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ON THE ZEROS OF GENERALIZED DERIVATIVE OF A POLYNOMIAL

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Abstract. In this paper, we obtain some results concerning the location of zeros of generalized derivatives of polynomials which are analogous to those for the ordinary derivative of polynomials.

1. Introduction and Main Results

Let $f(z)$ be a complex polynomial of degree n and $\mathbb{R}^n_+$ be set of all $n-$tuples $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$ of positive real numbers with $\gamma_1+\gamma_2+...+\gamma_n = n$. Recall, all points z where $f(z)$ vanishes are called the zeros of $f(z)$ and all points z where $f'(z)$ vanishes are called the critical points of $f(z)$. The relationship between zeros and critical points of a polynomial is given by following classical Gauss-Lucas theorem [5, p.71].

Theorem 1.1. *Every convex set containing all the zeros of a polynomial also contains all its critical points.*

By fundamental theorem of algebra, [5, Theorem 1.1.2], every polynomial can be written as $f(z) = c\prod_{\nu=1}^{n}(z - z_k)$, where $z_1, z_2, ..., z_n$ are the zeros of $f(z)$ repeated as per their multiplicity.

Definition 1.2. For $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n) \in \mathbb{R}^n_+$. Sz-Nagy [7] introduced generalized derivative $f^\gamma(z)$ of $f(z)$, defined by

$$f^\gamma(z) = f(z)\sum_{\nu=1}^{n}\frac{\gamma_\nu}{z - z_\nu}, \quad \sum_{\nu=1}^{n}\gamma_\nu = n$$

Taking $\gamma = (1, 1, ..., 1)$, we obviously obtain $f^\gamma(z) = f'(z)$.

N. A. Rather et al. [6] have extended Theorem 1.1 to the generalized derivative of a polynomial. However, we present alternative and simple proof. More precisely, we prove:

Theorem 1.3. *Every convex set K containing all the zeros of a polynomial $f(z)$ also contains all the zeros of $f^\gamma(z)$, for all $\gamma \in \mathbb{R}^n_+$.*

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Proof. Since K is the intersection of half planes. It is sufficient to show the claim when K is half plane, which we may assume to be

$$K = \{z : \Re(z) \leq 0\}$$

Let $z_1, z_2, \ldots, z_n$ be the zeros of $f(z)$, then $\Re(z_\nu) \leq 0$. Now if $z \notin K$, then $\Re(z) > 0$. Hence $\Re(z - z_\nu) > 0$, $\forall\ \nu$ and so

$$\Re\frac{1}{z - z_\nu} = \frac{\Re(z - z_\nu)}{|z - z_\nu|^2} > 0 \tag{1.1}$$

But

$$\frac{f^\gamma(z)}{f(z)} = \sum_{\nu=1}^{n} \frac{\gamma_\nu}{z - z_\nu}$$

We have

$$\Re\frac{f^\gamma(z)}{f(z)} = \sum_{\nu=1}^{n} \gamma_\nu \Re\frac{1}{z - z_\nu} > 0$$

shows that $f^\gamma(z) \neq 0$, for $z \notin K$. □

Next, we prove the following interesting result which includes Theorem 1.3 as a special case.

THEOREM 1.4. *If all the zeros of the polynomial $f(z)$ lie in $|z - c| \leq R$ and if w is any real or complex number satisfying the inequality*

$$|(w - c)f^\gamma(w)| \leq |(w - c)f^\gamma(w) - nf(w)|$$

for every $\gamma \in \mathbb{R}^n_+$, then $|w - c| \leq R$.

Proof. Let w be real or complex number satisfying the inequality

$$|(w - c)f^\gamma(w)| \leq |(w - c)f^\gamma(w) - nf(w)| \tag{1.2}$$

If $f(w) = 0$, then clearly $|w - c| \leq R$. So suppose $f(w) \neq 0$. From (1.2), we have

$$\left|\frac{(w - c)f^\gamma(w)}{nf(w)}\right| \leq \left|1 - \frac{(w - c)f^\gamma(w)}{nf(w)}\right|$$

which implies

$$\Re\frac{(w - c)f^\gamma(w)}{nf(w)} \leq \frac{1}{2}$$
$$\implies \Re\frac{(w - c)f^\gamma(w)}{f(w)} \leq \frac{n}{2}$$

Let $z_1, z_2, \ldots, z_n$ be the zeros of $f(z)$, then

$$\frac{(w - c)f^\gamma(w)}{f(w)} = \sum_{\nu=1}^{n} \frac{(w - c)\gamma_\nu}{w - z_\nu}.$$

Now

$$
\begin{aligned}
\sum_{\nu=1}^{n} \Re\left(\frac{w-c}{w-z_\nu}\right)\gamma_\nu &= \Re\sum_{\nu=1}^{n}\left(\frac{w-c}{w-z_\nu}\right)\gamma_\nu \\
&= \Re\frac{(w-c)f^{\gamma}(w)}{f(w)} \\
&\leq \frac{n}{2}
\end{aligned}
$$

So

$$\sum_{\nu=1}^{n} \Re\left(\frac{w-c}{w-z_\nu}\right)\gamma_\nu \leq \frac{n}{2}$$

which implies

$$\Re\left(\frac{w-c}{w-z_\nu}\right) \leq \frac{1}{2\gamma_\nu} \leq \frac{1}{2}$$

for at least one ν. This gives

$$\left|\frac{w-c}{w-z_\nu}\right| \leq \left|1-\frac{w-c}{w-z_\nu}\right| = \left|\frac{z_\nu-c}{w-z_\nu}\right|$$

which implies

$$|w-c| \leq |z_\nu - c|$$

for at least one ν. Using now the fact that

$$|z_\nu - c| \leq R, \quad \forall \quad \nu$$

We get

$$|w-c| \leq R$$

□

REMARK 1.5. If all the zeros of $f(z)$ lie in the circle $|z-c| \leq R$ and w is any zero of $f^{\gamma}(z)$, then $f^{\gamma}(w) = 0$, so that inequality (1.2) is trivally satisfied. Hence by above theorem $|w-c| \leq R$. This shows that all the zeros of $f^{\gamma}(z)$ lie in $|z-c| \leq R$.

Concerning the location of critical points of a non-constant polynomial with real coefficients, according to Rolle's theorem there is at least one real critical point between any two consecutive real zeros. Thus, for a polynomial with real coefficients the number of non-real critical points cannot exceed the number of non-real zeros. In this situation, the non-real zeros occur in conjugate pairs. As regards the location of the critical points, this information is being used to derive an interesting result called Jensen's theorem [4] which is not covered by Gauss– Lucas's theorem as the region, containing the critical points, obtained by using Jensen's theorem, is smaller than that given by Gauss–Lucas theorem.

DEFINITION 1.6. Let $f(z)$ be a polynomial with real coefficients. Denoting by $z_1, z_2, ..., z_n$, those zeros which lie in the upper half plane, the disks

$$D_\mu = \{z \in \mathbb{C} : |z - \Re z_\mu| \leq \Im z_\mu\}, \quad (\mu = 1, 2, ..., n)$$

are referred to as the Jensen disks of $f(z)$.

THEOREM 1.7 (Jensen). *Let $f(z)$ be a polynomial with real coefficients. Then the non-real critical points of $f(z)$ lie in the union of all the Jensen disks of $f(z)$.*

Next, we extend Jensen's theorem to the generalized derivative of polynomials with real coefficients. In fact, we prove

THEOREM 1.8. *Let $f(z)$ be a polynomial with real coefficients. For every $\gamma \in \mathbb{R}^n_+$, with $\gamma_\nu = \gamma_\mu$, if γ_ν is non real zero and γ_μ is its conjugate, non-real zeros of $f^\gamma(z)$ lie in the union of Jensen disks of $f(z)$.*

Proof. Let $z_\nu = \alpha + i\beta$ and $z_\mu = \alpha - i\beta$ be a pair of complex conjugate roots and let $z = x + iy$, then

$$\begin{aligned}
&\frac{\gamma_\nu}{z - z_\nu} + \frac{\gamma_\mu}{z - z_\mu} \\
&= \frac{\gamma_\nu}{(x + iy) - (\alpha + i\beta)} + \frac{\gamma_\mu}{(x + iy) - (\alpha - i\beta)} \\
&= \gamma_\nu \frac{2(x - \alpha)\left[(x - \alpha)^2 + y^2 + \beta^2\right] - i2y\left[(x - \alpha)^2 + y^2 - \beta^2\right]}{\left[(x - \alpha)^2 + (y - \beta)^2\right]\left[(x - \alpha)^2 + (y + \beta)^2\right]}
\end{aligned}$$

The coefficient of i is opposite in sign to y if $(x - \alpha)^2 + y^2 > \beta^2$, that is when z lies outside the Jensen circle $(x - \alpha)^2 + y^2 = \beta^2$.

In a simillar manner, for a real zero

$$\Im \frac{1}{z - \alpha} = \frac{-y}{|z - \alpha|^2}$$

which also has a sign opposite to that of y for the coefficient i.

Hence at any point exterior to all Jensen circles and not on real axis, the coefficient of i in $f^\gamma(z)/f(z)$ does not vanish.

□

REMARK 1.9. For $\gamma = (1, 1, ..., 1)$ in above theorem, we obtain Jensen Theorem.

Consider the following class of polynomials

$$P_n = \left\{ f(z) = z \prod_{\nu=1}^{n-1} (z - z_\nu), where \quad |z_\nu| \geq 1 \quad for \quad 1 \leq \nu \leq n - 1 \right\}$$

Sendov conjecture states that if all the zeros of $f(z)$ lie in $|z| \leq 1$, then for any zero z_0 of $f(z)$ the disk $|z - z_0| \leq 1$ contains at least one critical point of $f(z)$. In this connection, Brown [2] posed a problem that if $f(z) \in P_n$. Find the best constant C_n such that $f'(z)$ does not vanish in $|z| < C_n$ for all $f(z) \in P_n$. Brown himself observed that if $f(z) = z(z - 1)^{n-1}$, then $f'(\frac{1}{n}) = 0$ and conjectured that $C_n = \frac{1}{n}$. Aziz and Zargar [1] was able to solve this problem.

Next, in this paper, we prove the following result for generalized derivative of a polynomial which includes Brown's Conjecture as a special case.

THEOREM 1.10. *Let* $f(z) = z^m \prod_{\nu=1}^{n-m}(z - z_\nu)$, *where* $|z_\nu| \geq 1$ *for* $1 \leq \nu \leq n - m$, *then* $f^\gamma(z) \neq 0$, *for* $z \in \mathbb{C}$ *with* $0 < |z| < \frac{m}{n}$.

Proof. If $m = n$, the assertion is clearly true. Therefore assume that $m < n$, so $\frac{m}{n} < 1$.

We write $f(z) = z^m Q(z)$, where $Q(z) = \prod_{\nu=1}^{n-m}(z - z_\nu)$, then by definition of $f^\gamma(z)$, we obtain

$$
\begin{aligned}
f^\gamma(z) &= z^m Q(z)\left[\sum_{\nu=1}^{m}\frac{\gamma_\nu}{z} + \sum_{\nu=1}^{n-m}\frac{\gamma_{m+\nu}}{z - z_\nu}\right] \\
&= \sum_{\nu=1}^{m}\gamma_\nu z^{m-1}Q(z) + z^m Q^\delta(z)
\end{aligned}
\tag{1.3}
$$

where $Q^\delta(z)$ is generalized derivative of polynomial $Q(z)$ whose degree is $n - m$ and $\delta = (\gamma_{m+1}, \gamma_{m+2}, ..., \gamma_n)$ is $(n - m)$-tuple such that $\sum_{\nu=1}^{n-m}\gamma_{m+\nu} = n - m$ and so $\sum_{\nu=1}^{m}\gamma_\nu = m.$

Let z be such that $0 < |z| < \frac{m}{n}$, then $|z| < \frac{m}{n}$ implies that $m/|z| > n$. Since given that $Q(z)$ does not vanish in $0 < |z| < 1$. So $Q(z)$ does not vanish in $0 < |z| < \frac{m}{n}$.

Also, for the zeros of $Q^\delta(z)$, we have

$$
\begin{aligned}
\left|\sum_{\nu=1}^{n-m}\frac{\gamma_{m+\nu}}{z - z_\nu}\right| &\leq \sum_{\nu=1}^{n-m}\frac{\gamma_{m+\nu}}{|z - z_\nu|} \\
&< \sum_{\nu=1}^{n-m}\frac{\gamma_{m+\nu}}{1 - \frac{m}{n}} \\
&= \frac{n}{n-m}\sum_{\nu=1}^{n-m}\gamma_{m+\nu} = n
\end{aligned}
$$

Thus the factor on R.H.S of (1.3) does not vanish in $0 < |z| < \frac{m}{n}$.
Hence $f^\gamma(z) \neq 0$, for $z \in \mathbb{C}$ with $0 < |z| < \frac{m}{n}$. □

REMARK 1.11. For $\gamma = (1, 1, ..., 1)$ and $m = 1$ in above theorem, we obtain Brown's Conjecture.

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