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GENERAL SOLUTION AND ULAM STABILITY OF GENERALIZED CQ FUNCTIONAL EQUATION

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Abstract. In this paper, we introduce the following cubic-quartic functional equation of the form

$$f(x+4y)+f(x-4y)=16\left[f(x+y)+f(x-y)\right]\pm 30f(-x)+\frac{5}{2}\left[f(4y)-64f(y)\right].$$

Further, we investigate the general solution and the Ulam stability for the above functional equation in non-Archimedean spaces by using the direct method.

1. Introduction

Jun and Kim [7] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.1}$$

and they established the general solution and the Ulam stability for the functional equation (1.1). The function $f(x)=x^3$ satisfies the functional equation (1.1), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic mapping. Now we introduce the cubic functional equation and quartic functional equation

$$f(x+4y)+f(x-4y)=16\left[f(x+y)+f(x-y)\right]+30f(-x)+\frac{5}{2}\left[f(4y)-64f(y)\right] \tag{1.2}$$

and

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)+24f(y)-6f(x). \tag{1.3}$$

It is easy to see that the function $f(x)=x^4$ is a solution of the functional equation (1.3). Thus, it is natural that (1.3) is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic mapping.

In this section, we introduce the cubic-quartic functional equation of the form

$$f(x+4y)+f(x-4y)=16\left[f(x+y)+f(x-y)\right]\pm 30f(-x)+\frac{5}{2}\left[f(4y)-64f(y)\right] \tag{1.4}$$

Further, we investigate the general solution and the Ulam stability for the functional equation (1.4).

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By a non-Archimedean field we mean a field K equipped with a function (valuation) $|\cdot|$ from K into $[0,\infty)$ such that $|r|=0$ if and only if $r=0, |rs|=|r||s|$, and $|r+s|\le \max\{|r|,|s|\}$ for all $r,s\in K$. Clearly $|1|=|-1|=1$ and $|n|\le 1$ for all $n\in\mathbb{N}$.

DEFINITION 1.1. Let X be a vector space over a scalar field K with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\|: X\to K$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:
(i) $\|x\|=0$ if and only if $x=0$;
(ii) $\|rx\|=|r|\|x\|$ for all $r\in K, x\in X$;
(iii) The strong inequality (ultrametric); namely,

$$\|x+y\|\le \max\{\|x\|,\|y\|\}$$

for all $x,y\in X$. Then $(X,\|\cdot\|)$ is called a non-Archimedean space.
Due to the fact that

$$\|x_m-x_n\|\le \max\{\|x_{j+1}-x_j\| : m\le j\le n-1\} \qquad (n>m),$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1}-x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. Furthermore, some of the research papers related to non-Archimedean spaces are very useful to develop this article such as [1–4, 10, 15] and some of the other papers are used to build this section (see [5, 6, 8, 9, 11–14, 16]).

2. General solution for the cubic-quartic functional equation (1.4)

In this section, we find out the general solution of the cubic-quartic functional equation (1.4).

THEOREM 2.1. *If a mapping $f: X\to Y$ satisfies the functional equation (1.2), then the mapping $f: X\to Y$ satisfies the functional equation (1.1).*

Proof. Putting $x=y=0$ in (1.2), we get $f(0)=0$. Setting $y=0$ in (1.2), we obtain $f(-x)=-f(x)$ for all $x\in X$. Hence f is odd. Replacing (x,y) by $(0,x)$ in (1.2) we get

$$f(4x)=64f(x) \tag{2.5}$$

for all $x\in X$. So

$$f(x+4y)+f(x-4y)=16\,[f(x+y)+f(x-y)]-30f(x) \tag{2.6}$$

for all $x,y\in X$. Replacing x by $4x$ in (2.6), we obtain

$$f(4x+4y)+f(4x-4y)=16\,[f(4x+y)+f(4x-y)]-30f(4x) \tag{2.7}$$

for all $x,y\in X$. It follows from (2.5) and (2.7) that

$$f(4x+y)+f(4x-y)=4\,[f(x+y)+f(x-y)]+120f(4x) \tag{2.8}$$

for all $x,y\in X$. Replacing x by $x+y$ in (2.6), we obtain

$$f(x+5y)+f(x-3y)=16\,[f(x+2y)+f(x)]-30f(x+y) \tag{2.9}$$

for all $x,y\in X$. Replacing x by $x-y$ in (2.6), we obtain

$$f(x-5y)+f(x+3y)=16\,[f(x-2y)+f(x)]-30f(x-y) \tag{2.10}$$

for all $x,y\in X$. Adding (2.9) and (2.10), we get

$$\begin{aligned} &f(x+5y)+f(x-3y)+f(x-5y)+f(x+3y)\\ &=16\,[f(x+2y)+f(x-2y)]+32f(x)-30\,[f(x+y)+f(x-y)] \end{aligned} \tag{2.11}$$

for all $x, y \in X$. Further replacing y by $x + y$ in (2.6), we obtain

$$f(5x+4y) + f(-3x-4y) = 16\,[f(2x+y) - f(y)] - 30f(x) \tag{2.12}$$

for all $x, y \in X$ and replacing y by $-x + y$ in (2.6), we get

$$f(-3x+4y) + f(5x-4y) = 16\,[f(y) - f(2x-y)] - 30f(x) \tag{2.13}$$

for all $x, y \in X$. Adding (2.12) and (2.13), we get

$$\begin{aligned} &f(5x+4y) + f(5x-4y) + f(-3x+4y) + f(-3x-4y)\\ &= 16\,[f(2x+y) + f(2x-y)] - 60f(x) \end{aligned} \tag{2.14}$$

for all $x, y \in X$. Interchanging x by y in (2.14), we get

$$\begin{aligned} &f(4x+5y) + f(-4x+5y) + f(4x-3y) + f(-4x-3y)\\ &= 16\,[f(x+2y) - f(x-2y)] - 60f(y) \end{aligned} \tag{2.15}$$

for all $x, y \in X$. Simplifying (2.15) and using oddness, we have

$$\begin{aligned} &f(4x+5y) - f(4x-5y) + f(4x-3y) - f(4x+3y)\\ &= 16\,[f(x+2y) - f(x-2y)] - 60f(y) \end{aligned} \tag{2.16}$$

for all $x, y \in X$. It follows from (2.9) and (2.10) that

$$\begin{aligned} &f(x+5y) - f(x-5y) + f(x-3y) - f(x+3y)\\ &= 16\,[f(x+2y) - f(x-2y)] - 30\,[f(x+y) - f(x-y)] \end{aligned} \tag{2.17}$$

for all $x, y \in X$. Replacing x by $4x$ in (2.17), we obtain

$$\begin{aligned} &f(4x+5y) - f(4x-5y) + f(4x-3y) - f(4x+3y)\\ &= 16\,[f(4x+2y) - f(4x-2y)] - 30\,[f(4x+y) - f(4x-y)] \end{aligned} \tag{2.18}$$

for all $x, y \in X$. By comparing (2.16) and (2.18), we obtain

$$\begin{aligned} &16\,[f(x+2y) - f(x-2y) - 60f(x-2y)]\\ &= 16\,[f(4x+2y) - f(4x-2y)] - 30\,[f(4x+y) - f(4x-y)] \end{aligned} \tag{2.19}$$

for all $x, y \in X$. Now by interchanging x and y in (2.19), we get

$$\begin{aligned} &16\,[f(2x+y) - f(2x-y) - 60f(x)]\\ &= 16\,[f(2x+4y) + f(2x-4y)] - 30\,[f(x+4y) + f(x-4y)] \end{aligned} \tag{2.20}$$

for all $x, y \in X$. It follows from (2.6) and (2.20) that

$$\begin{aligned} &f(2x+y) + f(2x-y)\\ &= [f(2x+4y) + f(2x-4y)] - 30\,[f(x+y) + f(x-y)] + 60f(x) \end{aligned} \tag{2.21}$$

for all $x, y \in X$. Simplifying (2.21), we obtain

$$\begin{aligned} &f(2x+4y) + f(2x-4y)\\ &= [f(2x+y) + f(2x-y)] + 30\,[f(x+y) + f(x-y)] - 60f(x) \end{aligned} \tag{2.22}$$

for all $x, y \in X$. Replacing x by $2x$ in (2.6), we obtain

$$f(2x+4y) + f(2x-4y) = 16\,[f(2x+y) + f(2x-y)] - 2400f(x) \tag{2.23}$$

for all $x, y \in X$. From (2.22) and (2.23), we get the desired equation (1.1). □

THEOREM 2.2. *If an even mapping $f : X \to Y$ satisfies the functional equation (1.4), then the mapping $f : X \to Y$ satisfies the functional equation (1.3).*

Proof. Putting $x = y = 0$ in (1.4), we get $f(0) = 0$. Replacing (x, y) by $(0, x)$ in (1.4) and using the evenness of f, we get

$$f(4x) = 256f(x) \tag{2.24}$$

for all $x \in X$. It follows from (2.24) and (1.4) that

$$f(x+4y) + f(x-4y) = 16\,[f(x+y) + f(x-y)] - 30f(x) + 480f(y) \tag{2.25}$$

for all $x, y \in X$. Replacing x by $2x$ in (2.25), we have

$$f(2x+4y) + f(2x-4y) = 16\,[f(2x+y) + f(2x-y)] - 480f(x) + 480f(y) \tag{2.26}$$

for all $x, y \in X$. Replacing x by $x+y$ in (2.25), we obtain

$$f(x+5y) + f(x-3y) = 16\,[f(x+2y) + f(x)] - 30f(x+y) + 480f(y) \tag{2.27}$$

for all $x, y \in X$. Replacing x by $x-y$ in (2.25), we obtain

$$f(x-5y) + f(x+3y) = 16\,[f(x-2y) + f(x)] - 30f(x-y) + 480f(y) \tag{2.28}$$

for all $x, y \in X$. Adding (2.8) and (2.28), we get

$$\begin{aligned} &f(x+5y) + f(x-3y) + f(x-5y) + f(x+3y) \\ &= 16\,[f(x+2y) + f(x-2y)] + 32f(x) - 30\,[f(x+y) + f(x-y)] + 960f(y) \end{aligned} \tag{2.29}$$

for all $x, y \in X$. Replacing x by $4x$ in (2.29), we get

$$\begin{aligned} &f(4x+5y) + f(4x-3y) + f(4x-5y) + f(4x+3y) \\ &= 16\,[f(4x+2y) + f(4x-2y)] + 32f(4x) - 30\,[f(4x+y) + f(4x-y)] + 960f(y) \end{aligned} \tag{2.30}$$

for all $x, y \in X$. Replacing y by $x+y$ in (2.25), we obtain

$$f(5x+4y) + f(-3x-4y) = 16\,[f(2x+y) - f(y)] - 30f(x) + 480f(x+y) \tag{2.31}$$

for all $x, y \in X$. Replacing y by $x-y$ in (2.25), we have

$$f(5x-4y) + f(-3x+4y) = 16\,[f(2x-y) + f(y)] - 30f(x) + 480f(x-y) \tag{2.32}$$

for all $x, y \in X$. Adding (2.31) and (2.32), we obtain

$$\begin{aligned} &f(5x+4y) + f(-3x-4y) + f(5x-4y) + f(-3x+4y) \\ &= 16\,[f(2x+y) + f(2x-y)] + 32f(y) - 60f(x) + 480\,[f(x+y) + f(x-y)] \end{aligned} \tag{2.33}$$

for all $x, y \in X$. Interchanging x by y in (2.33) we get

$$\begin{aligned} &f(4x+5y) + f(4x+3y) + f(4x-5y) + f(4x-3y) \\ &= 16\,[f(x+2y) + f(x-2y)] + 32f(y) - 60f(x) + 480\,[f(x+y) + f(x-y)] \end{aligned} \tag{2.34}$$

for all $x, y \in X$. It follows from (2.30) and (2.34) that

$$\begin{aligned} &16\,[f(4x+2y) + f(4x-2y)] + 32f(4x) - 60\,[f(4x+y) + f(4x-y)] + 960f(y) \\ &= 16\,[f(x+2y) + f(x-2y)] + 32f(y) - 60f(x) + 480\,[f(x+y) + f(x-y)] \end{aligned} \tag{2.35}$$

for all $x, y \in X$. Simplifying (2.35), we have

$$\begin{aligned} &f(4x+2y) + f(4x-2y) - 60\,[f(x+y) + f(x-y)] - [f(x+2y) + f(x-2y)] \\ &= 774f(x) - 120f(y) \end{aligned} \tag{2.36}$$

for all $x, y \in X$. Interchanging x by y in (2.26), we have

$$f(4x+2y) - f(4x-2y) = 16\,[f(x+2y) - f(x-2y)] - 480f(y) + 480f(x) \tag{2.37}$$

for all $x, y \in X$. It follows from (2.37) and (2.36) that

$$\begin{aligned} &16\,[f(x+2y)-f(x-2y)]-480f(y)+480f(x)-60\,[f(x+y)+f(x-y)]\\ &\quad-[f(x+2y)-f(x-2y)]=774f(x)-120f(y) \end{aligned} \tag{2.38}$$

for all $x, y \in X$. Simplifying (2.38), we get

$$15\,[f(x+2y)+f(x-2y)]-60\,[f(x+y)+f(x-y)]=-90f(x)+360f(y) \tag{2.39}$$

for all $x, y \in X$. Dividing (2.39) by 15, we get the required equation (1.3). □

3. Stability of the cubic functional equation (1.2)

In this section, assume that G is an additive group and X is a complete non-Archimedean normed space. Now before taking up the main subject, for a given mapping $f : G \to X$, we define the difference operator

$$Df(x,y)=f(x+4y)+f(x-4y)-16\,[f(x+y)+f(x-y)]+30f(-x)-\frac{5}{2}\,[f(4y)-64f(y)]$$

for all $x, y \in G$. We consider the following function inequality

$$\|Df(x,y)\| \le \varphi(x,y)$$

for an upper bound $\varphi : G \times G \to [0,\infty)$.

Theorem 3.1. *Let $\varphi : G \times G \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty}\frac{\varphi(4^n x, 4^n y)}{|4|^{3n}}=0 \tag{3.40}$$

for all $x, y \in G$ and let for each $x \in G$ the following limit exists

$$\lim_{n\to\infty}\max\left\{\frac{\varphi(0,4^j x)}{|4|^{3j}} : 0 \le j < n\right\}, \tag{3.41}$$

which is denoted by $\varphi_C(x)$. Suppose that $f : G \to X$ is an odd mapping satisfying

$$\|Df(x,y)\| \le \varphi(x,y) \tag{3.42}$$

for all $x, y \in G$. Then there exists a cubic mapping $C : G \to X$ such that

$$\|C(x)-f(x)\| \le \frac{1}{|4|^3}\varphi_C(x) \tag{3.43}$$

for all $x \in G$, and if, in addition,

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(0,4^j x)}{|4|^{3j}} : i \le j < n+i\right\}=0$$

then C is the unique cubic mapping satisfying (3.43).

Proof. Replacing (x, y) by $(0, x)$ in (3.42), we get

$$\|f(4x)-64f(x)\| \le \varphi(0,x) \tag{3.44}$$

for all $x \in G$. It follows from (3.44) that

$$\|\frac{f(4x)}{4^3}-f(x)\| \le \frac{\varphi(0,x)}{4^3} \tag{3.45}$$

for all $x \in G$. Replacing x by $2^{2(n-1)}x$ in (3.45), we get

$$\|\frac{1}{4^{3n}}f(4^n x)-\frac{1}{4^{3(n-1)}}f(4^{n-1}x)\| \le \frac{\varphi(0,4^{n-1}x)}{|4^{3n}|} \tag{3.46}$$

for all $x \in G$. It follows from (3.46) and (3.40) that the sequence $\left\{\frac{f(4^n x)}{4^{3n}}\right\}$ is Cauchy. Since X is complete, we conclude that $\left\{\frac{f(4^n x)}{4^{3n}}\right\}$ is convergent. Set $C(x) := \lim_{n\to\infty} \frac{f(4^n x)}{4^{3n}}$. Using induction, one can show that

$$\left\|\frac{f(4^n x)}{4^{3n}} - f(x)\right\| \le \frac{1}{|4^3|} \max\left\{\frac{\varphi(0, 4^i x)}{|4|^{3i}} : 0 \le i < n\right\} \tag{3.47}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (3.47) and using (3.41) one obtains (3.43). By (3.40) and (3.42), we get

$$\|DC(x,y)\| = \lim_{n\to\infty} \frac{1}{|4^{3n}|}\|f(4^n x, 4^n y)\| \le \lim_{n\to\infty} \frac{\varphi(4^n x, 4^n y)}{|4|^{3n}} = 0$$

for all $x, y \in G$. Therefore the mapping $C : G \to X$ satisfies (1.2).

To prove the uniqueness property of C, let D be another cubic mapping satisfying (3.43). Then

$$\begin{aligned}\|C(x) - D(x)\| &= \lim_{i\to\infty} |4|^{-3i}\|C(4^i x) - D(4^i x)\| \\ &\le \lim_{i\to\infty} |4|^{-3i} \max\left\{\|C(4^i x) - f(4^i x)\|, \|f(4^i x) - D(4^i x)\|\right\} \\ &\le \frac{1}{|4|^3} \lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(0, 4^j x)}{|4|^{3j}} : i \le j < n+i\right\}\end{aligned}$$

for all $x \in G$. If

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(0, 4^j x)}{|4|^{3j}} : i \le j < n+i\right\} = 0,$$

then $C = D$, and the proof is complete. □

4. Stability of the quartic functional equation (1.4)

In this section, assume that G is an additive group and X is a complete non-Archimedean normed space. Now before taking up the main subject, for a given mapping $f : G \to X$, we define the difference operator

$$Df(x,y) = f(x+4y) + f(x-4y) - 16\,[f(x+y) + f(x-y)] - 30f(-x) - \frac{5}{2}\,[f(4y) - 64f(y)]$$

for all $x, y \in G$. We consider the following function inequality

$$\|Df(x,y)\| \le \varphi(x,y)$$

for an upper bound $\varphi : G \times G \to [0, \infty)$.

Theorem 4.1. *Let $\varphi : G \times G \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(4^n x, 4^n y)}{|4|^{4n}} = 0 \tag{4.48}$$

for all $x, y \in G$ and let for each $x \in G$ the following limit exists

$$\lim_{n\to\infty} \max\left\{\frac{\varphi(0, 4^j x)}{|4|^{4j}} : 0 \le j < n\right\}, \tag{4.49}$$

which is denoted by $\varphi_Q(x)$. Suppose that $f : G \to X$ is an even mapping satisfying

$$\|Df(x,y)\| \le \varphi(x,y) \tag{4.50}$$

for all $x, y \in G$. Then there exists a quartic mapping $Q : G \to X$ such that

$$\|Q(x) - f(x)\| \le \frac{1}{|4|^4}\varphi_Q(x) \tag{4.51}$$

for all $x \in G$, and if, in addition,

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(0,4^j x)}{|4|^{4j}} : i \le j < n+i\right\} = 0$$

then Q is the unique quartic mapping satisfying (4.51).

Proof. Replacing (x, y) by $(0, x)$ in (4.50), we get

$$\|f(4x) - 256f(x)\| \le \varphi(0, x) \tag{4.52}$$

for all $x \in G$. It follows from (4.52) that

$$\|\frac{f(4x)}{4^4} - f(x)\| \le \frac{\varphi(0,x)}{4^4} \tag{4.53}$$

for all $x \in G$. Replacing x by $2^{n-1}x$ in (4.53), we get

$$\left\|\frac{1}{4^{4n}}f(4^n x) - \frac{1}{4^{4(n-1)}}f(4^{n-1}x)\right\| \le \frac{\varphi(0,4^{n-1}x)}{|4^{4n}|} \tag{4.54}$$

for all $x \in G$. It follows from (4.54) and (4.48) that the sequence $\left\{\frac{f(4^n x)}{4^{4n}}\right\}$ is Cauchy. Since X is complete, we conclude that $\left\{\frac{f(4^n x)}{4^{4n}}\right\}$ is convergent. Set $Q(x) := \lim_{n\to\infty}\frac{f(4^n x)}{4^{4n}}$. Using induction, one can show that

$$\|\frac{f(4^n x)}{4^{4n}} - f(x)\| \le \frac{1}{|4^4|}\max\left\{\frac{\varphi(0,4^i x)}{|4|^{4i}} : 0 \le i < n\right\} \tag{4.55}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (4.55) and using (4.49) one obtains (4.51). By (4.48) and (4.50), we get

$$\|DQ(x,y)\| = \lim_{n\to\infty}\frac{1}{|4^{4n}|}\|f(4^n x, 4^n y)\| \le \lim_{n\to\infty}\frac{\varphi(4^n x, 4^n y)}{|4|^{4n}} = 0$$

for all $x, y \in G$. Since f is even, we can easily show that Q is even. Thus the mapping $Q : G \to X$ satisfies (1.4). To prove the uniqueness property of Q, let R be another quartic mapping satisfying (4.51). Then

$$\begin{aligned}\|Q(x) - R(x)\| &= \lim_{i\to\infty}|4|^{-4i}\|Q(4^i x) - R(4^i x)\| \\ &\le \lim_{i\to\infty}|4|^{-4i}\max\left\{\|Q(4^i x) - f(4^i x)\|, \|f(4^i x) - R(4^i x)\|\right\} \\ &\le \frac{1}{|4|^4}\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(0,4^j x)}{|4|^{4j}} : i \le j < n+i\right\}\end{aligned}$$

for all $x \in G$. If

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(0,4^j x)}{|4|^{4j}} : i \le j < n+i\right\} = 0,$$

then $Q = R$, and the proof is complete. □

5. Stability of the cubic-quartic functional equation (1.4)

In this section, assume that G is an additive group and X is a complete non-Archimedean normed space. Now before taking up the main subject, for a given mapping $f : G \to X$, we define the difference operator

$$Df(x,y) = f(x+4y)+f(x-4y)-16\left[f(x+y)+f(x-y)\right]\pm 30f(-x)-\frac{5}{2}\left[f(4y)-64f(y)\right]$$

for all $x, y \in G$. We consider the following function inequality

$$\|Df(x,y)\| \le \varphi(x,y)$$

for an upper bound $\varphi : G \times G \to [0,\infty)$.

THEOREM 5.1. *Let $\varphi : G \times G \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(4^n x, 4^n y)}{|4|^{3n}} = \lim_{n\to\infty} \frac{\varphi(4^n x, 4^n y)}{|4|^{4n}} = 0$$

for all $x, y \in G$ and let for each $x \in G$ the following limits exist

$$\lim_{n\to\infty} \max\left\{\frac{\varphi(0,4^j x)}{|4|^{3j}} : 0 \le j < n\right\}, \text{ and } \lim_{n\to\infty} \max\left\{\frac{\varphi(0,4^j x)}{|4|^{4j}} : 0 \le j < n\right\}$$

denoted by $\varphi_C(x)$ and denoted by $\varphi_Q(x)$, respectively. Suppose that $f : G \to X$ is a mapping satisfying

$$\|Df(x,y)\| \le \varphi(x,y)$$

for all $x, y \in G$. Then there exist a cubic mapping $C : G \to X$ and a quartic mapping $Q : G \to X$ such that

$$\begin{aligned} &\|f(x)-C(x)-Q(x)\| \\ &\le \max\left\{\frac{1}{|2||4|^3}\max\{\varphi_C(x),\varphi_C(-x)\}, \frac{1}{|2||4|^4}\max\{\varphi_Q(x),\varphi_Q(-x)\}\right\} \end{aligned} \tag{5.56}$$

for all $x \in G$, and if, in addition,

$$\begin{aligned} \lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\varphi(0,4^j x)}{|4|^{3j}} : i \le j < n+i\right\} &= \lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\varphi(0,4^j x)}{|4|^{4j}} : i \le j < n+i\right\} \\ &= 0, \end{aligned}$$

then C is the unique cubic mapping and Q is the unique quartic mapping.

Proof. Let $f_0(x) = \frac{1}{2}\left[f(x)-f(-x)\right]$ for all $x \in G$. Then $f_0(0) = 0, f_0(-x) = -f_0(x)$, and

$$\|Df_0(x,y)\| \le \frac{1}{|2|}\max\{\varphi(x,y),\varphi(-x,-y)\}$$

for all $x, y \in G$. From Theorem 3.1, it follows that there exists a unique cubic mapping $C : G \to X$ satisfying

$$\|f_0(x)-C(x)\| \le \frac{1}{|2||4|^3}\max\{\varphi_C(x),\varphi_C(-x)\}$$

for all $x \in G$. Let $f_e(x) = \frac{1}{2}\left[f(x)-f(-x)\right]$ for all $x \in G$. Then $f_e(0) = 0, f_e(-x) = f_e(x)$, and

$$\|Df_e(x,y)\| \le \frac{1}{2}\max\{\varphi(x,y),\varphi(-x,-y)\}$$

for all $x, y \in G$. From Theorem 4.1, it follows that there exists a unique quartic mapping $Q : G \to X$ satisfying

$$\|f_e(x) - Q(x)\| \leq \frac{1}{|2||4|^4} \max\{\varphi_Q(x), \varphi_Q(-x)\}$$

for all $x \in G$. Thus we get the desired inequality (5.56) □

6. Conclusion

In this paper, we have introduced the cubic-quartic functional equation (1.4) and we have investigated the general solution and have proved the Ulam stability for the functional equation (1.4) in non-Archimedean spaces by using the direct method.

Declarations

Availablity of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Fundings

Not applicable.

Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

References

[1] M. Eshaghi Gordji, R. Khodabakhsh, S. Jung and H. Khodaei, *AQCQ-Functional equation in non-Archimedean normed spaces*, Abstr.. Appl. Anal. **2010** (2010), Article ID 741942.

[2] M. Eshaghi Gordji, H. Khodaei and R. Khodabakhsh, *General quartic-cubic-quadratic functional equation in non-Archimedean normed spaces*, UPB Sci. Bull. Ser. A, **72** (2010), no. 3, 69–84.

[3] M. Eshaghi Gorgji and M. B. Savadkouhi, *Stability of cubic and quartic functional equations in non-Archimedean spaces,* Acta Appl. Math. **110** (2010), 1321–1329.

[4] M. Eshaghi Gordji and M. B. Savadkouhi, *Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces,* Appl. Math. Lett. **23** (2010), 1198–1202.

[5] A. Gilányi, *On a problem by K. Nikodem*, Math. Inequal. Appl. **5** (2002), 707–710.

[6] V. Govindan, S. Murthy and M. Saravanan, *Solution and stability of new type of (aaq,bbq,caq,daq) mixed type functional equation in various normed spaces: Using two different methods,* Int. J. Math. Appl. **5** (2017), 187–211.

[7] K. Jun and H. Kim, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl. **274** (2002), 267–278.

[8] S. Jung, D. Popa and M. Th. Rassias, *On the stability of the linear functional equation in a single variable on complete metric spaces*, J. Global Optim. **59** (2014), 13–16.

[9] Y. Lee, S. Jung and M. Th. Rassias, *Uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation*, J. Math. Inequal. **12** (2018), 43–61.

[10] D. Mihet, *The stability of the additive Cauchy functional equation in non- Archimedean fuzzy normed spaces*, Fuzzy Sets Syst. **161** (2010), 2206–2212.

[11] R. Murali, S. Pinelas and V. Vithya, *The stability of viginti unus functional equation in various spaces*, Global J. Pure Appl. Math. **13** (2017), 5735–5759.

[12] S. Murthy, V. Govindhan, *General solution and generalized HU (Hyers-Ulam) statbility of new dimension cubic functional equation in fuzzy ternary Banach algebras: Using two different methods*, Int. J. Pure Appl. Math. **113** (2017), no. 6, 394–403.

[13] P. Narasimman, K. Ravi and S. Pinelas, *Stability of Pythagorean mean functional equation,* Global J. Math. **4** (2015), 398–411.

[14] C. Park, *Additive ρ-functional inequalities and equations*, J. Math. Inequal. **9** (2015), 17–26.

[15] C. Park, *Additive ρ-functional inequalities in non-Archimedean normed spaces*, J. Math. Inequal. **9** (2015), 397–407.

[16] S. Pinelas, V. Govindan and K. Tamilvanan, *Stability of non-additive functional equation*, IOSR J. Math. **14** (2018), no. 2, 70–78.

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