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NORDHAUS-GADDUM TYPE RESULTS FOR CONNECTED DOMINATION NUMBER OF GRAPHS

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Abstract. Let $G = (V, E)$ be a graph. A subset S of V is called a *dominating set* of G if every vertex not in S is adjacent to some vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets of G. A dominating set S is called a *connected dominating set* if the subgraph induced by S is connected. The minimum cardinality taken over all connected dominating sets of G is called the *connected domination number* of G, and is denoted by $\gamma_c(G)$. In this paper, we investigate the Nordhaus-Gaddum type results for the connected domination number and its derived graphs like line graph, subdivision graph, power graph, block graph and total graph, and characterize the extremal graphs.

1. Introduction

By a graph, we mean a finite, simple graph $G = (V, E)$ with vertex set $V = V(G)$ of order $n = |V|$ and edge set $E = E(G)$ of size $m = |E|$. For basic definitions and notation, we follow [6, 9]. A vertex of degree one is called *an end* or *pendant vertex*. An *internal vertex* is a vertex that is not a pendant or end vertex. The *distance* between two vertices u and v is the length of the shortest u-v path and is denoted by $d(u, v)$. For any positive integer k, let $N_k(u) = \{v \in V \mid d(u, v) = k\}$. The *eccentricity* $e(v)$ of a vertex v is defined by $e(v) = max\{d(u, v) \mid u \in V(G)\}$. A *clique* in a graph G is a maximal complete subgraph of G. The *girth* of G is the length of the shortest cycle in G and is denoted by $g(G)$. A graph G is called *acyclic* if it has no cycles. A *tree* is a connected acyclic graph. A tree containing exactly two vertices that are not pendants is called a *double star*. A *spider* is a tree with one vertex of degree at least 3 and all others with degree at most 2. A connected graph G is said to be *unicyclic* if G has exactly one cycle. The *Cartesian Product* of simple graphs G and H is the simple graph $G \times H$ with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. A set S is called a *connected dominating set* if the subgraph induced by S is connected and if every vertex not in S is adjacent to some vertex in S. The minimum cardinality taken over all connected dominating sets in G is called the *connected domination number*, and is

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denoted by $\gamma_c(G)$. Moreover, a connected dominating set of G of cardinality $\gamma_c(G)$ is called a γ_c-set of G. A subset X of E is called *an edge dominating set* of G if every edge not in X is adjacent to some edge in X. An edge dominating set X is called a *connected edge dominating set* if the edge induced subgraph of X is connected. The *line graph* $L(G)$ of a graph G is a graph whose vertex set is $E(G)$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. The graphs *paw* and *diamond* are denoted by $K_{1,3}+e$, C_4+e respectively. The literature of domination in graphs and related results have been considered in [3, 5, 10, 21, 22]. The relation of Nordhaus-Gaddum type for domination in graphs were proved by Jaeger and Payan [12] in 1972 and are as follows.

Theorem 1.1 ([12]). *For any graph G with at least two vertices,*

$$3 \leq \gamma(G) + \gamma(\overline{G}) \leq n + 1 \text{ and}$$
$$2 \leq \gamma(G).\gamma(\overline{G}) \leq n.$$

This has been extended to other graph theoretic parameters. A survey of these results is published in [1]. Note that $\overline{G}$ is one of the derived graphs and there are several derived graphs in the literature. In [16–19], the authors obtained similar results for line graphs, total graphs, shadow graphs and block graphs. Hence, in this paper, we extend the Nordhaus-Gaddum type result to some derived graphs like line graph, subdivision graph, power graph, block graph and total graph for the parameter connected domination number.

2. Preliminary Results

For the sequal of the paper, we need the following of graphs as follows:

Theorem 2.1 ([22]). *5 If H is a connected spanning subgraph of G, then $\gamma_c(G) \leq \gamma_c(H)$.*

Theorem 2.2 ([22]). *For any connected graph G, $n/(\Delta(G)+1) \leq \gamma_c(G) \leq 2m-n$ with equality for the lower bound if and only if $\Delta(G) = n-1$ and equality for the upper bound if and only if G is a path.*

Definition 2.3. A *clique dominating set* [7] or a *dominating clique* is a dominating set that induces a complete subgraph.

In 2013, Wyatt J. Desormeaux et al. [23] gave the lower bound for connected domination number of a graph in terms of girth and characterized the equality. For this characterization, they defined the following family $\mathcal{F}_k$.

For $k \geq 3$, we define a family $\mathcal{F}_k$ of graphs as follows. Let $\mathcal{F}_3$ be the family of graphs with a dominating vertex (a vertex of full degree) and at least one triangle. Let $\mathcal{F}_4$ be the family of graphs that can be obtained from a double star $S(r, s)$, where $r, s \geq 1$, with central vertices x and y by adding at least one edge joining a leaf-neighbor of x and a leaf-neighbor of y.

For $k \geq 5$, let $\mathcal{F}_k$ be the family of graphs constructed from a k-cycle $v_1v_2 \dots v_kv_1$ as follows: For each i, $3 \leq i \leq k$, add zero or more pendant edges incident to v_i. Moreover, if $k \leq 6$, add zero or more edges joining v_3 and v_k and subdivide each such added edge twice.

THEOREM 2.4 ([23]). *Let G be a connected graph that contains a cycle. Then, $\gamma_c(G) \geq g(G) - 2$ with equality if and only if $G \in \mathcal{F}_k$.*

THEOREM 2.5 ([23]). *If G is a diameter two planar graph, then $\gamma_c(G) \leq 3$.*

THEOREM 2.6 ([11]). *If G is a connected graph, and $n \geq 3$, then $\gamma_c(G) = n - \epsilon_T(G) \leq n - 2$ where $\epsilon_T(G)$ denotes the maximum number of pendant edges in any spanning tree T of G.*

COROLLARY 2.7 ([22]). *If T is a tree with $n \geq 3$ vertices, then $\gamma_c(T) = n - p(T)$ where $p(T)$ denotes the number of end vertices of a tree T.*

THEOREM 2.8 ([15]). *If G is a 3-regular planar graph with diameter two, then G is isomorphic to the cartesian product $K_2 \times K_3$.*

THEOREM 2.9 ([15]). *If G is a 4-regular planar graph with diameter two, then G is isomorphic to any one of the graphs given in Fig. 1.*

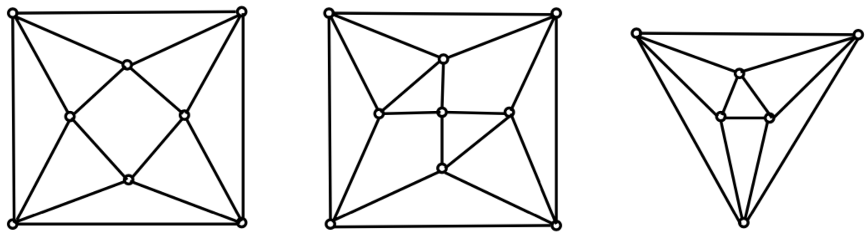


FIGURE 1. Regular Planar Graphs of diameter 2.

THEOREM 2.10 ([15]). *There exist no 5-regular planar graphs with diameter two.*

THEOREM 2.11 ([11]). *For any connected graph G of order $n \geq 3$, $\gamma_c(G) \leq n - \Delta(G)$.*

OBSERVATION 2.12. For any connected graph G of order $n \geq 3$, $\gamma_c(G) = n - 2$ if and only if G is either a path or a cycle.

THEOREM 2.13. *(i) For any connected graph G, $\gamma_c(G) \leq \gamma_c(T(G))$.*
(ii) If G is a tree, then $\gamma_c(G) = \gamma_c(T(G))$.

Proof. (i) Let S be a γ_c-set of $T(G)$. We consider three cases. a) If $S \subseteq V(G)$, then clearly S is a connected dominating set of G and hence $\gamma_c(G) \leq |S| = \gamma_c(T(G))$. b) If $S \subseteq E(G)$, then $|S| = n - 1 > n - 2$. Then by Theorem 2.6, $\gamma_c(G) \leq |S| = \gamma_c(T(G))$. c) If $S \subseteq V(G) \cup E(G)$, let $S = L \cup M$ and $L \subseteq V(G)$, $M \subseteq E(G)$ such that $|L| = l$ and $|M| = t$. If L is a connected dominating set of G, then the result is obvious. If L is not a connected dominating set of G, let $X = V(G) \setminus N_G(L) \subseteq V(G)$. Then some vertices of X are connected and is dominated by some edges $M' \subseteq M$ in $T(G)$ such that at least one edge of M' is incident with at least one vertex of L. Then clearly $|X| \leq |M'| \leq t$ and $L \cup X$ is a connected dominating set of G. Hence $\gamma_c(G) \leq |L \cup X| \leq |L| + |X| \leq |L| + |M'| \leq l + t = |S| = \gamma_c(T(G))$.
(ii) If G is a tree with p end vertices, then by Corollary 2.7, $\gamma_c(G) = n - p$. We claim that $\gamma_c(T(G)) = n - p$. By (i), $\gamma_c(T(G)) \geq n - p$. Further, it is clear that the set of all

internal vertices of G is a connected dominating set of $T(G)$. Hence $\gamma_c(T(G)) \leq n-p$. Thus, the required result follows. $\square$

COROLLARY 2.14. *For any path P_n, $\gamma_c(T(P_n)) = n - 2$.*

Proof. It follows from Observation 2.12 and Theorem 2.13 (ii). $\square$

PROPOSITION 2.15. *For any connected graph G, $\gamma_c(T(G)) \leq \gamma_c(S(G))$.*

Proof. Since $S(G)$ is a spanning subgraph of $T(G)$, the result follows from Theorem 2.1. $\square$

NOTATION 2.16 ([14]). Let G be a connected graph with n vertices $u_1, u_2, \ldots, u_n$. The graph obtained from G by attaching n_1 times an end vertex of P_{l_1} on the vertex u_1, n_2 times an end vertex of P_{l_2} on the vertex u_2, and so on, is denoted by $G(n_1P_{l_1},\ n_2P_{l_2},\ n_3P_{l_3}, \ldots, n_nP_{l_n})$, where $n_i,\ l_i \geq 0$ and $1 \leq i \leq n$. In particular, if $n_i = 1$ for each $i = 1$ to n, then it is denoted by $G(P_{l_1},\ P_{l_2},\ P_{l_3}, \ldots, P_{l_n})$. For example $C_3(P_4,\ P_3,\ P_1)$ and $C_4(P_3,\ P_1,\ P_4,\ P_1)$ are given in Fig. 2.

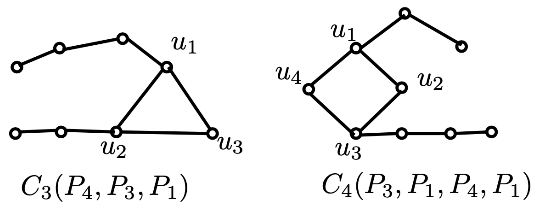


FIGURE 2. Illustrations

THEOREM 2.17. *Let G be a tree of order $n \geq 4$. Then $\gamma_c(G) = n - 3$ if and only if G is a spider with maximum degree three.*

Proof. Assume that $\gamma_c(G) = n-3$. Since $\gamma_c(G) = n-2$ if and only if G is either path or cycle, clearly $\Delta(G) \geq 3$. If at least two vertices are of degree 3, then the number of end vertices of G is strictly greater than 3, which gives $\gamma_c(G) \leq n - 4$ by Corollary 2.7. Therefore, there exists exactly one vertex $v \in V(G)$ such that $d_G(v) = 3$, and gives a spider graph. Converse is obvious. $\square$

THEOREM 2.18. *For any connected unicyclic graph G with cycle C, $\gamma_c(G) = n - 3$ if and only if G is one of the following;*
(i) If $C = C_3$, then $G \cong C(P_i,\ P_1,\ P_1)$ for $i \geq 2$ or when at least one of i and j is not equal to 1, $G \cong C(P_i,\ P_j,\ P_1)$ or when at least one of i, j and k is not equal to 1, $G \cong C(P_i,\ P_j,\ P_k)$.
(ii) If $C = C_4$, then $G \cong C(P_i,\ P_1,\ P_1,\ P_1)$ for $i \geq 2$ or when at least one of i and j is not equal to 1, $G \cong C(P_i,\ P_1,\ P_j,\ P_1)$.
(iii) If $C = C_k$ $(k \geq 5)$, then $G \cong C(P_{n-k+1})$.

Proof. Let G be any connected graph with cycle $C = (u_1u_2 \ldots u_k = u_1)$. Assume that $\gamma_c(G) = n - 3$. By Theorem 2.11 and Observation 2.12, $\Delta(G) = 3$. We claim that every vertex not on C is of degree less than or equal to two. Suppose there exists a vertex v not on C such that $d_G(v) \geq 3$. There is a spanning tree H of G with 4 end vertices. Let P be the set of end vertices of H. Then $V(H) - V(P)$ is a connected dominating set of G having $n-4$ vertices, that is, $\gamma_c(G) \leq n-4$, a contradiction. Hence every vertex not on C is of degree less than or equal to two. Clearly $V(G) - V(C)$ is a union of disjoint paths and exactly one end vertex of each path is adjacent to a vertex of C. Then we consider the three cases. If $C = C_3$, then $G \cong C(P_i,\ P_1,\ P_1)$ or $C(P_i,\ P_j,\ P_1)$ or $C(P_i,\ P_j,\ P_k)$. Now let $C = C_4$. We observe that three or four vertices of degree three in G is not possible. If G has exactly one vertex of degree three, then $G \cong C(P_i,\ P_1,\ P_1,\ P_1)$. If G has two vertices of degree three, then they are adjacent or not adjacent. If they are adjacent, then we can get a spanning tree with these two adjacent vertices having 4 end vertices, and hence $\gamma_c(G) \leq n-4$, a contradiction. If they are not adjacent, then every spanning tree of G has at most 3 end vertices. Hence $G \cong C(P_i,\ P_1,\ P_j,\ P_1)$. If $C = C_k (k \geq 5)$, then G cannot have two vertices of degree three in G, since otherwise, there is a spanning tree of G with at least 4 end vertices. Hence $G \cong C(P_{n-k+1})$. Converse is obvious by verification. □

3. Line Graphs

Definition 3.1. The *line graph* $L(G)$ of a graph G is a graph whose vertex set is $E(G)$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G.

Definition 3.2. The *degree of an edge* $e = uv$ of G is defined by $deg\ e = d_G(u) + d_G(v) - 2$ and maximum degree of an edge is denoted by $\Delta'(G)$.

In this section, we obtain the lower and upper bounds for the sum of connected domination number of a graph and its line graph in terms of the order of a graph.

Theorem 3.3 ([3]). *For any connected graph G of order $n \geq 4$, $\gamma_c'(G) \leq n-2$ and equality holds if and only if G is either K_n or C_n where $\gamma_c'(G)$ is the connected edge domination number of G.*

Observation 3.4. (i) For any cycle C_n, $\gamma_c(L(C_n)) = n-2$.
(ii) For any path P_n, $\gamma_c(L(P_n)) = n-3$.

Theorem 3.5. *For any connected graph G of order $n \geq 3$ and size $m \geq 2$, $2 \leq \gamma_c(G) + \gamma_c(L(G)) \leq 2n-4$ with equality for the lower bound if and only if $\Delta(G) = n-1$ and $\Delta'(G) = m-1$ and equality for the upper bound if and only if $G \cong C_n$.*

Proof. By the definition, the lower bound is obvious. Since $\gamma_c'(G) = \gamma_c(L(G))$, by Theorems 2.6 and 3.3, the required upper bound holds. If $\gamma_c(G) + \gamma_c(L(G)) = 2$, then $\gamma_c(G) = \gamma_c(L(G)) = 1$. Then, clearly G has $\Delta(G) = n-1$ and $\Delta'(G) = m-1$. If $\gamma_c(G) + \gamma_c(L(G)) = 2n-4$, then by Theorems 2.6 and 3.3, $\gamma_c(G) = \gamma_c(L(G)) = n-2$. Since $\gamma_c(K_n) = 1$, by Observation 2.12 and Theorem 3.3, $G \cong C_n$. Converse is obvious. □

Theorem 3.6. *If G is a connected graph of order $n \geq 3$, then $\gamma_c(G) + \gamma_c(L(G)) = 2n-5$ if and only if G is either K_4 or P_n.*

Proof. Assume that $\gamma_c(G) + \gamma_c(L(G)) = 2n-5$. Then by Theorems 2.6 and 3.3, $\gamma_c(G) = n-3$ and $\gamma_c(L(G)) = n-2$ (or) $\gamma_c(G) = n-2$ and $\gamma_c(L(G)) = n-3$. In the former case, by Theorem 3.3, G is either K_n or C_n. Since $\gamma_c(C_n) = n-2 \neq n-3$

and $\gamma_c(K_n) = 1 = n - 3$ which gives $n = 4$, and hence $G \cong K_4$. In the latter case, by Observation 2.12, G is either P_n or C_n, and by Observation 3.4, $G \cong P_n$. Converse is obvious. □

THEOREM 3.7. *Let G be a connected graph of order n with at most one cycle C_k. Then $\gamma_c(G) + \gamma_c(L(G)) = 2n - 6$ if and only if G is either a claw or $G \cong C_k(P_{n-k+1}),\ k \geq 3$.*

Proof. Assume that $\gamma_c(G)+\gamma_c(L(G)) = 2n-6$. Then by Theorems 2.6 and 3.3, we have three cases. (i) $\gamma_c(G) = n - 4$ and $\gamma_c(L(G)) = n - 2$
(ii) $\gamma_c(G) = n - 2$ and $\gamma_c(L(G)) = n - 4$
(iii) $\gamma_c(G) = \gamma_c(L(G)) = n - 3$.
From (i), by Theorem 3.3, $G \cong K_n$ or C_n. If $G = C_n$, then $\gamma_c(G) = n - 2 \neq n - 4$. If $G = K_n$, then $\gamma_c(G) = 1 = n - 4$ which implies $n = 5$ and so $G \cong K_5$. But from hypothesis no such graph exists. From (ii), by Observation 2.12 and Lemma 3.4, no such graph exists. Consider the case (iii). If G is a tree, then by Theorem 2.17, G is a spider with maximum degree three. Let $d_G(v) = 3$. We claim that $d(v,u) = 1$ for every vertex $u \neq v$ in G. If $d(v,u) \geq 2$ for some vertex u in G, then $\gamma_c'(G) = \gamma_c(L(G)) = m - 3 = (n - 1) - 3 = n - 4$, a contradiction. Hence $G \cong K_{1,3}$(claw). If G is a unicyclic graph, then by Theorem 2.18, G is one of the graphs (i) or (ii) or (iii). We claim that G has one vertex of degree three. If G has more than one vertex of degree three, then $\gamma_c'(G) = \gamma_c(L(G)) = m - 4 = n - 4$, a contradiction. Hence $G \cong C_k(P_{n-k+1}),\ k \geq 3$. Converse can be easily verified. □

4. Subdivision Graphs

DEFINITION 4.1. The *subdivision graph* $S(G)$ of a graph G is a graph which is obtained by subdividing each edge of G exactly once.

In this section, we obtain some bounds for the sum of connected domination number of a graph and its subdivision graph. For this purpose, we need the following results.

THEOREM 4.2 ([2]). *For any connected graph G of order $n \geq 3,\ \gamma_c(S(G)) \leq 2n-2$ and equality holds if and only if $G \cong K_n$ or C_n.*

THEOREM 4.3 ([2]). *For any tree T of order $n \geq 3,\ \gamma_c(S(T)) = 2n - p(T) - 1$ where $p(T)$ denotes the number of end vertices of T.*

THEOREM 4.4 ([2]). *For any star $K_{1,n-1},\ \gamma_c(S(K_{1,n-1})) = n$.*

THEOREM 4.5. *For any tree T of order $n \geq 3,\ \gamma_c(T) + \gamma_c(S(T)) = 3n - 2p(T) - 1$ where $p(T)$ denotes the number of end vertices of T.*

Proof. It follows from Corollary 2.7 and Theorem 4.3. □

LEMMA 4.6. *For any connected graph $G,\ \gamma_c(G) \leq \gamma_c(S(G))$.*

Proof. It follows from Theorem 2.13 (i) and Proposition 2.15. □

LEMMA 4.7. *For any path $P_n,\ \gamma_c(S(P_n)) = 2n - 3$.*

Proof. Since $S(P_n) = P_{2n-1},\ \gamma_c(S(P_n)) = 2n - 3$. □

LEMMA 4.8. *Let G be a connected graph with $\Delta(G) = n - 1$. Then $\gamma_c(S(G)) = 3$ if and only if $G \cong P_3$.*

Proof. By Theorem 4.13, G contains no cycle so that G is a tree. Since $\Delta(G) = n-1$, G is a star. Also, by Theorem 4.4, $\gamma_c(S(K_{1,n-1})) = n = 3$. Hence $G \cong K_{1,n-2} \cong P_3$. Converse follows by verification. □

OBSERVATION 4.9. There exist no connected graph G with $\gamma_c(S(G)) = 2$.

THEOREM 4.10. *For any connected graph G of order $n \geq 3$, $4 \leq \gamma_c(G)+\gamma_c(S(G)) \leq 3n-4$ with equality for the upper bound if and only if $G \cong C_n$ and equality for the lower bound if and only $G \cong P_3$.*

Proof. The required upper and lower bounds follow from Lemma 4.8 and Theorems 2.6, 4.2. If $\gamma_c(G)+\gamma_c(S(G)) = 4$, then by Observation 4.9, $\gamma_c(G) = 1$ and $\gamma_c(S(G)) = 3$. Hence by Lemma 4.8, $G \cong P_3$. If $\gamma_c(G) + \gamma_c(S(G)) = 3n-4$, then $\gamma_c(G) = n-2$ and $\gamma_c(S(G)) = 2n-2$. By Theorem 4.2, $G \cong K_n$ or C_n. Since $\gamma_c(C_n) = n-2$ and $\gamma_c(K_n) = 1 = n-2$ which implies $n = 3$, $G \cong C_n$. Converses are obvious by verification. □

THEOREM 4.11. *For any connected graph G of order $n \geq 3$, $\gamma_c(G) + \gamma_c(S(G)) = 3n-5$ if and only if $G \cong P_n$ or K_4.*

Proof. Assume that $\gamma_c(G)+\gamma_c(S(G)) = 3n-5$. Then $\gamma_c(G) = n-3$ and $\gamma_c(S(G)) = 2n-2$ (or) $\gamma_c(G) = n-2$ and $\gamma_c(S(G)) = 2n-3$. In the former case, by Theorem 4.2, G is either K_n or C_n. Since $\gamma_c(K_n) = 1 = n-3$ which gives $n = 4$, and $\gamma_c(C_n) = n-2 \neq n-3$, $G \cong K_4$. In the latter case, by Observation 2.12, G is either P_n or C_n. If $G \cong C_n$, then by Theorem 4.2, $\gamma_c(S(G)) = 2n-2 \neq 2n-3$. If $G \cong P_n$, then by Lemma 4.7, $\gamma_c(S(P_n)) = 2n-3$. Converse can be easily verified. □

THEOREM 4.12. *Let G be a connected graph of order n with at most one cycle C_k. Then $\gamma_c(G) + \gamma_c(S(G)) = 3n-6$ if and only if $G \cong C_k(P_{n-k+1})$.*

Proof. Assume that $\gamma_c(G)+\gamma_c(S(G)) = 3n-6$. Then by Theorems 2.6 and 4.2, we have three cases. (i) $\gamma_c(G) = n-4$ and $\gamma_c(S(G)) = 2n-2$
(ii) $\gamma_c(G) = n-2$ and $\gamma_c(S(G)) = 2n-4$
(iii) $\gamma_c(G) = n-3$ and $\gamma_c(S(G)) = 2n-3$.
From (i), by Theorem 4.2, $G \cong K_n$ or C_n. If $G \cong C_n$, then $\gamma_c(C_n) = n-2 \neq n-4$. If $G \cong K_n$, then $\gamma_c(K_n) = 1 = n-4$ which implies $n = 5$. Hence $G \cong K_5$. As K_5 has more than one cycle, it contradicts the hypothesis. From (ii), by Observation 2.12, $G \cong P_n$ or C_n. By Lemma 4.7 and Theorem 4.2, $\gamma_c(S(G)) \neq 2n-4$. Now consider the case (iii). If G is a tree, then $S(G)$ is a tree. By Corollary 2.7, $\gamma_c(S(G)) = n+m-p = n+(n-1)-p = 2n-1-p$, where p denotes the number of end vertices of G. By Theorem 2.17, G is a spider with $\Delta(G) = 3$ and so $p = 3$. Hence $\gamma_c(S(G)) = 2n-1-3 = 2n-4$, a contradiction. If G is a unicyclic graph with cycle C, then G is one of the graphs from Theorem 2.18. We claim that exactly one vertex has degree three. If not, then there are at least two vertices of degree three in G. Then C must be either C_3 or C_4. If $C = C_3$ or C_4, let $S(H)$ be the spanning tree of $S(G)$ which has at least 4 end vertices, and hence $\gamma_c(S(G)) \leq 2n-4$, a contradiction. Hence, $G \cong C_k(P_{n-k+1})$. Converses are obvious by verification. □

THEOREM 4.13. *Let G be a connected graph of order n and size m that contains a cycle. Then $\gamma_c(S(G)) \geq 2g(G)-2$, and equality holds if and only if $G \cong C_n$.*

Proof. By Theorem 2.4, $\gamma_c(S(G)) \geq 2(g(G))-2$. Assume that $\gamma_c(S(G)) = 2(g(G))-2$. Let S be any minimum connected dominating set of $S(G)$. We claim that $g(G) = n$.

Suppose $g(G)$ is at most $n-1$. If the subgraph induced by S in $S(G)$ contains a cycle, then $g(S(G)) \leq |S| = g(S(G)) - 2 = 2g(G) - 2$, a contradiction. Hence the subgraph induced by S in $S(G)$ is a tree. Let $v \in V(S(G)) \setminus S$. If v has two neighbors in S, then the subgraph induced by $S \cup \{v\}$ in $S(G)$ contains a cycle of length at most $|S|+1 \leq 2g(G)-1$, a contradiction. Hence v has at most one neighbor in S. If l and k be the number of cycles and pendant vertices of G respectively, then it is evident that $\gamma_c(S(G)) \geq n+m-(2l+k)$ and further by hypothesis, $n+m-(2l+k) > 2g(G)-2$. Hence $\gamma_c(S(G)) > 2g(G) - 2$, a contradiction. Hence $g(G) = n$ so that $G \cong C_n$. Converse is obvious. $\square$

Observation 4.14. Let G be a connected graph that contains a cycle. Then, $\gamma_c(G) + \gamma_c(S(G)) \geq 3g(G) - 4$ and equality holds if and only if $G \cong C_n$.

Theorem 4.15. *Let G be a connected graph that contains a cycle. Then, $\gamma_c(G) + \gamma_c(S(G)) = 3g(G)-3$ if and only if G is either a paw or a diamond or $K_{2,3}$ or $C_{n-1}(P_2)$.*

Proof. Assume that $\gamma_c(G) + \gamma_c(S(G)) = 3g(G) - 3$. Then we have two cases.
Case : 1 $\gamma_c(G) = g(G) - 1$ and $\gamma_c(S(G)) = 2g(G) - 2$.
By Theorem 4.13, $G \cong C_n$. But $\gamma_c(G) = n - 2 = g(G) - 2 \neq g(G) - 1$.
Case : 2 $\gamma_c(G) = g(G) - 2$ and $\gamma_c(S(G)) = 2g(G) - 1$.
By Theorem 2.4, $G \in \mathcal{F}_k$. Let $g(G) = k$.
Subcase : 2.1 $k = 3$
Then $G \in \mathcal{F}_3$. By the definition of $\mathcal{F}_3$, G contains at least one triangle with a dominating vertex, say $v_0(\Delta(G) = n-1)$. Let $V(G) = \{v_0,\ v_1,\ \ldots, v_{n-1}\}$ with $deg_G(v_0) = n-1$. Let S be a minimum connecetd dominating set of $S(G)$. Suppose v_0 is not a cut vertex in G. Let w_{ij} be the vertex of $S(G)$ adjacent to v_i and v_j. Suppose $v_0 \notin S$. Since w_{0j} is adjacent to v_0 and v_j, S contains $\{v_1,\ v_2, \ldots, v_{n-1}\}$ and w_{0j} for some j. Since $\langle S \rangle$ is connected, S contains $\{w_{12},\ w_{23}, \ldots, w_{(n-2)(n-1)}\}$. Hence $|S| \geq n + (n-2) = 2n-2$. Suppose $v_0 \in S$. Since w_{ij} is adjacent to v_i and v_j, S contains $\{w_{12}, w_{23}, \ldots, w_{(n-2)(n-1)}\}$. If $w_{0j} \notin S$ for some j, then S contains w_{0j}, so that S contains at least $n-2$ vertices of $\{v_2, v_3, \ldots, v_j\}$. Hence $|S| \geq 1 + (n-2) + (n-2)$. In both cases, by hypothesis, $n = 4$. Suppose v_0 is a cut vertex of G. If $n \geq 5$, then $\Delta(G) \geq 4$. If $\Delta(G) = 4$, then G has $K_{1,4}$ as a subgraph and v_0 be a vertex of full degree in G. Since v_0 is a cut vertex of G, then v_0 must be in S. Since w_{0j} is adjacent to v_0 and v_j, S contains $\{w_{01}, w_{02}, w_{03}, w_{04}\}$. Also G contains at least one triangle and $\langle S \rangle$ is connected, so that S contains at least one vertex, say v_j. Hence $|S| \geq 1+4+1 = 6$, a contradiction. Hence $n = 4$ and so G must be either a paw or a diamond or K_4. But by Theorem 4.2, $\gamma_c(S(K_4)) = 6 \neq 2g(G) - 1$. Hence G is isomorphic to either a paw or a diamond.
Subcase : 2.2 $k = 4$
Let $C : v_1v_2v_3v_4v_1$ be a shortest cycle in G. We claim that G has at most one end vertex. If G has at least two end vertices, then we observe that $\gamma_c(S(G)) \geq 8$, a contradiction. Hence G has at most one end vertex.
Case : 2.2.1 G has one end vertex
By the definition of $\mathcal{F}_4$, we observe that G has exactly one C_4 and so $G \cong C(P_2)$. Hence $G \cong C_4(P_2)$.
Case : 2.2.2 G has no end vertex
By the definition of $\mathcal{F}_4$, we observe that, if $r + s \geq 4$ with no end vertices, then $\gamma_c(S(G)) > 7$ and hence by assumption, $r + s = 3$. Hence either $r = 2$ and $s = 1$ (or)

$r = 1$ and $s = 2$. Without loss of generality, we may take $r = 2$ and $s = 1$. Hence x has two leaves, say x_1, x_2 and y has one leaf, say y_1. Since G has no end vertices, by the definition of $\mathcal{F}_4$, both $x_1,\ x_2$ are adjacent to y_1 and hence $G \cong K_{2,3}$ which satisfy the hypothesis.

Subcase : 2.3 $k \geq 5$

Let $C : v_1v_2 \dots v_kv_1$ be a shortest cycle in G. If G has no pendant edge added to the vertices of C, then $G \cong C_n$ so that $\gamma_c(S(G)) = 2n - 2 \neq 2g(G) - 1$. We claim that G has exactly one pendant edge. If G has two pendant edges, let the two pendant vertices be $u_1,\ u_2$. Let w_{ij} be the vertex of $S(G)$ adjacent to v_i and u_j. If both u_1 and u_2 are adjacent to a vertex v_i for some $3 \leq i \leq k$, then $2k - 2$ vertices of $S(C)$ including v_i and the vertices $w_{i1},\ w_{i2}$ is a minimum connected dominating set of cardinality $2n - 4$. But our assumption, $\gamma_c(S(G)) = 2n - 5$, a contradiction. If the vertices u_1 and u_2 are adjacent to two distinct vertices of C, then by a similar argument, we get a minimum connected dominating set of cardinality $2n - 4$, a contradiction. Hence $G \cong C_{n-1}(P_2)$. Converse can be easily verified. □

5. Square Graphs

Definition 5.1. For any integer $k \geq 2$, *the power* G^k of a graph G is a graph whose vertex set is $V(G)$ and two distinct vertices of G^k are adjacent if their distance in G is at most k.

We observe that $\gamma_c(G^2) \leq \gamma_c(G)$ (5.1).

In this section, we obtain some bounds for the sum of connected domination number of a graph and its power graph. The following observations are used in this section.

(i) If G is a connected graph of order $n \leq 5$, then $\gamma_c(G^2) = 1$.

(ii) $\gamma_c(G^2) = 1$ if and only if $e(v) \leq 2$ for some $v \in V(G)$.

(iii) Equality of (5.1) holds if and only if both of them must be one.

Observation 5.2. If $G \cong P_n$ or C_n with $n \geq 5$, then $\gamma_c(G^2) = \lfloor \frac{n}{2} \rfloor - 1$.

Observation 5.3. If G is a connected graph, then $\gamma_c(G) + \gamma_c(G^2) = 3$ if and only if G has a dominating clique K_2.

Theorem 5.4. *For any connected graph G of order n, $2 \leq \gamma_c(G) + \gamma_c(G^2) \leq 2n - 4$ with equality for the lower bound if and only if $\Delta(G) = n - 1$ and equality for the upper bound if and only if $G \cong C_3$ or P_3.*

Proof. Since $\gamma_c(G) \geq 1$, by Eq.(5.1), $2 \leq \gamma_c(G) + \gamma_c(G^2)$. By Theorem 2.6 and Eq.(5.1), the required upper bound holds. If $\gamma_c(G) + \gamma_c(G^2) = 2$, then $\gamma_c(G) = \gamma_c(G^2) = 1$ and hence $\Delta(G) = n - 1$. If $\gamma_c(G) + \gamma_c(G^2) = 2n - 4$, then $\gamma_c(G) = n - 2$ and $\gamma_c(G^2) = n - 2$. By Observation 2.12, $G \cong P_n$ or C_n. Further, by Observation 5.2, $n = 3$. Hence $G \cong P_3$ or C_3. Converse is obvious. □

Theorem 5.5. *For any connected graph G, $\gamma_c(G) + \gamma_c(G^2) = 2n - 5$ if and only if G is either P_4 or C_4.*

Proof. If $\gamma_c(G) + \gamma_c(G^2) = 2n - 5$, then by Theorem 2.6 and Eq.(5.1), $\gamma_c(G) = n - 2$ and $\gamma_c(G^2) = n - 3$. By Observation 2.12, $G \cong P_n$ or C_n and by Observation 5.2, $n = 4$. Hence $G \cong P_4$ or C_4. Converse can be easily verified. □

Theorem 5.6. *Let G be a connected graph of order n with at most one cycle. Then $\gamma_c(G) + \gamma_c(G^2) = 2n - 6$ if and only if G is either $P_5,\ C_5,\ P_6,\ C_6,\ K_{1,3}$ or $K_{1,3} + e$.*

Proof. Assume that $\gamma_c(G) + \gamma_c(G^2) = 2n - 6$. Then by Theorem 2.6 and Eq.(5.1), $\gamma_c(G) = n - 2$ and $\gamma_c(G^2) = n - 4$ (or) $\gamma_c(G) = \gamma_c(G^2) = n - 3$. In the former case, by Observation 2.12, $G \cong P_n$ or C_n. Then by Observation 5.2, $n = 5$ or 6. Hence $G \cong P_5,\ C_5,\ P_6,\ C_6$. Now consider the latter case. If G is a tree, then by Theorem 2.17, G is a spider with maximum degree 3. By observation (iii), $G \cong K_{1,3}$. If G is a unicyclic graph, then by Theorem 2.18 and observation (iii), $G \cong K_{1,3} + e$(paw). Converse is evident by verification. □

Theorem 5.7. *If G is any connected graph of order n and size m, then $\gamma_c(G) + \gamma_c(G^2) \leq 4m - 2n$ and equality holds if and only if $G \cong P_3$.*

Proof. The required upper bound follows from Theorem 2.2 and Eq.(5.1). If $\gamma_c(G)+\gamma_c(G^2) = 4m - 2n$, then $\gamma_c(G) = \gamma_c(G^2) = 2m - n$. By Theorem 2.2, $G \cong P_n$ and hence by Observation 5.2, $G \cong P_3$. Converse is obvious. □

Theorem 5.8. *For any connected graph G, $\gamma_c(G) + \gamma_c(G^2) = 4m - 2n - 1$ if and only if $G \cong P_4$.*

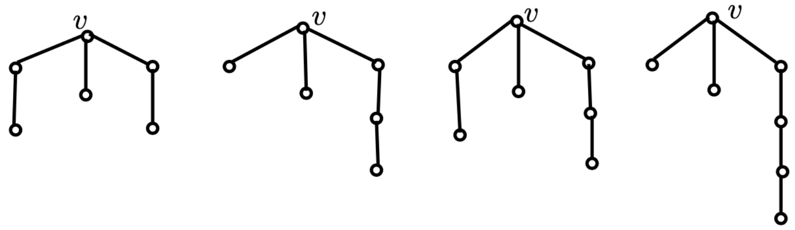
Proof. Assume that $\gamma_c(G) + \gamma_c(G^2) = 4m - 2n - 1$. Then by Theorem 2.2 and Eq.(5.1), $\gamma_c(G) = 2m - n$ and $\gamma_c(G^2) = 2m - n - 1$, and hence by Theorem 2.2 and Observation 5.2, $G \cong P_4$. Converse is obvious. □

Theorem 5.9. *Let G be a connected graph of order n and size m. Then $\gamma_c(G) + \gamma_c(G^2) = 4m - 2n - 2$ if and only if $G \cong P_5,\ P_6$ or $K_{1,3}$.*

Proof. If $\gamma_c(G) + \gamma_c(G^2) = 4m - 2n - 2$, then by Eq.(5.1) and Theorem 2.2, $\gamma_c(G) = 2m - n$ and $\gamma_c(G^2) = 2m - n - 2$ (or) $\gamma_c(G) = 2m - n - 1$ and $\gamma_c(G^2) = 2m - n - 1$. In the former case, by Theorem 2.2 and Observation 5.2, $n = 5$ or 6. Hence $G \cong P_5,\ P_6$. In the latter case, it is clear that $m = n - 1$ that because of this, G is a tree by Theorem 2.6. Then $\gamma_c(G) = \gamma_c(G^2) = n - 3$. By Theorem 2.17, G is a spider with maximum degree three. By observation (iii), $G \cong K_{1,3}$. Converse can be easily verified. □

Theorem 5.10. *Let G be a connected graph of order $n \geq 5$ and size m. Then $\gamma_c(G) + \gamma_c(G^2) = 4m - 2n - 3$ if and only if G is either $P_7,\ P_8$ or a double star with three end vertices.*

Proof. If $\gamma_c(G) + \gamma_c(G^2) = 4m - 2n - 3$, then by Eq.(5.1) and Theorem 2.2, $\gamma_c(G) = 2m - n$ and $\gamma_c(G^2) = 2m - n - 3$ (or) $\gamma_c(G) = 2m - n - 1$ and $\gamma_c(G^2) = 2m - n - 2$. In the former case, by Theorem 2.2 and Observation 5.2, $n = 7$ or 8, and hence $G \cong P_7$ or P_8. In the latter case, it is clear that $m = n - 1$ that because of this, G is a tree by Theorem 2.6. Then $\gamma_c(G) = n - 3$ and $\gamma_c(G^2) = n - 4$. By Theorem 2.17, G is a spider with maximum degree three. If $diam(G) \geq 4$, then G contains an induced P_5, say $v_1v_2v_3v_4v_5$. Since $\Delta(G) = 3$, either v_2 or v_3 is of degree three in G. Now consider G^2. Since $\Delta(G^2) \geq 5$, there is a spanning subgraph of G^2 with at least 5 end vertices and so $\gamma_c(G^2) \leq n - 5$, a contradiction. Thus, $diam(G) \leq 3$ and by hypothesis, G is a double star with three end vertices. Converse can be easily verified. □



$P_2[v(2P_3)]$ $P_2[v(P_2, P_4)]$ $P_2[v(P_3, P_4)]$ $P_2[v(P_2, P_5)]$

FIGURE 3. Graphs satisfying $\gamma_c(G) + \gamma_c(G^2) = 4m - 2n - 4$.

NOTATION 5.11 ([13]). If G is a graph with the vertex set $V = \{u_1, u_2, \dots\}$, then the graph obtained by identifying one of the end vertices of n_2 copies of P_2, n_3 copies of $P_3 \dots$ at u_1, m_2 copies of P_2, m_3 copies of $P_3 \dots$ at $u_2 \dots$ is denoted by $G[u_1(n_2P_2, n_3P_3, \dots);\ u_2(m_2P_2, m_3P_3, \dots); \dots]$.

THEOREM 5.12. *For any connected graph G of order $n \geq 3$ and size m, $\gamma_c(G) + \gamma_c(G^2) = 4m - 2n - 4$ if and only if G either C_3, P_9, P_{10}, $K_{1,4}$ or one of the graphs given in Fig. 3.*

Proof. If $\gamma_c(G) + \gamma_c(G^2) = 4m - 2n - 4$. Then we have three cases.

$$\gamma_c(G) = 2m - n \text{ and } \gamma_c(G^2) = 2m - n - 4 \tag{5.2}$$
$$\text{(or) } \gamma_c(G) = 2m - n - 1 \text{ and } \gamma_c(G^2) = 2m - n - 3 \tag{5.3}$$
$$\text{(or) } \gamma_c(G) = 2m - n - 2 \text{ and } \gamma_c(G^2) = 2m - n - 2 \tag{5.4}$$

From Eq.(5.2), by Theorem 2.2 and Observation 5.2, $G \cong P_9$, P_{10}. From Eq.(5.3), it is clear that $m = n - 1$ that because of this, G is a tree by Theorem 2.6. Then $\gamma_c(G) = n - 3$ and $\gamma_c(G^2) = n - 5$. By Theorem 2.17, G is a spider with maximum degree three. Let v be a vertex of degree three in G. We claim that $d(v, x) \leq 4$ for all end vertices x in G. If $d(v, x) \geq 5$ for some end vertex x in G, then since $\Delta(G^2) \geq 6$, there is a spanning subgraph of G^2 with at least 6 end vertices. Then $\gamma_c(G^2) \leq \gamma_c(H) \leq n - 6$, a contradiction. Hence $d(v, x) \leq 4$ for all end vertices x in G.

Case : 1 $N_4(v) \neq \emptyset$.
We claim that $G - N[v] = P_3$. If $G - N[v]$ has two P_3's or P_1 or P_2, then there is a spanning subgraph of G^2 which has at least 6 end vertices and so $\gamma_c(G^2) \leq n - 6$, a contradiction. Hence $G \cong P_2[v(P_2, P_5)]$.

Case : 2 $N_4(v) = \emptyset$ and $N_3(v) \neq \emptyset$.
We claim that $G - N[v] = P_2$ or $P_1 \cup P_2$. If $G - N[v]$ is union of two or more P_2's or $P_2 \cup P_1 \cup P_1$, then H is a spanning subgraph of G^2 which has at least 6 end vertices and so $\gamma_c(G^2) \leq n - 6$, a contradiction. Hence $G \cong P_2[v(P_2, P_4)]$ or $P_2[v(P_3, P_4)]$.

Case : 3 $N_3(v) = \emptyset$ and $N_2(v) \neq \emptyset$.
We claim that $G - N[v] = 2P_1$. If $G - N[v]$ is union of three or more P_1's, then by the similar argument, $\gamma_c(G^2) \leq n - 6$, a contradiction. Hence $G \cong P_2[v(2P_3)]$.

From Eq.(5.4), it is clear that $m = n - 1$ or n by Theorem 2.6. If $m = n$, then $\gamma_c(G) = n - 2 = \gamma_c(G^2)$. By Observation 2.12, $G \cong C_n$ and by Observation 5.2, $n = 3$. Hence $G \cong C_3$. If $m = n - 1$, then $\gamma_c(G) = n - 4 = \gamma_c(G^2)$, and we observe that by observation (iii), $G \cong K_{1,4}$. Converses are obvious by verification. $\square$

Theorem 5.13. *Let G be a connected graph with diameter two. Then the following holds.*

(a) $\gamma_c(G^2) = 1$.

(b) *If G is planar, then $\gamma_c(G) + \gamma_c(G^2) \leq 4$.*

(c) *Equality of (b) holds for regular planar graphs if and only if $G \cong C_5$ or the graphs $F_1,\ F_2$ given in Fig. 1.*

Proof. a) It follows immediately from Eq.(5.1) and result (ii).
b) It follows from Theorem 2.5 and Part (a).
c) To prove equality of Part (b) for regular graphs. If $\gamma_c(G) + \gamma_c(G^2) = 4$, then by Eq.(5.1) and Theorem 2.5, $\gamma_c(G) = 3$ and $\gamma_c(G^2) = 1$. Since G is planar and regular, let $d_G(v) = k$ and so $k \leq 5$. If G is 2-regular, then $G \cong C_n$. By Observation 2.12 and hypothesis, $n = 5$ and hence $G \cong C_5$. If G is 3-regular, then by Theorem 2.8, G is isomorphic to the Cartesian product $K_2 \times K_3$ for which $\gamma_c(G) = 2 \neq 3$. If G is 4-regular graph, then G is isomorphic to one of the graphs in Theorem 2.9, and $F_1,\ F_2$ are the only graphs satisfying. If G is 5-regular, then by Theorem 2.10, no such graph exist. Converse can be easily verified. □

6. Total Graphs

Definition 6.1. The *total graph* $T(G)$ of a graph G is a graph whose vertex set is $V(T(G)) = V(G) \cup E(G)$ and two distinct vertices x and y of $T(G)$ are adjacent if x and y are adjacent vertices of G or adjacent edges of G or a vertex and an edge incident with it in G.

In this section, we obtain some bounds for the sum of connected domination number of a graph and its total graph. We need the following results.

Theorem 6.2 ([4]). *Total graph $T(G)$ of a graph G is nothing but the square of the subdivision graph of G.*

Theorem 6.3. *(i) For any star $K_{1,n-1},\ \gamma_c(T(K_{1,n-1})) = 1$.*
(ii) For any complete graph $K_n,\ \gamma_c(T(K_n)) = n - 1$.

Proof. (i) It follows from Theorem 2.13 (ii). (ii) It is observed that by [8], $T(K_n) \cong L(K_{n+1})$. By Theorem 3.3, $\gamma_c(T(K_n)) = n - 1$. □

Theorem 6.4. *For any cycle $C_n,\ \gamma_c(T(C_n)) = n - 1$.*

Proof. It follows from Theorem 6.2 and Observation 5.2. □

Theorem 6.5. *For any connected graph G of order $n \geq 3,\ 1 \leq \gamma_c(T(G)) \leq n - 1$ and the bounds are sharp.*

Proof. By definition, $\gamma_c(T(G)) \geq 1$. Let u be any vertex of G. Since $T(G) \cong S(G)^2,\ S = V(G) - \{u\}$ is a connected dominating set of $T(G)$, and hence $\gamma_c(T(G)) \leq n - 1$. By Theorem 6.3, the bounds are sharp. □

Theorem 6.6. *If G is a connected graph of order $n \geq 3$, then $\gamma_c(T(G)) = 1$ if and only if G is a star.*

Proof. Assume that $\gamma_c(T(G)) = 1$. Let D be a γ_c-set of $T(G)$. We claim that $D \neq \{e\}$ for all $e \in E(G)$. If $D = \{e\}$ for some $e \in E(G)$, then D can dominate exactly two vertices which are incident with e, and hence $n = 2$, a contradiction. If $D = \{v\}$ for some $v \in V(G)$, then v is adjacent to all the remaining vertices of G and $L(G)$. Hence v is a full vertex which is incident with all edges. Thus, it is a star. Converse follows from Theorem 6.3 (i). □

THEOREM 6.7. *For any connected graph G of order n, $2 \leq \gamma_c(G) + \gamma_c(T(G)) \leq 2n-3$ with equality for the lower bound if and only if G is a star and equality for the upper bound if and only if $G \cong C_n$.*

Proof. Since $\gamma_c(G) \geq 1$, $\gamma_c(T(G)) \geq 1, \gamma_c(G) + \gamma_c(T(G)) \geq 2$. By Theorem 2.6, $\gamma_c(G) \leq n-2$, and by Theorem 6.5, the required upper bound holds. If $\gamma_c(G) + \gamma_c(T(G)) = 2$, then $\gamma_c(G) = 1$ and $\gamma_c(T(G)) = 1$. By Theorem 6.6, G is a star for which $\gamma_c(G) = 1$. Converse is obvious. If $\gamma_c(G) + \gamma_c(T(G)) = 2n-3$, then $\gamma_c(G) = n-2$ and $\gamma_c(T(G)) = n-1$. By Observation 2.12 and Theorem 6.4, $G \cong C_n$. Converse follows from Theorem 6.4, Corollary 2.14 and Observation 2.12. □

7. Block Graphs

DEFINITION 7.1. The *block graph* $B(G)$ of a graph G is a graph whose vertex set is the set of blocks in G and two vertices of $B(G)$ are adjacent if and only if the corresponding blocks have a common cut vertex in G.

In this section, we obtain some bounds for the sum of connected domination number of a graph and its block graph.

THEOREM 7.2. *Let G be a connected graph with n' blocks. Then $1 \leq \gamma_c(B(G)) \leq n'-2$.*

Proof. The lower bound is evident. Clearly, $B(G)$ is a graph of order n' and by Theorem 2.6, $\gamma_c(B(G)) \leq n'-2$. Hence $1 \leq \gamma_c(B(G)) \leq n'-2$. □

REMARK 7.3. If G is a block, then the lower bound of Theorem 7.2 is sharp. Also, if G has exactly two end blocks, then $\gamma_c(B(G)) = n'-2$, and so the upper bound of Theorem 7.2 is sharp.

THEOREM 7.4. *If G is a connected graph of order n and n' blocks, then $2 \leq \gamma_c(G) + \gamma_c(B(G)) \leq n + n' - 4$.*

Proof. It follows from Theorems 2.6 and 7.2. □

THEOREM 7.5. *For any connected graph G of order n and n' blocks, $\gamma_c(G) + \gamma_c(B(G)) = n + n' - 4$ if and only if $G \cong P_n$.*

Proof. If $\gamma_c(G)+\gamma_c(B(G)) = n+n'-4$, then $\gamma_c(G) = n-2$ and $\gamma_c(B(G)) = n'-2$. By Observation 2.12, $G \cong P_n$ or C_n. If $G \cong C_n$, then $\gamma_c(B(G)) = \gamma_c(K_1) = 1 \neq n'-2$. If $G \cong P_n$, then $\gamma_c(B(G)) = \gamma_c(P_{n'}) = n'-2$. Converse is obvious by verification. □

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