

INEQUALITIES FOR B-OPERATOR

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ABSTRACT. Let \mathcal{P}_n denote the space of all complex polynomials $P(z) = \sum_{j=0}^n a_j z^j$ of degree n . Let $P \in \mathcal{P}_n$, for any complex number α , $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$, denote the polar derivative of the polynomial $P(z)$ with respect to α and B_n denote a family of operators that maps \mathcal{P}_n into itself. In this paper, we combine the operators B and D_α and establish certain operator preserving inequalities concerning polynomials, from which a variety of interesting results can be obtained as special cases.

1. Introduction

Let \mathcal{P}_n denotes the space of all complex polynomials $p(z) = \sum_{j=0}^n a_j z^j$ of degree at most n . We write $T = \{z \in \mathbb{C} : |z| = 1\}$, $D_- = \{z \in \mathbb{C} : |z| < 1\}$ and $D_+ = \{z \in \mathbb{C} : |z| > 1\}$. If $p \in \mathcal{P}_n$, then

$$(1) \quad \max_{z \in T} |p'(z)| \leq n \max_{z \in T} |p(z)|.$$

Equality holds in (1) for the polynomial $p(z) = \zeta z^n$ where $\zeta \in \mathbb{C}$. Inequality (1) is an immediate consequence of S. Bernstein's Theorem (see [4], [14], [16]) on the derivative of a trigonometric polynomial. For the class of polynomials $P \in \mathcal{P}_n$ which do not vanish in D_- , we have

$$(2) \quad \max_{z \in T} |p'(z)| \leq \frac{n}{2} \max_{z \in T} |p(z)|.$$

Equality in (2) holds for $p(z) = \eta z^n + \zeta$; $|\eta| = |\zeta| = 1$. Inequality (2) was conjectured by P. Erdős and later verified by P. D. Lax [11]. Aziz and Dawood [2] used $\min_{|z|=1} |p(z)|$ to obtain refinement of inequality (2) by proving that if $p \in \mathcal{P}_n$ and $p(z) \neq 0$ for $z \in D_-$, then

$$(3) \quad \max_{z \in T} |p'(z)| \leq \frac{n}{2} \left\{ \max_{z \in T} |p(z)| - \min_{z \in T} |p(z)| \right\}.$$

Equality holds in (3) for the polynomial $p(z) = \eta z^n + \zeta$; $|\eta| = |\zeta| = 1$.

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Aziz was among the first to extend these results by replacing the derivative with the polar derivative of the polynomial. For a complex number α and for $p \in \mathcal{P}_n$, let

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

$D_\alpha p(z)$ is a polynomial of degree at most $n - 1$. This is the so-called polar derivative of $p(z)$ with respect to point α [14]. It generalizes the ordinary derivative in the sense that:

$$\lim_{\alpha \rightarrow \infty} \frac{D_\alpha p(z)}{\alpha} = p'(z).$$

Now corresponding to a given n^{th} degree polynomial $p(z)$, we construct a sequence of polar derivatives

$$\begin{aligned} D_\alpha p(z) &= np(z) + (\alpha - z)p'(z) \\ D_{\alpha_k} D_{\alpha_{k-1}} \dots D_{\alpha_1} p(z) &= (n - k + 1)D_{\alpha_{k-1}} \dots D_{\alpha_1} p(z) \\ &\quad + (\alpha_k - z)(D_{\alpha_{k-1}} \dots D_{\alpha_1} p(z))' \quad \text{for } k = 2, 3, \dots, n. \end{aligned}$$

The points $\alpha_1, \alpha_2, \dots, \alpha_k, k = 1, 2, \dots, n$, may be equal or unequal. Like the k^{th} ordinary derivative $p^{(k)}(z)$ of $p(z)$, the k^{th} polar derivative $D_{\alpha_k} D_{\alpha_{k-1}} \dots D_{\alpha_1} p(z)$ of $p(z)$ is a polynomial of degree $n - k$. For $p_j(z) = D_{\alpha_j} D_{\alpha_{j-1}} \dots D_{\alpha_1} p(z)$, we have

$$(4) \quad \begin{aligned} p_j(z) &= (n - j + 1)p_{j-1} + (\alpha_j - z)p'_{j-1}(z), \quad j = 1, 2, \dots, k, \\ p_0(z) &= p(z). \end{aligned}$$

As an extension of (1) for the polar derivative Aziz and Shah [3] proved that

$$(5) \quad |D_\alpha p(z)| \leq |\alpha z^{n-1}| \max_{z \in T} |p(z)| \quad \text{for } z \in T \cup D_+.$$

Aziz [1] extended (5) to the j^{th} polar derivative and proved the following theorem

THEOREM 1.1. *If $p(z)$ is a polynomial of degree n such that $\alpha_1, \alpha_2, \dots, \alpha_t, (t < n)$, are complex numbers with $|\alpha_i| \geq 1$ for all $i = 1, 2, \dots, t$ then for $|z| \geq 1$,*

$$|p_t(z)| \leq n(n - 1) \dots (n - t + 1) |\alpha_1 \alpha_2 \dots \alpha_t| |z|^{n-t} \max_{|z|=1} |p(z)|.$$

In the literature [8], [9], there exists many generalisations and refinements of (5).

Let \mathbb{T} be a linear operator from \mathcal{P}_n into \mathcal{P}_n . We shall say that \mathbb{T} is a B_n -operator if, for every polynomial f of degree n having all its zeros in the closed unit disc, $\mathbb{T}[f]$ has all its zeros in the closed unit disc.

It is interesting to mention that Professor Q. I. Rahman has pointed out to characterize all such operators. As an attempt to this characterization, it was proved [16] that the operator B which carries a polynomial $p(z)$ into the polynomial

$$B[p](z) := \lambda_0 p(z) + \lambda_1 \left(\frac{nz}{2}\right) \frac{p'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{p''(z)}{2!},$$

is a B_n -operator if all the zeros of

$$(6) \quad u(z) := \lambda_0 + n\lambda_1 z + \frac{n(n-1)}{2} \lambda_2 z^2,$$

lie in the half plane

$$(7) \quad |z| \leq \left| z - \frac{n}{2} \right|.$$

where λ_0, λ_1 and λ_2 are real or complex numbers. As an extension of Bernstein's inequality, it was observed by Rahman [15], that if $|p(z)| \leq M$ for $z \in T$, then

$$|B[p(z)]| \leq M|B[z^n]|, \quad z \in T \cup D_+.$$

Bidkham and Mezerji [6] have generalized some of the above inequalities by combining B and D_α operators and proved that if $p \in \mathcal{P}_n$ does not vanish in D_- , then for every real or complex number α with $|\alpha| \geq 1$ and for $z \in T \cup D_+$:

$$(8) \quad |B[D_\alpha p(z)]| \leq \frac{n}{2}[\{|\alpha||B[z^{n-1}]| + |\lambda_0|\}M - \{|\alpha||B[z^{n-1}]| - |\lambda_0|\}m].$$

2. Main Results

In this paper, we combine the two operators B and D_α and obtain an improvement and generalisations of the above inequalities.

THEOREM 2.1. *If $p(z)$ is a polynomial of degree at most n having all zeros in $|z| \leq 1$, then for every $\alpha_1, \alpha_2, \dots, \alpha_t$ with $|\alpha_i| \geq 1, \quad i = 1, 2, \dots, t(t < n)$ and $z \in T \cup D_+$*

$$(9) \quad |B[p_t(z)]| \geq n_t|\alpha_1||\alpha_2|\dots|\alpha_t||B[z^{n-t}]| \min_{z \in T} |p(z)|$$

where $n_t = n(n-1)\dots(n-t+1)$.

Substituting the value of $B[p_t(z)]$ and $B[z^{n-t}]$ in (9) we have for $z \in T \cup D_+$

$$\begin{aligned} & \left| \lambda_0 p_t(z) + \lambda_1 \frac{nz}{2} (p_t(z))' + \frac{\lambda_2}{2!} \left(\frac{nz}{2}\right)^2 (p_t(z))'' \right| \\ & \geq n_t |\alpha_1| |\alpha_2| \dots |\alpha_t| \left| \lambda_0 z^{n-t} + \lambda_1 \frac{nz}{2} (n-t) z^{n-t-1} + \frac{\lambda_2}{2!} \left(\frac{nz}{2}\right)^2 (z-t)(z-t-1) z^{n-t-2} \right| \\ & \quad \times \min_{z \in T} |p(z)| \end{aligned}$$

where $\lambda_0, \lambda_1, \lambda_2$ are such that all the zeros of $u(z)$ defined by (6) lie in the half plane $|z| \leq |z - \frac{n}{2}|$. If we choose $\lambda_1 = 0 = \lambda_2$ in (9), we get the following result:

COROLLARY 2.2. *If $p(z)$ is a polynomial of degree at most n having all zeros in $|z| \leq 1$, then for every $\alpha_1, \alpha_2, \dots, \alpha_t$ with $|\alpha_i| \geq 1, \quad i = 1, 2, \dots, t(t < n)$ and $z \in T \cup D_+$*

$$|p_t(z)| \geq n_t |\alpha_1| |\alpha_2| \dots |\alpha_t| |z^{n-t}| \min_{z \in T} |p(z)|$$

where $n_t = n(n-1)\dots(n-t+1)$.

Let $\alpha_1 = \alpha_2 = \dots = \alpha_t = \alpha$. Dividing both sides of (9) by $|\alpha|^t$ and letting $|\alpha| \rightarrow \infty$ we get the following result:

COROLLARY 2.3. *If $p(z)$ is a polynomial of degree at most n having all zeros in $|z| \leq 1$, then for $z \in T \cup D_+$*

$$|B[p^{(t)}(z)]| \geq n_t |B[z^{n-t}]| \min_{z \in T} |p(z)|$$

where $n_t = n(n-1)\dots(n-t+1)$.

THEOREM 2.4. *If $p(z)$ is a polynomial of degree at most n having no zeros in $|z| < 1$, then for every $\alpha_1, \alpha_2, \dots, \alpha_t$ with $|\alpha_i| \geq 1, \quad i = 1, 2, \dots, t(t < n)$ and $z \in T \cup D_+$*

$$(10) \quad \begin{aligned} |B[p_t(z)]| \leq & \frac{n_t}{2} \left\{ \prod_{i=1}^t |\alpha_i| |B[z^{n-t}]| + |\lambda_0| \right\} \max_{z \in T} |p(z)| \\ & - \frac{n_t}{2} \left\{ \prod_{i=1}^t |\alpha_i| |B[z^{n-t}]| - |\lambda_0| \right\} \min_{z \in T} |p(z)| \end{aligned}$$

where $n_t = n(n - 1) \dots (n - t + 1)$.

REMARK 2.5. For $t = 1$ we get inequality (8).

Let $\alpha_1 = \alpha_2 = \dots = \alpha_t = \alpha$. Dividing both sides of (9) by $|\alpha|^t$ and letting $|\alpha| \rightarrow \infty$ we get the following result:

COROLLARY 2.6. *If $p(z)$ is a polynomial of degree at most n having no zeros in $|z| < 1$ and $z \in T \cup D_+$*

$$(11) \quad |B[p^{(t)}(z)]| \leq \frac{n_t}{2} \left\{ |B[z^{n-t}]| \max_{z \in T} |p(z)| - |B[z^{n-t}]| \min_{z \in T} |p(z)| \right\}$$

where $n_t = n(n - 1) \dots (n - t + 1)$.

Substituting the value of $B[p_t(z)]$ and $B[z^{n-t}]$ in (10) we have for $z \in T \cup D_+$

$$\begin{aligned} & \left| \lambda_0 p_t(z) + \lambda_1 \frac{nz}{2} (p_t(z))' + \frac{\lambda_2}{2!} \left(\frac{nz}{2}\right)^2 (p_t(z))'' \right| \\ & \leq \frac{n_t}{2} \left\{ \prod_{i=1}^t |\alpha_i| \left| \lambda_0 z^{n-t} + \lambda_1 \frac{nz}{2} (n-t) z^{n-t-1} + \frac{\lambda_2}{2!} \left(\frac{nz}{2}\right)^2 (z-t)(z-t-1) z^{n-t-2} \right| + |\lambda_0| \right\} \\ & \quad \times \max_{z \in T} |p(z)| \\ & \quad - \frac{n_t}{2} \left\{ \prod_{i=1}^t |\alpha_i| \left| \lambda_0 z^{n-t} + \lambda_1 \frac{nz}{2} (n-t) z^{n-t-1} + \frac{\lambda_2}{2!} \left(\frac{nz}{2}\right)^2 (z-t)(z-t-1) z^{n-t-2} \right| - |\lambda_0| \right\} \\ & \quad \times \min_{z \in T} |p(z)| \end{aligned}$$

where $\lambda_0, \lambda_1, \lambda_2$ are such that all the zeros of $u(z)$ defined by (6) lie in the half plane $|z| \leq |z - \frac{n}{2}|$. If we choose $\lambda_1 = 0 = \lambda_2$ in (10), we get the following result in terms of t^{th} polar derivative:

COROLLARY 2.7. *If $p(z)$ is a polynomial of degree at most n having no zeros in $|z| < 1$, then for every $\alpha_1, \alpha_2, \dots, \alpha_t$ with $|\alpha_i| \geq 1, \quad i = 1, 2, \dots, t(t < n)$ and $z \in T \cup D_+$*

$$|[p_t(z)]| \leq \frac{n_t}{2} \left\{ \left(\prod_{i=1}^t |\alpha_i| |z^{n-t}| + 1 \right) \max_{z \in T} |p(z)| - \left(\prod_{i=1}^t |\alpha_i| |z^{n-t}| - 1 \right) \min_{z \in T} |p(z)| \right\}$$

where $n_t = n(n - 1) \dots (n - t + 1)$.

Substituting the value of $B[p_t(z)]$ and $B[z^{n-t}]$ in (11) and choosing $|\lambda_1| = |\lambda_2| = 0$ we get the result in terms of t^{th} derivative as follows:

COROLLARY 2.8. *If $p(z)$ is a polynomial of degree at most n having no zeros in $|z| < 1$, then for $z \in T \cup D_+$*

$$(12) \quad |[p^{(t)}(z)]| \leq \frac{n_t}{2} \left\{ |[z^{n-t}]| \max_{z \in T} |p(z)| - |[z^{n-t}]| \min_{z \in T} |p(z)| \right\}$$

where $n_t = n(n - 1) \dots (n - t + 1)$.

3. Lemmas

For the proof of these theorems, we need the following Lemmas. The first Lemma is due to Laguerre [10].

LEMMA 3.1. *If all the zeros of an n^{th} degree polynomial $p(z)$ lie in a circular region C and if none of the points $\alpha_1, \alpha_2, \dots, \alpha_t, (t < n)$ lie in the region C then each of the polar derivatives $p_1(z), p_2(z), \dots, p_t(z)$, (defined by (4)) has all its zeros in region C .*

The following Lemma which we need is in fact implicit in [16] (Rahman and Schmeisser, 2002, Lemma 14.5.7, p. 540)

LEMMA 3.2. *If all the zeros of polynomial $p(z)$ of degree n lie in $|z| \leq 1$, then all the zeros of the polynomial $B[p(z)]$ also lie in $|z| \leq 1$.*

LEMMA 3.3. *If all the zeros of the polynomial $p(z)$ lie in $|z| \leq 1$, then for every $\alpha_1, \alpha_2, \dots, \alpha_t$ with $|\alpha_i| \geq 1, i = 1, 2, \dots, t (t < n)$, the polynomial $B[p_t(z)]$ also has zeros in $|z| \leq 1$.*

Proof. Since all the zeros of $p(z)$ lie in $|z| \leq 1$, then by Lemma 3.1, for every $\alpha_1, \alpha_2, \dots, \alpha_t$ with $|\alpha_i| \geq 1, i = 1, 2, \dots, t (t < n)$, the polynomial $p_t(z)$ has all its zeros in $|z| \leq 1$. Hence by Lemma 3.2, the polynomial $B[p_t(z)]$ has all its zeros in $|z| \leq 1$. □

LEMMA 3.4. *If the polynomial $p(z)$ of degree at most n has no zeros in $|z| < 1$, then for every $\alpha_1, \alpha_2, \dots, \alpha_t$ with $|\alpha_i| \geq 1, i = 1, 2, \dots, t (t < n)$,*

$$|B[p_t(z)]| \leq |B[q_t(z)]| \quad \text{for } |z| \geq 1$$

where $q(z) = z^n \overline{p(\frac{1}{z})}$.

Proof. Since $p(z)$ does not vanish in D_- , so $q(z)$ has all zeros in D_- . Also $|p(z)| = |q(z)|$ for $z \in T$. By *Rouche's Theorem* the polynomial $p(z) - \gamma q(z)$ has all zeros in D_- for every γ with $|\gamma| > 1$. By Lemma 3.3, the polynomial $B[p_t(z) - \gamma q_t(z)] = B[p_t(z)] - \gamma B[q_t(z)]$ has all zeros in D_- . This gives

$$|B[p_t(z)]| \leq |B[q_t(z)]| \quad \text{for } |z| \geq 1.$$

□

LEMMA 3.5. *If $p(z)$ is a polynomial of degree at most n , then for every $\alpha_1, \alpha_2, \dots, \alpha_t$ with $|\alpha_i| \geq 1, i = 1, 2, \dots, t (t < n)$,*

$$|B[p_t(z)]| + |B[q_t(z)]| \leq n_t \left\{ \prod_{i=1}^t |\alpha_i| |B[z^{n-t}]| + |\lambda_0| \right\} \max_{|z|=1} |p(z)|$$

where n_t is defined in Theorem 2.1.

Proof. Let $M = \max_{|z|=1} |p(z)|$. By *Rouche's Theorem* the polynomial $g(z) = p(z) - \beta M$ does not vanish in $|z| \leq 1$. Applying Lemma 3.4 to the polynomial $g(z)$ we have for $z \in T \cup D_+$

$$|B[g_t(z)]| \leq |B[h_t(z)]|$$

where $h(z) = z^n \overline{g\left(\frac{1}{z}\right)} = q(z) - \bar{\beta} M z^n$. Substitute for $g_t(z)$ and $h_t(z)$ we get

$$|B[p_t(z) - n_t \beta M]| \leq |B[q_t(z) - n_t \bar{\beta} M \alpha_1 \alpha_2 \dots \alpha_t z^{n-t}]|$$

Equivalently,

$$(13) \quad |B[p_t(z)] - n_t \beta M \lambda_0| \leq |B[q_t(z)] - n_t \bar{\beta} M \alpha_1 \alpha_2 \dots \alpha_t B[z^{n-t}]|$$

Since $h(z) = q(z) - \bar{\beta} M z^n$ has all zeros in $|z| < 1$, so by Lemma 3.3 the polynomial $B[h_t(z)] = B[q_t(z)] - n_t \bar{\beta} M \alpha_1 \alpha_2 \dots \alpha_t B[z^{n-t}]$ has all zeros in $|z| < 1$ which gives for $|z| \geq 1$

$$(14) \quad |B[q_t(z)]| < n_t M |\alpha_1| |\alpha_2| \dots |\alpha_t| |B[z^{n-t}]|.$$

Choosing argument of β in right hand side of (13) which is possible by (14), we get for $z \in T \cup D_+$

$$(15) \quad |B[p_t(z)]| - n_t |\beta| M |\lambda_0| \leq n_t |\beta| M |\alpha_1| |\alpha_2| \dots |\alpha_t| |B[z^{n-t}]| - B[q_t(z)]$$

Letting $\beta \rightarrow 1$ in (15) we get

$$|B[p_t(z)]| + B[q_t(z)] \leq n_t \left\{ \prod_{i=1}^t |\alpha_i| |B[z^{n-t}]| + |\lambda_0| \right\} \max_{|z|=1} |p(z)|.$$

□

4. Proofs of the Theorems

Proof of Theorem 2.1: If $p(z)$ has no zero on $|z| = 1$, then there is nothing to prove. Suppose that all the zeros of $p(z)$ lie in D_- , then $\min_{z \in T} |p(z)| = m > 0$, so we have $m \leq |p(z)|$ for $z \in T$. It follows by *Rouche's Theorem* that all the zeros of $f(z) = p(z) - m \zeta z^n$ lie in D_- with $|\zeta| < 1$. Therefore by Lemma 3.3 all the zeros of $B[f_t(z)] = B[p_t(z)] - mn_t \zeta \alpha_1 \alpha_2 \dots \alpha_t z^{n-t}$ also lie in D_- which gives for $z \in T \cup D_+$

$$(16) \quad |B[p_t(z)]| \geq mn_t |\alpha_1| |\alpha_2| \dots |\alpha_t| |B[z^{n-t}]|.$$

If this is not true then there exist z_0 with $|z_0| \geq 1$ such that

$$(17) \quad |B[p_t(z_0)]| < mn_t |\alpha_1| |\alpha_2| \dots |\alpha_t| |B[z_0^{n-t}]|.$$

Take $\zeta = \frac{|B[p_t(z_0)]|}{mn_t |\alpha_1| |\alpha_2| \dots |\alpha_t| |B[z_0^{n-t}]|}$ so that $|\zeta| < 1$ by (17). With this choice of ζ , $B[f_t(z_0)] = 0$ which is a contradiction as all zeros of $B[f_t(z)]$ lie in D_- . □

Proof of Theorem 2.4: If $p(z)$ has a zero on T , then $m = \min_{z \in T} |p(z)| = 0$ and result follows by combining Lemma 3.4 and Lemma 3.5. We suppose that all the zeros of $p(z)$ lie in D_+ so that $m > 0$. By direct application of *Rouche's Theorem*, for any complex number δ with $|\delta| < 1$, the polynomial $R(z) = p(z) - \delta m$ does not vanish in D_- . Let $S(z) = z^n \overline{R\left(\frac{1}{z}\right)} = q(z) - \bar{\delta} m z^n$, then all the zeros of $S(z)$ lie in D_- and $|R(z)| = |S(z)|$ for $z \in T$. Again by *Rouche's Theorem* all the zeros

of $R(z) - \rho S(z)$ lie in D_- for every ρ with $|\rho| > 1$. By Lemma 3.3 the polynomial $B[R_t(z) - \rho S_t(z)] = B[R_t(z)] - \rho B[S_t(z)]$ has all zeros in D_- , which gives for $z \in T \cup D_+$

$$|B[R_t(z)]| \leq B[S_t(z)]$$

Equivalently we get $|z| \geq 1$,

$$(18) \quad |B[p_t(z)] - n_t \delta m \lambda_0| \leq |B[q_t(z)] - n_t \bar{\delta} m \alpha_1 \alpha_2 \dots \alpha_t B[z^{n-t}]|$$

Or

$$(19) \quad |B[p_t(z)]| - n_t |\delta| m |\lambda_0| \leq |B[q_t(z)] - n_t \bar{\delta} m \alpha_1 \alpha_2 \dots \alpha_t B[z^{n-t}]|$$

Since $p(z)$ does not vanish in D_- , so $q(z)$ has all the zeros in D_- and $\min_{z \in T} |p(z)| = \min_{z \in T} |q(z)| = m$. Therefore choosing argument of δ in right hand side of (19) which is possible by (9), we get

$$(20) \quad |B[p_t(z)]| - n_t |\delta| m |\lambda_0| \leq |B[q_t(z)]| - n_t |\delta| m |\alpha_1| |\alpha_2| \dots |\alpha_t| |B[z^{n-t}]|$$

Letting $\delta \rightarrow 1$ in (20), we get

$$(21) \quad |B[p_t(z)]| - |B[q_t(z)]| \leq -n_t \left\{ \prod_{i=1}^t |\alpha_i| |B[z^{n-t}]| - |\lambda_0| \right\} \min_{z \in T} |p(z)|.$$

Combining (21) with Lemma 3.5 we get the desired result. □

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