0) = \mathcal{C}_{\Sigma}$ for the class of bi-close-to-convex functions. For recent works on bi-close-to-convex functions, please see [7–9, 12, 13, 21–23, 26].

In a very recent paper, the author introduced Libera type bi-close-to-convex functions as follows.

Definition 1.1. [8] Let $0 \leq \alpha, \beta < 1$. A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{C}_{\Sigma}(\alpha, \beta)$ of *bi-close-to-convex functions of order* α *and type* β (or *Libera type bi-close-to-convex functions*) if there exists the functions $g, G \in \mathcal{S}^*(\beta)$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha \qquad \text{and} \qquad \Re\left(\frac{wF'(w)}{G(w)}\right) > \alpha \qquad (z, w \in \mathbb{U}),$$

where the function $F = f^{-1}$ is defined by (2).

In particular, we get the class $\mathcal{C}_{\Sigma}(\alpha, 0) = \mathcal{C}_{\Sigma}(\alpha)$ of bi-close-to-convex functions of order α.

Remark 1.2. We note that when $\beta = \alpha$, $g = f$ and $G = F$, the class $\mathcal{C}_{\Sigma}(\alpha, \beta)$ reduces to the class $\mathcal{S}_{\Sigma}^{*}(\alpha)$ of bi-starlike functions of order α $(0 \leq \alpha < 1)$ which consists of functions $f \in \Sigma$ satisfying

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad \text{and} \qquad \Re\left(\frac{wF'(w)}{F(w)}\right) > \alpha \qquad (z, w \in \mathbb{U}),$$

where the function $F = f^{-1}$ is defined by (2).

Now, we introduce a new generalization of Libera type bi-close-to-convex functions of complex order as follows.

Definition 1.3. Let $0 \le \alpha, \beta < 1$, $0 \le \lambda, \delta \le 1$ and $\gamma, \tau \in \mathbb{C}^*$. A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{SC}_{\Sigma}^{\gamma;\tau}(\lambda, \alpha; \delta, \beta)$ if there exists the functions $g, G \in \mathcal{SC}(\tau, \delta, \beta)$ such that

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{z\left[(1-\lambda) f(z) + \lambda z f'(z)\right]'}{(1-\lambda) g(z) + \lambda z g'(z)} - 1\right)\right) > \alpha \qquad (z \in \mathbb{U}) \tag{3}$$

and

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{w\left[(1-\lambda) F(w) + \lambda w F'(w)\right]'}{(1-\lambda) G(w) + \lambda w G'(w)} - 1\right)\right) > \alpha \qquad (w \in \mathbb{U}), \tag{4}$$

where the function $F = f^{-1}$ is defined by (2).

Remark 1.4. If we set $\beta = 0$, $\delta = 0$ and $\gamma = \tau = 1$ in Definition 1.3, then the class $\mathcal{SC}_{\Sigma}^{\gamma;\tau}(\lambda, \alpha; \delta, \beta)$ reduces to the class $\mathcal{T}_{\Sigma}(\lambda, \alpha)$ which consists of functions $f \in \Sigma$ satisfying

$$\Re\left(\frac{z\left[(1-\lambda) f(z) + \lambda z f'(z)\right]'}{(1-\lambda) g(z) + \lambda z g'(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

and

$$\Re\left(\frac{w\left[(1-\lambda) F(w) + \lambda w F'(w)\right]'}{(1-\lambda) G(w) + \lambda w G'(w)}\right) > \alpha \qquad (w \in \mathbb{U}),$$

where $g, G \in \mathcal{S}^*$ and the function $F = f^{-1}$ is defined by (2). This class is introduced by Sakar and Güney [21]. The authors investigated the coefficient bounds for a_n of functions belong to the class $\mathcal{T}_{\Sigma}(\lambda, \alpha)$. They proved their main result by making use of the assertion: if an analytic function f of the form (1) is in the class $\mathcal{T}(\lambda, \alpha)$, that is, if it satisfies the condition

$$\Re\left(\frac{z\left[(1-\lambda) f(z) + \lambda z f'(z)\right]'}{(1-\lambda) g(z) + \lambda z g'(z)}\right) > \alpha, \qquad g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^* \qquad (z \in \mathbb{U}),$$

and if $a_k = 0 \ (2 \le k \le n-1)$, then the coefficients $b_k = 0 \ (2 \le k \le n-1)$. But we can provide a counterexample to illuminate the above assertion is wrong. For example, by choosing the functions f and g as

$$f(z) = z \quad \text{and} \quad g(z) = z - \frac{z^2}{2},$$

clearly, we see that $g \in \mathcal{S}^*$ and $f \in \mathcal{T}(1/2, 1/2)$. It is worthy to note that for these functions $a_2 = 0$ but $b_2 = -1/2 \ne 0$ (see Figure 1).

Remark 1.5. If we set $\lambda = \delta = 0$ and $\gamma = \tau = 1$, then the class $\mathcal{SC}_{\Sigma}^{\gamma;\tau}(\lambda, \alpha; \delta, \beta)$ reduces to the class $\mathcal{C}_{\Sigma}(\alpha, \beta)$ of Libera type bi-close-to-convex functions defined in Definition 1.1.

2. Preliminary Lemmas

Let the class $\mathcal{P}$ be defined by

$$\mathcal{P} = \{p \in \mathcal{H} : p(0) = 1 \quad \text{and} \quad \Re(p(z)) > 0 \ (z \in \mathbb{U})\}.$$

Assume that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \ (z \in \mathbb{U}). \tag{5}$$

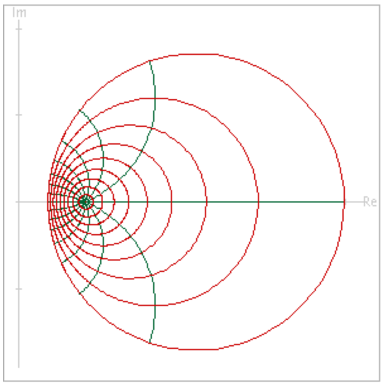

FIGURE 1.

LEMMA 2.1. (Carathéodory Lemma [19]) *Let* $p \in \mathcal{P}$ *given by* (5). *Then*

$$|c_n| \leq 2 \qquad (n \in \mathbb{N}).$$

LEMMA 2.2. [10] *If* $p \in \mathcal{P}$ *given by* (5) *and* $\mu \in \mathbb{C}$, *then*

$$\left|c_2 - \mu c_1^2\right| \leq 2 \max\{1, |2\mu - 1|\}.$$

LEMMA 2.3. [1] *If* $g \in \mathcal{SC}(\tau, \delta, \beta)$ $(0 \leq \beta < 1,\ 0 \leq \delta \leq 1,\ \tau \in \mathbb{C}^*)$ *with* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, *then*

$$|b_n| \leq \frac{\prod_{j=0}^{n-2} [j + 2|\tau|(1-\beta)]}{(n-1)!\ [1 + \delta(n-1)]} \qquad (n \in \mathbb{N}^* := \mathbb{N} \backslash \{1\} = \{2, 3, \ldots\}).$$

LEMMA 2.4. *If* $g \in \mathcal{SC}(\tau, \delta, \beta)$ $(0 \leq \beta < 1,\ 0 \leq \delta \leq 1,\ \tau \in \mathbb{C}^*)$ *with* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, *then for* $\mu \in \mathbb{C}$

$$\left|b_3 - \mu b_2^2\right| \leq \frac{|\tau|(1-\beta)}{1+2\delta} \max\left\{1, \left|1 + 2\tau(1-\beta)\left(1 - \frac{2(1+2\delta)}{(1+\delta)^2}\mu\right)\right|\right\}.$$

Proof. Let $0 \leq \beta < 1$, $0 \leq \delta \leq 1$ and $\tau \in \mathbb{C}^*$. If $g \in \mathcal{SC}(\tau, \delta, \beta)$, then we have

$$\Re\left(1 + \frac{1}{\tau}\left(\frac{zG_\delta'(z)}{G_\delta(z)} - 1\right)\right) > \beta \qquad (z \in \mathbb{U}),$$

where

$$G_\delta(z) = (1-\delta)\, g(z) + \delta z g'(z).$$

Then there exist a positive real part function $h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n \in \mathcal{P}$ in $\mathbb{U}$ such that

$$1 + \frac{1}{\tau}\left(\frac{zG_\delta'(z)}{G_\delta(z)} - 1\right) = \beta + (1-\beta)\, h(z) = 1 + (1-\beta)\sum_{n=1}^{\infty} h_n z^n. \tag{6}$$

From (6), we have

$$b_2 = \frac{\tau(1-\beta)}{1+\delta} h_1, \tag{7}$$

$$b_3 = \frac{\tau(1-\beta)}{2(1+2\delta)}\left[h_2 + \tau(1-\beta)\, h_1^2\right]. \tag{8}$$

Taking into account (7) and (8), we obtain

$$b_3-\mu b_2^2=\frac{\tau(1-\beta)}{2(1+2\delta)}\left(h_2-\nu h_1^2\right), \tag{9}$$

where

$$\nu=-\tau(1-\beta)\left(1-\frac{2(1+2\delta)}{(1+\delta)^2}\mu\right).$$

Our result now follows by an application of Lemma 2.2. This completes the proof of Lemma 2.4. □

3. Main Results

THEOREM 3.1. *For* $0\le\alpha,\beta<1$, $0\le\lambda,\delta\le1$ *and* $\gamma,\tau\in\mathbb{C}^*$, *let* $f\in\mathcal{SC}_\Sigma^{\gamma,\tau}(\lambda,\alpha;\delta,\beta)$. *If* $a_k=0$ $(2\le k\le n-1)$, *then for* $n\ge3$,

$$\begin{aligned}|a_n| \le\ & \frac{\prod_{j=0}^{n-2}\left[j+2|\tau|(1-\beta)\right]}{n!\ \left[1+\delta(n-1)\right]}+\frac{2|\gamma|(1-\alpha)}{n\left[1+(n-1)\lambda\right]}\\ &+\frac{1}{n\left[1+(n-1)\lambda\right]}\sum_{l=1}^{n-2}\frac{\left[1+(n-l-1)\lambda\right]\prod_{j=0}^{n-l-2}\left[j+2|\tau|(1-\beta)\right]}{(n-l-1)!\ \left[1+\delta(n-l-1)\right]}\Omega_l^\lambda.\end{aligned}$$

where

$$\Omega_l^\lambda=\min\left\{\left|K_l^{-1}\left((1+\lambda)b_2,\ldots,(1+l\lambda)b_{l+1}\right)\right|,\ \left|K_l^{-1}\left((1+\lambda)B_2,\ldots,(1+l\lambda)B_{l+1}\right)\right|\right\}. \tag{10}$$

Proof. For $0\le\alpha,\beta<1$, $0\le\lambda,\delta\le1$ and $\gamma,\tau\in\mathbb{C}^*$, let the function f given by (1) satisfies the hypothesis of the theorem, that is, let f belongs to the class $\mathcal{SC}_\Sigma^{\gamma,\tau}(\lambda,\alpha;\delta,\beta)$. Then there exist the functions

$$g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{SC}(\tau,\delta,\beta)\quad\text{and}\quad G(w)=w+\sum_{n=2}^{\infty}B_nw^n\in\mathcal{SC}(\tau,\delta,\beta), \tag{11}$$

such that (3) and (4) hold. The Faber polynomial expansion for

$$1+\frac{1}{\gamma}\left(\frac{z\left[(1-\lambda)f(z)+\lambda zf'(z)\right]'}{(1-\lambda)g(z)+\lambda zg'(z)}-1\right)$$

and

$$1+\frac{1}{\gamma}\left(\frac{w\left[(1-\lambda)F(w)+\lambda wF'(w)\right]'}{(1-\lambda)G(w)+\lambda wG'(w)}-1\right)$$

are obtained by

$$\begin{aligned}1+\frac{1}{\gamma}\left(\frac{z\left[(1-\lambda)f(z)+\lambda zf'(z)\right]'}{(1-\lambda)g(z)+\lambda zg'(z)}-1\right)&=1+\sum_{n=2}^{\infty}\left\{\frac{1+(n-1)\lambda}{\gamma}(na_n-b_n)\right.\\ +\sum_{l=1}^{n-2}\frac{1+(n-l-1)\lambda}{\gamma}K_l^{-1}\left((1+\lambda)b_2,\ldots,(1+l\lambda)b_{l+1}\right)&\left.\left[(n-l)a_{n-l}-b_{n-l}\right]\right\}z^{n-1}\end{aligned} \tag{12}$$

and

$$1+\frac{1}{\gamma}\left(\frac{w\left[(1-\lambda)F(w)+\lambda wF'(w)\right]'}{(1-\lambda)G(w)+\lambda wG'(w)}-1\right)=1+\sum_{n=2}^{\infty}\left\{\frac{1+(n-1)\lambda}{\gamma}(nA_n-B_n)\right.$$

(13)

$$\left.+\sum_{l=1}^{n-2}\frac{1+(n-l-1)\lambda}{\gamma}K_l^{-1}\left((1+\lambda)B_2,\dots,(1+l\lambda)B_{l+1}\right)\left[(n-l)A_{n-l}-B_{n-l}\right]\right\}w^{n-1},$$

respectively. On the other hand by (3) and (4), we see that there exist two positive real part functions

$$p(z)=1+\sum_{n=1}^{\infty}c_nz^n\in\mathcal{P}\quad\text{and}\quad q(w)=1+\sum_{n=1}^{\infty}d_nw^n\in\mathcal{P}$$

in $\mathbb{U}$ such that

$$\begin{aligned}1+\frac{1}{\gamma}\left(\frac{z\left[(1-\lambda)f(z)+\lambda zf'(z)\right]'}{(1-\lambda)g(z)+\lambda zg'(z)}-1\right)&=\alpha+(1-\alpha)p(z)\\&=1+(1-\alpha)\sum_{n=1}^{\infty}c_nz^n,\end{aligned}\tag{14}$$

and

$$\begin{aligned}1+\frac{1}{\gamma}\left(\frac{w\left[(1-\lambda)F(w)+\lambda wF'(w)\right]'}{(1-\lambda)G(w)+\lambda wG'(w)}-1\right)&=\alpha+(1-\alpha)q(w)\\&=1+(1-\alpha)\sum_{n=1}^{\infty}d_nw^n.\end{aligned}\tag{15}$$

We note that

$$|c_n|\le 2\qquad\text{and}\qquad|d_n|\le 2\qquad(n\in\mathbb{N})\tag{16}$$

by Lemma 2.1. Comparing the corresponding coefficients of (12) and (14), for any $n\geq 2$, yields

(17)

$$\begin{aligned}&\frac{1+(n-1)\lambda}{\gamma}(na_n-b_n)\\&\quad+\sum_{l=1}^{n-2}\frac{1+(n-l-1)\lambda}{\gamma}K_l^{-1}\left((1+\lambda)b_2,\dots,(1+l\lambda)b_{l+1}\right)\left[(n-l)a_{n-l}-b_{n-l}\right]\\&=(1-\alpha)c_{n-1}.\end{aligned}$$

Similarly, it follows from (13) and (15) that

(18)

$$\begin{aligned}&\frac{1+(n-1)\lambda}{\gamma}(nA_n-B_n)\\&\quad+\sum_{l=1}^{n-2}\frac{1+(n-l-1)\lambda}{\gamma}K_l^{-1}\left((1+\lambda)B_2,\dots,(1+l\lambda)B_{l+1}\right)\left[(n-l)A_{n-l}-B_{n-l}\right]\\&=(1-\alpha)d_{n-1}.\end{aligned}$$

By the hypothesis $a_k = 0 \ (2 \leq k \leq n-1)$, we find from (17) and (18) that

$$\begin{aligned} & \frac{1+(n-1)\lambda}{\gamma}(na_n - b_n) \\ & -\sum_{l=1}^{n-2} \frac{1+(n-l-1)\lambda}{\gamma} b_{n-l} K_l^{-1}((1+\lambda) b_2, \ldots, (1+l\lambda) b_{l+1}) \\ = & \ (1-\alpha) c_{n-1}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} & \frac{1+(n-1)\lambda}{\gamma}(nA_n - B_n) \\ & -\sum_{l=1}^{n-2} \frac{1+(n-l-1)\lambda}{\gamma} B_{n-l} K_l^{-1}((1+\lambda) B_2, \ldots, (1+l\lambda) B_{l+1}) \\ = & \ (1-\alpha) d_{n-1}, \end{aligned} \tag{20}$$

respectively. Also the equality $a_k = 0 \ (2 \leq k \leq n-1)$ implies that $A_n = -a_n$. Thus (19) and (20) gives

$$\begin{aligned} \frac{n[1+(n-1)\lambda]}{\gamma} a_n &= \frac{1+(n-1)\lambda}{\gamma} b_n + (1-\alpha) c_{n-1} \\ & +\sum_{l=1}^{n-2} \frac{1+(n-l-1)\lambda}{\gamma} b_{n-l} K_l^{-1}((1+\lambda) b_2, \ldots, (1+l\lambda) b_{l+1}) \end{aligned} \tag{21}$$

and

$$\begin{aligned} -\frac{n[1+(n-1)\lambda]}{\gamma} a_n &= \frac{1+(n-1)\lambda}{\gamma} B_n + (1-\alpha) d_{n-1} \\ & +\sum_{l=1}^{n-2} \frac{1+(n-l-1)\lambda}{\gamma} B_{n-l} K_l^{-1}((1+\lambda) B_2, \ldots, (1+l\lambda) B_{l+1}), \end{aligned} \tag{22}$$

respectively.

On the other hand, by the hypothesis (11), since $g, G \in \mathcal{SC}(\tau, \delta, \beta)$ we obtain the coefficient inequalities

$$|b_n| \leq \frac{\prod_{j=0}^{n-2} [j + 2|\tau|(1-\beta)]}{(n-1)! \ [1+\delta(n-1)]} \qquad \text{and} \qquad |B_n| \leq \frac{\prod_{j=0}^{n-2} [j + 2|\tau|(1-\beta)]}{(n-1)! \ [1+\delta(n-1)]}$$

from Lemma 2.3. Considering the above coefficient bounds and the inequalities in (16), from (21) and (22) we get

$$\begin{aligned} & \frac{n[1+(n-1)\lambda]}{|\gamma|} |a_n| \\ & \leq \frac{1+(n-1)\lambda}{|\gamma|} |b_n| + (1-\alpha)|c_{n-1}| \\ & +\sum_{l=1}^{n-2} \frac{1+(n-l-1)\lambda}{|\gamma|} |b_{n-l}| \ \left| K_l^{-1}((1+\lambda) b_2, \ldots, (1+l\lambda) b_{l+1}) \right| \end{aligned}$$

$$\leq \frac{[1+(n-1)\lambda] \prod_{j=0}^{n-2} [j+2|\tau|(1-\beta)]}{|\gamma|\ (n-1)!\ [1+\delta(n-1)]} + 2(1-\alpha)$$

(23)

$$+\sum_{l=1}^{n-2} \frac{[1+(n-l-1)\lambda] \prod_{j=0}^{n-l-2} [j+2|\tau|(1-\beta)]}{|\gamma|\ (n-l-1)!\ [1+\delta(n-l-1)]} \left|K_l^{-1}((1+\lambda)b_2,\ldots,(1+l\lambda)b_{l+1})\right|$$

and

$$\frac{n[1+(n-1)\lambda]}{|\gamma|}|a_n|$$

$$\leq \frac{1+(n-1)\lambda}{|\gamma|}|B_n| + (1-\alpha)|d_{n-1}|$$

$$+\sum_{l=1}^{n-2} \frac{1+(n-l-1)\lambda}{|\gamma|}|B_{n-l}|\ \left|K_l^{-1}((1+\lambda)B_2,\ldots,(1+l\lambda)B_{l+1})\right|$$

$$\leq \frac{[1+(n-1)\lambda] \prod_{j=0}^{n-2} [j+2|\tau|(1-\beta)]}{|\gamma|\ (n-1)!\ [1+\delta(n-1)]} + 2(1-\alpha)$$

(24)

$$+\sum_{l=1}^{n-2} \frac{[1+(n-l-1)\lambda] \prod_{j=0}^{n-l-2} [j+2|\tau|(1-\beta)]}{|\gamma|\ (n-l-1)!\ [1+\delta(n-l-1)]} \left|K_l^{-1}((1+\lambda)B_2,\ldots,(1+l\lambda)B_{l+1})\right|,$$

respectively. Consequently, by comparing (23) and (24), we get the coefficient bounds for $|a_n|$ as asserted in Theorem 3.1. □

By setting $\delta = 0$, $\beta = 0$ and $\gamma = \tau = 1$ in Theorem 3.1, we get the following result.

Corollary 3.2. *For $0 \leq \alpha < 1$ and $0 \leq \lambda \leq 1$, let $f \in \mathcal{T}_\Sigma(\lambda, \alpha)$. If $a_k = 0$ $(2 \leq k \leq n-1)$, then for $n \geq 3$,*

$$\begin{aligned} |a_n| \ &\leq \ 1 + \frac{2(1-\alpha)}{n[1+(n-1)\lambda]} \\ &+ \frac{1}{n[1+(n-1)\lambda]} \sum_{l=1}^{n-2} [1+(n-l-1)\lambda]\ (n-l)\,\Omega_l^\lambda. \end{aligned}$$

where Ω_l^λ is defined by (10).

By setting $\lambda = \delta = 0$ and $\gamma = \tau = 1$ in Theorem 3.1, we have the following consequence.

Corollary 3.3. *[8] For* $0 \leq \alpha, \beta < 1$, *let* $f \in \mathcal{C}_{\Sigma}(\alpha, \beta)$. *If* $a_k = 0 \ (2 \leq k \leq n-1)$, *then for* $n \geq 3$,

$$\begin{aligned} |a_n| &\leq \frac{\prod_{j=0}^{n-2}[j+2(1-\beta)]}{n!} + \frac{2(1-\alpha)}{n} \\ &+ \frac{1}{n}\sum_{l=1}^{n-2}\frac{\prod_{j=0}^{n-l-2}[j+2(1-\beta)]}{(n-l-1)!}\min\left\{\left|K_l^{-1}(b_2,\ldots,b_{l+1})\right|, \left|K_l^{-1}(B_2,\ldots,B_{l+1})\right|\right\}. \end{aligned}$$

By setting $\beta = 0$, $\lambda = \delta = 0$ and $\gamma = \tau = 1$ in Theorem 3.1, we get the following result.

Corollary 3.4. *[26] For* $0 \leq \alpha < 1$, *let* $f \in \mathcal{C}_{\Sigma}(\alpha)$. *If* $a_k = 0 \ (2 \leq k \leq n-1)$, *then for* $n \geq 3$,

$$|a_n| \leq 1 + \frac{2(1-\alpha)}{n} + \frac{1}{n}\sum_{l=1}^{n-2}(n-l)\min\left\{\left|K_l^{-1}(b_2,\ldots,b_{l+1})\right|, \left|K_l^{-1}(B_2,\ldots,B_{l+1})\right|\right\}.$$

By setting $b_k = B_k = 0 \ (2 \leq k \leq n-1)$ in Theorem 3.1, we get the following result.

Corollary 3.5. *For* $0 \leq \alpha, \beta < 1$, $0 \leq \lambda, \delta \leq 1$ *and* $\gamma, \tau \in \mathbb{C}^*$, *let* $f \in \mathcal{SC}_{\Sigma}^{\gamma,\tau}(\lambda, \alpha; \delta, \beta)$. *If* $a_k = b_k = B_k = 0 \ (2 \leq k \leq n-1)$, *then for* $n \geq 3$,

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}[j+2|\tau|(1-\beta)]}{n!\ [1+\delta(n-1)]} + \frac{2|\gamma|(1-\alpha)}{n[1+(n-1)\lambda]}.$$

By setting $b_k = B_k = 0 \ (2 \leq k \leq n-1)$, $\beta = 0$, $\delta = 0$ and $\gamma = \tau = 1$ in Theorem 3.1, we get the following result. It corrects the errors of [21, Theorem 2.1]. More precisely, Theorem 2.1 in [21] holds only with the additional condition $b_k = B_k = 0 \ (2 \leq k \leq n-1)$.

Corollary 3.6. (*Correction of [21, Theorem 2.1]*) *For* $0 \leq \alpha < 1$ *and* $0 \leq \lambda \leq 1$, *let* $f \in \mathcal{T}_{\Sigma}(\lambda, \alpha)$. *If* $a_k = b_k = B_k = 0 \ (2 \leq k \leq n-1)$, *then for* $n \geq 3$,

$$|a_n| \leq 1 + \frac{2(1-\alpha)}{n[1+(n-1)\lambda]}.$$

Corollary 3.7. *For* $0 \leq \alpha, \beta < 1$, $0 \leq \lambda, \delta \leq 1$ *and* $\gamma, \tau \in \mathbb{C}^*$, *let* $f \in \mathcal{SC}_{\Sigma}^{\gamma,\tau}(\lambda, \alpha; \delta, \beta)$. *Also suppose that*

$$G(w) = g^{-1}(w) = w - b_2w^2 + \left(2b_2^2 - b_3\right)w^3 - \left(5b_2^3 - 5b_2b_3 + b_4\right)w^4 + \cdots. \tag{25}$$

If $a_k = b_k = 0 \ (2 \leq k \leq n-1)$, *then for* $n \geq 3$,

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}[j+2|\tau|(1-\beta)]}{n!\ [1+\delta(n-1)]} + \frac{2|\gamma|(1-\alpha)}{n[1+(n-1)\lambda]}.$$

As a special case to Theorem 3.1, we determine the initial coefficient bounds of functions belonging to the class $\mathcal{SC}_{\Sigma}^{\gamma,\tau}(\lambda, \alpha; \delta, \beta)$ of bi-close-to-convex functions of order α and type β.

Theorem 3.8. *For $0 \leq \alpha, \beta < 1$, $0 \leq \lambda, \delta \leq 1$ and $\gamma, \tau \in \mathbb{C}^*$, let the function f given by* (1) *be in the function class $\mathcal{SC}_{\Sigma}^{\gamma,\tau}(\lambda, \alpha; \delta, \beta)$ and suppose that the function G be defined by* (25)*. Then one has the following*

$$|a_2| \leq \min\left\{ \frac{|\gamma|(1-\alpha)}{1+\lambda} + \frac{|\tau|(1-\beta)}{1+\delta}, \right.$$

$$\left. \sqrt{\frac{4|\tau|(1-\beta)}{3(1+\delta)}\left(\frac{|\gamma|(1+\lambda)(1-\alpha)}{1+2\lambda} + \frac{|\tau|(1-\beta)}{1+\delta}\right) + \frac{2|\gamma|(1-\alpha)}{3(1+2\lambda)}} \right\} \tag{26}$$

and

$$|a_3| \leq \min\left\{ \frac{|\gamma|(1-\alpha)}{3(1+2\lambda)}[1+\max\{1,|\mu|\}] + \frac{|\tau|(1-\beta)}{3(1+2\delta)}\max\{1,|\rho|\} + \frac{2|\gamma|(1-\alpha)|\tau|(1-\beta)}{(1+\lambda)(1+\delta)}, \right.$$

$$\left. \frac{2|\gamma|(1-\alpha)}{3(1+2\lambda)} + \frac{4|\gamma|(1+\lambda)(1-\alpha)|\tau|(1-\beta)}{3(1+2\lambda)(1+\delta)} + \frac{|\tau|(1-\beta)[1+2|\tau|(1-\beta)]}{3(1+2\delta)} \right\}, \tag{27}$$

where

$$\mu = 1 + \frac{3(1+2\lambda)}{(1+\lambda)^2}\gamma(1-\alpha), \qquad \rho = 1 + \frac{1+2\delta+2\delta^2}{(1+\delta)^2}\tau(1-\beta).$$

Proof. If we set $n = 2$ and $n = 3$ in (17) and (18), respectively, we get

$$2a_2 = \frac{\gamma(1-\alpha)}{1+\lambda}c_1 + b_2 \tag{28}$$

$$3a_3 = \frac{\gamma(1-\alpha)}{1+2\lambda}c_2 + \frac{\gamma(1+\lambda)(1-\alpha)}{1+2\lambda}c_1 b_2 + b_3 \tag{29}$$

$$-2a_2 = \frac{\gamma(1-\alpha)}{1+\lambda}d_1 - b_2 \tag{30}$$

$$6a_2^2 - 3a_3 = \frac{\gamma(1-\alpha)}{1+2\lambda}d_2 - \frac{\gamma(1+\lambda)(1-\alpha)}{1+2\lambda}d_1 b_2 + 2b_2^2 - b_3. \tag{31}$$

From (28) and (30), we find

$$c_1 = -d_1 \tag{32}$$

and

$$a_2 = \frac{\gamma(1-\alpha)}{2(1+\lambda)}c_1 + \frac{1}{2}b_2. \tag{33}$$

On the other hand, from (29) and (31), we obtain

$$a_2^2 = \frac{\gamma(1-\alpha)}{6(1+2\lambda)}(c_2+d_2) + \frac{\gamma(1+\lambda)(1-\alpha)}{3(1+2\lambda)}c_1 b_2 + \frac{1}{3}b_2^2. \tag{34}$$

Therefore by applying triangle inequality to (33) and (34), using (16) and the fact that

$$|b_2| \leq \frac{2|\tau|(1-\beta)}{1+\delta} \tag{35}$$

obtained from Lemma 2.3 for $n=2$, we get the desired estimate on the coefficient bound for $|a_2|$ as asserted in (26).

Next, in order to find the bound for $|a_3|$, we subtract (31) from (29). We thus get

$$6a_3-6a_2^2=\frac{\gamma(1-\alpha)}{1+2\lambda}(c_2-d_2)+\frac{\gamma(1+\lambda)(1-\alpha)}{1+2\lambda}b_2(c_1+d_1)-2b_2^2+2b_3.$$

By (32), we obtain

$$a_3=a_2^2+\frac{\gamma(1-\alpha)}{6(1+2\lambda)}(c_2-d_2)+\frac{b_3-b_2^2}{3}. \tag{36}$$

If we set the value of a_2^2 from (33) in (36), then we have

$$\begin{aligned} a_3 &= \frac{\gamma^2(1-\alpha)^2}{4(1+\lambda)^2}c_1^2+\frac{1}{4}b_2^2+\frac{\gamma(1-\alpha)}{2(1+\lambda)}c_1b_2 \\ &\quad +\frac{\gamma(1-\alpha)}{6(1+2\lambda)}c_2-\frac{\gamma(1-\alpha)}{6(1+2\lambda)}d_2+\frac{1}{3}b_3-\frac{1}{3}b_2^2 \\ &= \frac{\gamma(1-\alpha)}{6(1+2\lambda)}\left(c_2+\frac{3\gamma(1+2\lambda)(1-\alpha)}{2(1+\lambda)^2}c_1^2\right)+\frac{1}{3}\left(b_3-\frac{1}{4}b_2^2\right) \\ &\quad +\frac{\gamma(1-\alpha)}{2(1+\lambda)}c_1b_2-\frac{\gamma(1-\alpha)}{6(1+2\lambda)}d_2. \end{aligned}$$

So using Lemma 2.2, Lemma 2.4, (16) and (35), we get

$$\begin{aligned} |a_3| &\le \frac{|\gamma|(1-\alpha)}{3(1+2\lambda)}[1+\max\{1,|\mu|\}]+\frac{|\tau|(1-\beta)}{3(1+2\delta)}\max\{1,|\rho|\} \\ &\quad +\frac{2|\gamma|(1-\alpha)|\tau|(1-\beta)}{(1+\lambda)(1+\delta)}, \end{aligned} \tag{37}$$

where

$$\mu=1+\frac{3(1+2\lambda)}{(1+\lambda)^2}\gamma(1-\alpha), \qquad \rho=1+\frac{1+2\delta+2\delta^2}{(1+\delta)^2}\tau(1-\beta).$$

If we set the value of a_2^2 from (34) in (36), then we have

$$a_3=\frac{\gamma(1-\alpha)}{3(1+2\lambda)}c_2+\frac{\gamma(1+\lambda)(1-\alpha)}{3(1+2\lambda)}c_1b_2+\frac{1}{3}b_3.$$

So using (16), (35) and the fact that

$$|b_3|\le\frac{|\tau|(1-\beta)[1+2|\tau|(1-\beta)]}{1+2\delta}$$

obtained from Lemma 2.3 for $n=3$, we obtain

$$|a_3|\le\frac{2|\gamma|(1-\alpha)}{3(1+2\lambda)}+\frac{4|\gamma|(1+\lambda)(1-\alpha)|\tau|(1-\beta)}{3(1+2\lambda)(1+\delta)}+\frac{|\tau|(1-\beta)[1+2|\tau|(1-\beta)]}{3(1+2\delta)}. \tag{38}$$

Hence (37) and (38) give the desired estimate on the coefficient $|a_3|$ as asserted in (27). □

By setting $\delta=0$, $\beta=0$ and $\gamma=\tau=1$ in Theorem 3.8, we get the following result.

Corollary 3.9. *For $0 \leq \alpha < 1$ and $0 \leq \lambda \leq 1$, let the function f given by (1) be in the function class $\mathcal{T}_{\Sigma}(\lambda, \alpha)$ and suppose that the function G be defined by (25). Then one has the following*

$$|a_2| \leq \min\left\{ \frac{2-\alpha+\lambda}{1+\lambda} \, , \, \sqrt{\frac{2(1-\alpha)(3+2\lambda)}{3(1+2\lambda)} + \frac{4}{3}} \right\}$$

and

$$|a_3| \leq \min\left\{ \frac{2(2-\alpha+2\lambda)}{3(1+2\lambda)} + \frac{(1-\alpha)(3-\alpha+2\lambda)}{(1+\lambda)^2} \, , \, \frac{2(1-\alpha)(3+2\lambda)}{3(1+2\lambda)} + 1 \right\}.$$

By setting $\lambda = \delta = 0$ and $\gamma = \tau = 1$ in Theorem 3.8, we get the following result.

Corollary 3.10. *[8] For $0 \leq \alpha, \beta < 1$, let the function f given by (1) be in the function class $\mathcal{C}_{\Sigma}(\alpha, \beta)$ and suppose that the function G be defined by (25). Then one has the following*

$$|a_2| \leq \min\left\{ (2-\alpha-\beta) \, , \, \sqrt{\frac{4(1-\beta)(2-\alpha-\beta) + 2(1-\alpha)}{3}} \right\}$$

and

$$|a_3| \leq \frac{1}{3} \begin{cases} (3-2\beta)(3-2\alpha-\beta) & , \quad 0 \leq \alpha \leq \frac{2+\beta}{3} \\ (1-\alpha)(5-3\alpha) + (1-\beta)(2-\beta) + 6(1-\alpha)(1-\beta) & , \quad \frac{2+\beta}{3} \leq \alpha < 1 \end{cases}.$$

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