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REMARKS ON LOCALLY HALF-FACTORIAL DOMAINS

Seok Ryong Yoon

Abstract. In this paper, we study Dedekind domains D such that each proper localization D_S of D is an half-factorial domain.

1. Introduction

Let D be an integral domain. As in [5], we say that a saturated multiplicative set S of D is a *splitting multiplicative set* if for each nonzero $d \in D$, $d = sa$ for some $s \in S$ and $a \in D$ with $s'D \bigcap aD = s'aD$ for all $s' \in S$. Then $T = \{0 \neq t \in D | sD \bigcap tD = stD$ for all $s \in S\}$ is also a splitting multiplicative set, $ST = D - \{0\}$, and $S \bigcap T = U(D)$, where $U(D)$ is the group of units of D. We call T the *m-complement set* for S. We say that a saturated multiplicative set $S \neq U(D)$ is a GCD-set if each pair of elements $a, b \in S$ has a $gcd(a, b)$ in D (and hence in S). Thus D^* is a GCD-set if and only if D is a GCD-domain(recall that D is a GCD-domain if any two elements of D have a GCD in D, or equivalently, the intersection of any two principal ideals of D is principal).

An integral domain D is *atomic* if each nonzero nonunit of D is a product of irreducible elements. Following Zaks [13], we define D to be a *half-factorial domain* (HFD) if D is atomic and for any irreducible elements $x_1, ..., x_m, y_1, ..., y_n$ of D with $x_1 \cdots x_m = y_1 \cdots y_n$, then $m = n$. Following Valenza [12], [8], we define the *elasticity* of an atomic integral domain D as

$$\rho(D) = sup\{ \frac{m}{n} | x_1 \cdots x_m = y_1 \cdots y_n \text{ for irreducible } x_i, y_j \in D\}.$$

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(Define $\rho(D) = 1$ if D is a field.) Notice that $1 \leq \rho(D) \leq \infty$, and $\rho(D) = 1$ if and only if D is an HFD. Thus $\rho(D)$ measures how far D is from being an HFD.

Throughout, we will assume that D is a Dedekind domain with $\mathcal{C}l(D)$ its divisor class group, $[I]$ the ideal class of I in $\mathcal{C}l(D)$, $U(D)$ its group of units, D^* its set of nonzero elements, $S \subseteq D^*$ a multiplicative subset of D, $X^{(1)}(D)$ its set of nonzero (maximal) prime ideals, and $\mathcal{I}(D)$ its set of irreducible elements. A multiplicative set S is generated by $C \subseteq D^*$, and written $\langle C \rangle$, if $S = \{uc_1 \cdots c_n | u \in U(D), \text{each } c_i \in C, n \geq 1\}$. For a group G and $C \subseteq G$, we also denote by $\langle C \rangle$ the subgroup of G generated by C. To avoid trivialities, we will assume that D is *not* a UFD (PID), i.e., $\mathcal{C}l(D) \neq \{0\}$. For general references on factorization in integral domains, see [5].

If for a given abelian group G and subset $\mathcal{A} \subseteq G - \{0\}$ there exists a Dedekind domain D such that $\mathcal{C}l(D) = G$ and $\mathcal{A} = \{[P] | P$ is prime ideal of D and $[P] \neq 0\}$, then the pair $\{G, \mathcal{A}\}$ is called *realizable* [11], [10]. For D a Dedekind domain with realizable pair $\{\mathcal{C}l(D), \mathcal{A}\}$ and S a saturated multiplicative subset of D, set $\mathcal{A}[S] = \{[P] \mid P \cap S \neq \emptyset\} \subseteq \mathcal{A}$. Let $G[S]$ be the subgroup of $\mathcal{C}l(D)$ generated by $\mathcal{A}[S]$. It is possible that $\mathcal{A}[S] = \emptyset$ (for example, if S is generated by principal primes, or if $S = U(D)$). Note that $\mathcal{A}[S] = \emptyset$ if and only if $G[S] = \{0\}$. By Nagata's Theorem [9, Corollary 7.2], $G[S] = ker\, \varphi$, where $\varphi : \mathcal{C}l(D) \to \mathcal{C}l(D_S)$ is the natural homomorphism.

If P is a prime ideal of a Dedekind domain D with $|[P]| < \infty$, then set $S[P] = \{x \in D^* | xD = P_1 P_2 \cdots P_n$ with each $P_i \in [P]\} \cup U(D)$.

2. Main results

An integral domain D is said to be a *locally half-factorial domain* (LHFD) if each localization D_S of D (including D itself) is an HFD [6]. Any direct sum of cyclic groups is the divisor class group of a Dedekind LHFD[6, Example 4]. In [1], an integral domain D is said to be *locally factorial* if $D_f = D[1/f]$ is factorial (a UFD) for each nonzero nonunit $f \in D$. An integral domain D is said to be a *proper locally half-factorial domain* (PLHFD)[7] if every proper localization of D is an HFD. Thus any locally factorial domain is obviously a PLHFD.

For future reference, we include a result from[7, Theorem 2.4].

THEOREM 2.1. *Let D be a Dedekind domain such that every nonzero ideal class of D contains a prime ideal.*

(1) *If D contains a principal prime, then D is a PLHFD if and only if $Cl(D)$ is either $\{0\}$ or Z_2*
(2) *If D contains no principal primes, then D is a PLHFD if and only if $Cl(D)$ is either $Z_2 \bigoplus Z_2$, Z_4, or Z_p, p a prime*

Proof. (1) If every nonzero ideal class of a Dedekind domain D contains a prime ideal, then the same holds true for any localization of D. Also, a Dedekind domain D with the property that each nonzero ideal class contains a prime ideal is an HFD if and only if $|\mathcal{C}l(D)| \leq 2$.
(2) [7, Theorem 2.7]. □

Let G be an abelian group. The *Davenport constant* of G, denoted by $D(G)$, is the least positive integer d such that for each sequence $S \subseteq G$ with $|S| = d$, some nonempty subsequence of S has sum 0. In general, there is no known formula for $D(G)$. However, $D(Z_n) = n$, and if p is prime and $G = Z_{p^{n_1}} \bigoplus \cdots \bigoplus Z_{p^{n_r}}$, then $D(G) = 1 + \sum_{i=1}^{r}(p^{n_i} - 1)$.

Let D be an atomic integral domain with $\rho(D)$ a rational number. We say that $\rho(D)$ is *realized by a factorization* if there is a factorization $r_1 \cdots r_n = t_1 \cdots t_m$ with each $r_i, t_j \in D$ irreducible such that $\rho(D) = m/n$. If D is a Krull domain with finite divisor class group, then $\rho(D)$ is realized by a factorization [3, Theorem 10]. Next, we show that if D is a PLHFD with $\mathcal{C}l(D)$ noncyclic, then $\rho(D)$ is realized by a factorization by the computation of ideal classes of $\mathcal{C}l(D)$ directly.

THEOREM 2.2. *Let D be a Dedekind domain such that every nonzero ideal class of D contains a prime ideal. If D contains no principal primes and D is a PLHFD with $\mathcal{C}l(D)$ noncyclic, then*

(1) $\rho(D) = 3/2$,
(2) *$\rho(D)$ is realized by a factorization.*

Proof. (1) Since $\mathcal{C}l(D)$ is noncyclic by Theorem 2.1, $\mathcal{C}l(D) = Z_2 \bigoplus Z_2$. Note that the Davenport constant of $Z_2 \bigoplus Z_2$, $D(Z_2 \bigoplus Z_2) = 1 + (2 - 1) + (2 - 1) = 3$. Thus $\rho(R) = D(Z_2 \bigoplus Z_2)/2 = 3/2$ [2, Corollary 2.3(b)].

(2) Now, let P_1, P_2, and P_3 be prime ideals of D such that $[P_1] = (1, 0), [P_2] = (0, 1)$ and $[P_3] = (1, 1)$. Let x, y, z and w be irreducible

elements of D such that $xD = P_1^2$, $yD = P_2^2$, $zD = P_3^2$, and $wD = P_1P_2P_3$. Then $w^2D = xyzD$. Hence $\rho(D)$ is realized by a factorization. □

Let G be an abelian group and $A \subseteq G$. A is called an *independent set* in G if $n_1a_1 + \cdots + n_ka_k = 0, n_i \in Z$, distinct $a_i \in A$, implies that each $n_ia_i = 0$.

Example 2.3.

(1) *Let R be a Dedekind domain with class group $\mathcal{C}l(R)$ and let $D = R_S$, where S is the multiplicative set generated by the principal primes of R. Then $\mathcal{C}l(D) = \mathcal{C}l(R)$ by Nagata's Theorem, and D has no principal primes.*

(2) *Let D be a Dedekind domain such that every nonzero ideal class of D contains a prime ideal. Suppose that D has no principal primes. If $\mathcal{C}l(D) = Z_2 \bigoplus Z_2$, then since D is not an HFD, D has no nontrivial splitting sets and $\{(1,0),(0,1),(1,1)\} \subset \mathcal{C}l(D)$ is not an independent set. On the other hand, if $\mathcal{C}l(D) = Z_3$, then $\rho(D) = D(Z_3)/2 = 3/2$. Also, let P_1, P_2 be prime ideals of D such that $[P_1] = 1$ and $[P_2] = 2$. Let x, y be irreducible elements of D such that $xD = P_1^3, yD = P_2^3$ and $zD = P_1P_2$. Then $z^3D = xyD$. Hence $\rho(D)$ is realized by a factorization.*

(3) *Let D be as in Theorem 2.1 and let S be a splitting multiplicative set with T the $m-$complement for S. If $\mathcal{C}l(D) \neq Z_2 \bigoplus Z_2$, then $\mathcal{C}l(D)$ is indecomposable and so we may assume that $G[T] = \{0\}$. Thus $T = U(D)$; so D has no nontrivial splitting multiplicative sets.*

Example 2.4.

(1) Let $G = Z_4$. For $C = \{2,3\}$, we denote that $\{H_i\}$ is the family of subgroups of G generated by subsets of C. Then $\{G/H_i\} = \{Z_2, \{0\}\}$. Then there exists a Dedekind domain D such that $\mathcal{C}l(D) = G$ and the set of divisor class groups of overrings of D is $\{Z_2, \{0\}\}$ (and hence D is a PLHFD)(such a Dedekind domain exists by [11, Theorem 2.3]). If there exists a nontrivial splitting multiplicative set of D, then $\mathcal{C}l(D) \simeq \mathcal{C}l(D_S) \bigoplus \mathcal{C}l(D_T)$ given by $[I] \to ([ID_S],[ID_T])$, where T is the $m-$complement for S. Since $\mathcal{C}l(D) = Z_4$, we may assume that $\mathcal{C}l(D_T) = \{0\}$. Hence D_T is a UFD; S is generated by prime elements

of D and $\mathcal{C}l(D) \simeq \mathcal{C}l(D_S)$, but $\mathcal{C}l(D) = Z_4$ and $\mathcal{C}l(D_S) = Z_2$ or $\{0\}$, a contradiction. Hence D^* and $U(D)$ are the only splitting multiplicative sets of D.

(2) As in (1), let $G = Z_p$, p a prime and let $C = \{1, 2, ..., p-2\}$. Let $\{H_i\}$ be the family of subgroups of G generated by subsets of C. Then $\{G/H_i\} = \{0\}$. Then there exists a Dedekind domain D such that $\mathcal{C}l(D) = G$ and the set of divisor class groups of overrings of D is $\{0\}$ (and hence D is a LHFD)(such a Dedekind domain exists by [11, Theorem 2.3]). Suppose that there exists a splitting multiplicative set S of D with $S \neq D^*, U(D)$. By the observation in (1), we have $\mathcal{C}l(D) \simeq \mathcal{C}l(D_S)$, a contradiction.

Let D be a Dedekind domain with divisor class group G. In [10], let $\Delta(g) \in \{0, 1, 2, ...\} \cup \{\infty\}$ denote the number of prime ideals of D in the class $g \in G$. Let G be a finitely generated torsion abelian group generated by $\mathcal{A}$ as a monoid. If $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$ is a partition of $\mathcal{A}$ such that $\mathcal{B}' \cup \mathcal{C}$ generates G as a monoid for each cofinite subset $\mathcal{B}'$ of $\mathcal{B}$, then there exists a Dedekind domain D such that $\{G, \mathcal{A}\}$ is realizable, $\Delta(b) = 1$ for each $b \in \mathcal{B}$ and $\Delta(c) = \infty$ [10, Theorem 8].

THEOREM 2.5. *Let D be a Dedekind domain such that every nonzero ideal class of D contains a prime ideal and D contains no principal primes. If D is a PLHFD but not an HFD such that $[P] \in \mathcal{C}l(D)$ has exactly one prime ideal of D for some P, then*

(1) *$S[P]$ is a GCD-set and $\mathcal{H}_{[P]}$ is not an HF-set;*
(2) *Each $x \in S[P] \bigcap \mathcal{I}(D)$ is P-primary.*
(3) *$S[P]$ is not a splitting multiplicative set.*

Proof. (1), (2) Since D is a PLHFD and $[P]$ contains exactly one prime ideal with $|[P]| < \infty$, $S[P]$ is a GCD-set and each $X \in S[P] \bigcap \mathcal{I}(D)$ is P-primary[4, Theorem 3.2].

(3) Since D is not an HFD; so D has no nontrivial multiplicative sets. But, if $S[P] = D^*$ is trivial, then D is an atomic GCD-domain. Thus D is a UFD. Hence $S[P]$ is not a splitting multiplicative set. □

We conclude this paper with some more examples.

EXAMPLE 2.6.

(1) As in Example 2.4, we can not construct D with $\mathcal{C}l(D) = Z_2$ such that $S[P]$ is a GCD-set by partition method. Let R be a Dedekind

domain with divisor class group $\mathcal{C}l(R) = Z_2$. Let T be the multiplicative set generated by all principal primes of R. Then $D = R_T$ has no principal primes and $\mathcal{C}l(D) = \mathcal{C}l(R) = Z_2$. Thus $D^* = S[P]$ for each nonprincipal prime P of D. If $S[P] = D^*$ is a GCD-set, then D is an atomic GCD-domain. Hence D ia s UFD.

(2) As in Theorem 2.1, there exist a PLHFD D such that D has no principal primes, $\mathcal{C}l(D) = Z_p, p \geq 3$ such that $\Delta(1) = 1, \Delta(2) = \infty$. Then $S[P]$ is a GCD-set, where $[P] = 1$.

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Department of Mathematics
Inha University
Incheon 402-751 Korea
E-mail: drys1018@inha.ac.kr