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STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS IN RANDOM NORMED SPACES

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability of the following quadratic functional equations

$$\begin{aligned} & cf\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n} x_i - (n+c-1)x_j\right) \\ = \; & (n+c-1)\left(f(x_1) + c\sum_{i=2}^{n} f(x_i) + \sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} f(x_i - x_j)\right)\right), \end{aligned}$$

$$Q\left(\sum_{i=1}^{n} d_i x_i\right) + \sum_{1\le i<j\le n} d_i d_j Q(x_i - x_j) = \left(\sum_{i=1}^{n} d_i\right)\left(\sum_{i=1}^{n} d_i Q(x_i)\right)$$

in random normed spaces.

1. Introduction

The stability problem of functional equations was originated from a question of Ulam [29] in 1940, concerning the stability of group homomorphisms. Let $(G_1, .)$ be a group and let $(G_2, *, d)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, under what

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condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Hyers [13] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$ and some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear. Th.M. Rassias [23] provided a generalization of the Hyers' theorem which allows the Cauchy difference to be unbounded. Gajda [8] answered the question for the case $p > 1$, which was raised by Th.M. Rassias. This new concept is known as generalized Hyers-Ulam stability of functional equations (see [1]–[3], [6, 9], [14]–[16], [24, 25]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation (1.1) is said to be a *quadratic mapping*. The generalized Hyers-Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for mappings $f : A \to B$, where A is a normed space and B is a Banach space (see [28]). Cholewa [5] noticed that the theorem of Skof is still true if relevant domain A is replaced by an abelian group. In [7], Czerwik proved the generalized Hyers-Ulam stability of the functional equation (1.1). Grabiec [10] has generalized these results mentioned above.

The generalized Hyers-Ulam stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in [17] and [20]–[22]. It should be noticed that in all these papers the triangle inequality is expressed by using the strongest triangular norm T_M.

The aim of this paper is to investigate the generalized Hyers-Ulam stability of the following quadratic functional equations

$$(1.2)\ cf\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n} x_i - (n+c-1)x_j\right)$$
$$= (n+c-1)\left(f(x_1) + c\sum_{i=2}^{n} f(x_i) + \sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} f(x_i - x_j)\right)\right),$$

$$(1.3)\ Q\left(\sum_{i=1}^{n} d_i x_i\right) + \sum_{1\le i<j\le n} d_i d_j Q(x_i - x_j) = \left(\sum_{i=1}^{n} d_i\right)\left(\sum_{i=1}^{n} d_i Q(x_i)\right)$$

in random normed spaces in the sense of Sherstnev under arbitrary continuous t-norms.

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [4, 18, 19, 26, 27]. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.1. ([26]) *A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:*
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a, 1) = a$ for all $a \in [0, 1]$;
(d) $T(a, b) \le T(c, d)$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous t-norms are $T_P(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ (the Lukasiewicz t-norm). Recall (see [11, 12]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, then $T_{i=1}^{n}x_i$ is defined recurrently by $T_{i=1}^{1}x_i = x_1$ and $T_{i=1}^{n}x_i = T(T_{i=1}^{n-1}x_i, x_n)$ for $n \geq 2$. $T_{i=n}^{\infty}x_i$ is defined as $T_{i=1}^{\infty}x_{n+i-1}$. It is known ([12]) that for the Lukasiewicz t-norm the following implication holds:

$$\lim_{n\to\infty}(T_L)_{i=1}^{\infty}x_{n+i-1} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty.$$

Definition 1.2. ([27]) *A* random normed space *(briefly, RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm and μ is a mapping from X into D^+ such that the following conditions hold:*
(RN_1) *$\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;*
(RN_2) *$\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \neq 0$;*
(RN_3) *$\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.*

Every normed space $(X, \|.\|)$ defines a random normed space (X,μ,T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 1.3. *Let (X,μ,T) be an RN-space.*
(1) *A sequence $\{x_n\}$ in X is said to be* convergent *to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\epsilon) > 1-\lambda$ whenever $n \geq N$.*
(2) *A sequence $\{x_n\}$ in X is called a* Cauchy sequence *if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1-\lambda$ whenever $n \geq m \geq N$.*
(3) *An RN-space (X,μ,T) is said to be* complete *if and only if every Cauchy sequence in X is convergent to a point in X.*

Theorem 1.4. ([26]) *If (X,μ,T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty}\mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

This paper is organized as follows: In Section 2, we prove the generalized Hyers-Ulam stability of the quadratic functional equation (1.2) in RN-spaces. In Section 3, we prove the generalized Hyers-Ulam stability of the quadratic functional equation (1.3) in RN-spaces.

Throughout this paper, assume that X is a real vector space and that (Y,μ,T) is a complete RN-space.

2. Generalized Hyers-Ulam stability of the quadratic functional equation (1.2) in random normed spaces

For a given mapping $f : X \to Y$, consider the mapping $Pf : X^n \to Y$, defined by

$$Pf(x_1, x_2, \cdots, x_n) = cf\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n} x_i - (n+c-1)x_j\right)$$
$$-(n+c-1)\left(f(x_1) + c\sum_{i=2}^{n} f(x_i) + \sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} f(x_i - x_j)\right)\right)$$

for all $x_1, \cdots, x_n \in X$.

In this section, we prove the generalized Hyers-Ulam stability of the functional equation $Pf(x_1, x_2, \cdots, x_n) = 0$ in complete RN-spaces.

Theorem 2.1. Let $f : X \to Y$ be an even mapping for which there is a $\rho : X^n \to D^+$
($\rho(x_1, x_2, \cdots, x_n)$ is denoted by $\rho_{x_1,x_2,\cdots,x_n}$) such that

$$\mu_{Pf(x_1,x_2,\cdots,x_n)}(t) \ge \rho_{x_1,x_2,\cdots,x_n}(t) \tag{2.1}$$

for all $x_1, x_2, \cdots, x_n \in X$ and all $t > 0$. *Let* $v = 2 - c - n > 1$. If

$$\lim_{n\to\infty} T_{k=1}^{\infty}\left(\rho_{0,v^{n+k-1}x,0,0,\cdots,0}\left(v^{2n+k}(v-1)t\right)\right) = 1, \tag{2.2}$$

$$\lim_{n\to\infty} \rho_{v^n x_1, v^n x_2,\cdots,v^n x_n}(v^{2n}t) = 1 \tag{2.3}$$

hold for all $x, y \in X$ and all $t > 0$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \ge T_{k=1}^{\infty}\left(\rho_{0,v^{k-1}x,0,0,\cdots,0}(v^k(v-1)t)\right) \tag{2.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $x_2 = x$ and $x_1 = x_3 = x_4 = \cdots = x_n = 0$ in (2.1), we get

$$\mu_{f((2-c-n)x)-(2-c-n)^2 f(x)}(t) \ge \rho_{0,x,0,0,\cdots,0}(t) \tag{2.5}$$

for all $y \in X$ and all $t > 0$. Replacing $2 - c - n$ by v in (2.5), we get

$$\mu_{f(vx)-v^2 f(x)}(t) \ge \rho_{0,x,0,0,\cdots,0}(t) \tag{2.6}$$

for all $y \in X$ and all $t > 0$. Thus we have

$$\mu_{\frac{f(vx)}{v^2}-f(x)}(t) \ge \rho_{0,x,0,0,\cdots,0}\left(v^2 t\right)$$

for all $x \in X$ and all $t > 0$. Hence

$$\mu_{\frac{f(v^{k+1}x)}{v^{2(k+1)}} - \frac{f(v^k x)}{v^{2k}}}(t) \geq \rho_{0,v^k x,0,0,\cdots,0}\left(v^{2(k+1)}t\right)$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. From $\frac{1}{v-1} > \frac{1}{v} + \frac{1}{v^2} + \cdots + \frac{1}{v^n}$ ($v > 1$), it follows that

$$\begin{aligned} \mu_{\frac{f(v^n x)}{v^{2n}} - f(x)}(t) &\geq T_{k=1}^n \left(\mu_{\frac{f(v^k x)}{v^{2k}} - \frac{f(v^{(k-1)}x)}{v^{2(k-1)}}}\left(\frac{(v-1)\,t}{v^k}\right)\right) \\ &\geq T_{k=1}^n \left(\rho_{0,v^{k-1}x,0,0,\cdots,0}\left(v^k\,(v-1)\,t\right)\right) \end{aligned} \tag{2.7}$$

for all $x \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\{\frac{f(v^n x)}{v^{2n}}\}$, replacing x with $v^m x$ in (2.7), we obtain that

$$\begin{aligned} &\mu_{\frac{f(v^{n+m}x)}{v^{2(n+m)}} - \frac{f(v^m x)}{v^{2m}}}(t) \\ &\quad \geq T_{k=1}^n \left(\rho_{0,v^{k+m-1}x,0,0,\cdots,0}\left(v^{k+2m}\,(v-1)\,t\right)\right). \end{aligned} \tag{2.8}$$

Since the right hand side of the inequality (2.8) tends to 1 as m and n tend to infinity, the sequence $\{\frac{f(v^n x)}{v^{2n}}\}$ is a Cauchy sequence. Thus we may define $Q(x) = \lim_{n\to\infty} \frac{f(v^n x)}{v^{2n}}$ for all $x \in X$.

Now we show that Q is an quadratic mapping. Replacing x_i with $v^n x_i$ $(i = 1, 2, \cdots, n)$ in (2.1), respectively, we get

$$\mu_{\frac{Pf(v^n x_1, v^n x_2, \cdots, v^n x_n)}{v^2 n}}(t) \geq \rho_{v^n x_1, v^n x_2, \cdots, v^n x_n}(v^{2n}t). \tag{2.9}$$

Taking the limit as $n \to \infty$, we find that $\frac{Pf(v^n x_1, v^n x_2, \cdots, v^n x_n)}{v^2 n}(t)$ tends to 0, which implies that the mapping $Q : X \to Y$ is quadratic. Letting the limit as $n \to \infty$ in (2.8), we get (2.4).

Next, we prove the uniqueness of the quadratic mapping $Q : X \to Y$ subject to (2.4). Let us assume that there exists another quadratic mapping $R : X \to Y$ which satisfies (2.4). Since $Q(v^n x) = v^{2n}Q(x)$, $R(v^n x) = v^{2n}R(x)$ for all $x \in X$ and all $n \in \mathbb{N}$, from (2.4), it follows that

$$\begin{aligned} \mu_{Q(x)-R(x)}(vt) &= \mu_{Q(v^n x)-R(v^n x)}(v^{2n+1}t) \\ &\geq T(\mu_{Q(v^n x)-f(v^n x)}(v^{2n}t), \mu_{f(v^n x)-R(v^n x)}(v^{2n}t)) \\ &\geq T\left(T_{k=1}^{\infty}\left(\rho_{0,v^{n+k-1}x,0,0,\cdots,0}\left(v^{2n+k}\,(v-1)\,t\right)\right),\right. \\ &\qquad \left. T_{k=1}^{\infty}\left(\rho_{0,v^{n+k-1}x,0,0,\cdots,0}\left(v^{2n+k}\,(v-1)\,t\right)\right)\right) \end{aligned} \tag{2.10}$$

for all $x \in X$ and all $t > 0$. Letting $n \to \infty$ in (2.10), we conclude that $Q = R$. □

THEOREM 2.2. Let $f : X \to Y$ be an even mapping for which there is a $\rho : X^n \to D^+$
($\rho_{(x_1,x_2,\cdots,x_n)}$ is denoted by $\rho_{x_1,x_2,\cdots,x_n}$) such that

$$\mu_{Pf(x_1,x_2,\cdots,x_n)}(t) \geq \rho_{x_1,x_2,\cdots,x_n}(t) \tag{2.11}$$

for all $x_1, x_2, \cdots, x_n \in X$ and all $t > 0$. (Let $v = 2 - c - n$, $0 < v < 1$) If

$$\lim_{n\to\infty} T_{k=1}^{\infty} \left(\rho_{0,\frac{x}{v^{n+k}},0,0,\cdots,0} \left(\frac{(v-1)\,t}{v^{2n+k-1}} \right) \right) = 1, \tag{2.12}$$

$$\lim_{n\to\infty} \rho_{\frac{x_1}{v^n},\frac{x_2}{v^n},\cdots,\frac{x_n}{v^n}} \left(\frac{t}{v^{2n}} \right) = 1 \tag{2.13}$$

hold for all $x \in X$ and all $t > 0$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq T_{k=1}^{\infty} \left(\rho_{0,\frac{x}{v^k},0,0,\cdots,0} \left(\frac{(v-1)\,t}{v^{k-1}} \right) \right) \tag{2.14}$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $x_2 = x$ and $x_1 = x_3 = x_4 = \cdots = x_n = 0$ in (2.11), we get

$$\mu_{f((2-c-n)x)-(2-c-n)^2 f(x)}(t) \geq \rho_{0,x,0,0,\cdots,0}(t) \tag{2.15}$$

for all $x \in X$ and all $t > 0$. Replacing $2 - c - n$ by v and x by $\frac{x}{v}$ in (2.15), we get

$$\mu_{f(x)-v^2 f\left(\frac{x}{v}\right)}(t) \geq \rho_{0,\frac{x}{v},0,0,\cdots,0}(t) \tag{2.16}$$

for all $x \in X$ and all $t > 0$. Hence

$$\mu_{v^{2k} f\left(\frac{x}{v^k}\right) - v^{2(k+1)} f\left(\frac{x}{v^{k+1}}\right)}(t) \geq \rho_{0,\frac{x}{v^{k+1}},0,0,\cdots,0} \left(\frac{t}{v^{2k}} \right)$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. From $\frac{1}{1-v} > 1 + v + \cdots + v^{n-1}$ ($0 < v < 1$), it follows that

$$\begin{aligned} \mu_{f(x)-v^{2n} f\left(\frac{x}{v^n}\right)}(t) &\geq T_{k=1}^{n} \left(\mu_{v^{2(k-1)} f\left(\frac{x}{v^{k-1}}\right) - v^{2k} f\left(\frac{x}{v^k}\right)} \left(v^{k-1}\,(1-v)\,t \right) \right) \\ &\geq T_{k=1}^{n} \left(\rho_{0,\frac{x}{v^k},0,0,\cdots,0} \left(\frac{(v-1)\,t}{v^{k-1}} \right) \right) \end{aligned} \tag{2.17}$$

for all $x \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\{v^{2n} f\left(\frac{x}{v^n}\right)\}$, replacing x with $\frac{x}{v^m}$ in (2.17), we obtain that

$$\mu_{v^{2m} f\left(\frac{x}{v^m}\right) - v^{2(m+n)} f\left(\frac{x}{v^{m+n}}\right)}(t) \geq T_{k=1}^{n} \left(\rho_{0, \frac{x}{v^{k+m}}, 0, 0, \cdots, 0}\left(\frac{(v-1)\,t}{v^{k+2m-1}}\right)\right). \tag{2.18}$$

Since the right hand side of the inequality (2.18) tends to 1 as m and n tend to infinity, the sequence $\{v^{2n} f\left(\frac{x}{v^n}\right)\}$ is a Cauchy sequence. Thus we may define $Q(x) = \lim_{n\to\infty} v^{2n} f\left(\frac{x}{v^n}\right)$ for all $x \in X$.

Now we show that Q is a quadratic mapping. Replacing x_i with $\frac{x_i}{v^n}$ $(i = 1, 2, \cdots, n)$ in (4.1), respectively, we get

$$\mu_{v^{2n} Pf\left(\frac{x_1}{v^n}, \frac{x_2}{v^n}, \cdots, \frac{x_n}{v^n}\right)}(t) \geq \rho_{\frac{x_1}{v^n}, \frac{x_2}{v^n}, \cdots, \frac{x_n}{v^n}}\left(\frac{t}{v^{2n}}\right). \tag{2.19}$$

Taking the limit as $n \to \infty$, we find that $v^{2n} Pf\left(\frac{x_1}{v^n}, \frac{x_1}{v^n}, \cdots, \frac{x_1}{v^n}\right)$ tends to 0, which implies that the mapping $Q : X \to Y$ is quadratic. Letting the limit as $n \to \infty$ in (2.18), we get (2.14).

The rest of the proof is similar to the proof of Theorem 2.1. □

3. Generalized Hyers-Ulam stability of the quadratic functional equation (1.3) in random normed spaces

For a given mapping $Q : X \to Y$, consider the mapping $DQ : X^n \to Y$, defined by

$$\begin{aligned} DQ(x_1, x_2, \cdots, x_n) : &= Q\left(\sum_{i=1}^{n} d_i x_i\right) + \sum_{1\leq i<j\leq n} d_i d_j Q(x_i - x_j) \\ &\quad - \left(\sum_{i=1}^{n} d_i\right)\left(\sum_{i=1}^{n} d_i Q(x_i)\right) \end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in X$ and let $d = \sum_{i=1}^{n} d_i$

In this section, we prove the generalized Hyers-Ulam stability of the functional equation $DQ(x_1, x_2, \cdots, x_n) = 0$ in complete RN-spaces.

THEOREM 3.1. *Let $Q : X \to Y$ be an even mapping for which there is a $\rho : X^n \to D^+$ ($\rho(x_1, x_2, \cdots, x_n)$ is denoted by $\rho_{x_1, x_2, \cdots, x_n}$) satisfying $Q(0) = 0$ and $d > 1$. If*

$$\mu_{DQ(x_1, x_2, \cdots, x_n)} \geq \rho_{x_1, x_2, \cdots, x_n}(t) \tag{3.1}$$

for all $x_1, x_2, \cdots, x_n \in X$ *and all* $t > 0$, *and*

$$\lim_{n\to\infty} \rho_{d^{(n-1)}x,\cdots,d^{(n-1)}x}((d^2-1)t) = 1 \tag{3.2}$$

for all $x, y \in X$ *and all* $t > 0$, *then there exists a unique quadratic mapping* $R : X \to Y$ *such that*

$$\mu_{R(x)-Q(x)}(t) = T_{k=1}^{\infty}(\rho_{d^{k-1}x,d^{k-1}x,\cdots,d^{k-1}x}(d^2-1)) \tag{3.3}$$

for all $x \in X$ *and all* $t > 0$.

Proof. Putting $x_1 = x_2 = \cdots = x_n = x$ in (3.1), we get

$$\mu_{Q(dx)-d^2Q(x)}(t) \geq \rho_{x,x,\cdots,x}(t) \tag{3.4}$$

and so

$$\mu_{Q(dx)/d^2-Q(x)}(\frac{t}{d^2}) \geq \rho_{x,x,\cdots,x}(t) \tag{3.5}$$

for all $x \in X$ and all $t > 0$. Let

$$\psi_x(t) = \rho_{x,x,\cdots,x}(t)$$

Replacing x by $d^n x$ and t by $d^{2(n+1)}t$ in (3.5), we get

$$\begin{aligned} \mu_{Q(d^{n+1}x)/d^2-Q(d^nx)}(d^{2n}t) &\geq \psi_x(d^{2(n+1)}t), \\ \mu_{Q(d^{n+1}x)/d^{2(n+1)}-Q(d^nx)/d^{2n}}(t) &\geq \psi_x(d^{2(n+1)}t) \end{aligned} \tag{3.6}$$

for all $n \in N$, $x \in X$ and all $t > 0$. It follows from (3.6) and $1 \geq (d^2-1)(\frac{1}{d^2} + \frac{1}{d^4} + \cdots + \frac{1}{d^{2n}})$ that

$$\begin{aligned} \mu_{Q(d^nx)/d^{2n}-Q(x)}(t) &= \mu_{(Q(d^nx)/d^{2n}-Q(d^{(n-1)}x)/d^{2(n-1)}+\cdots+Q(dx)/d^2-Q(x))}(t) \\ &\geq T_{k=1}^{n}\left(\mu_{(Q(d^kx)/d^{2k}-Q(d^{(k-1)}x)/d^{2(k-1)}}\left(\frac{(d^2-1)t}{d^{2k}}\right)\right) \\ &\geq T_{k=1}^{n}\left(\psi_{d^{(k-1)}x}((d^2-1)t)\right) \\ &= T_{k=1}^{n}\left(\rho_{d^{(k-1)}x,d^{(k-1)}x,\cdots,d^{(k-1)}x}((d^2-1)t)\right) \end{aligned} \tag{3.7}$$

for all $x \in X$ and all $n \in N$. Thus we have

$$\begin{aligned} &\mu_{Q(d^nx)/d^{2n}-Q(d^m)x/d^{2m}}(t) \\ &\qquad \geq T_{k=m}^{n}\left(\rho_{d^{(k-1)}x,d^{(k-1)}x,\cdots,d^{(k-1)}x}((d^2-1)t)\right). \end{aligned} \tag{3.8}$$

Since the right hand side of the inequality (3.8) tends to 1 as m, n tend to infinity, the sequence $(\frac{Q(d^nx)}{d^{2n}})$ is a Cauchy sequence. Thus we may define $R(x) = \lim_{n\to\infty} \frac{Q(d^nx)}{d^{2n}}$ for all $x \in X$. Then

$$\mu_{R(x)-Q(x)}(t) = T_{k=1}^{\infty}\left(\rho_{d^{(k-1)}x,d^{(k-1)}x,\cdots,d^{(k-1)}x}((d^2-1)t)\right) \tag{3.9}$$

Now we show that R is a quadratic mapping. Putting $x_1 = x_2 = \cdots = x_n = d^n x$ in (3.1), we get

$$\mu_{\frac{DQ(d^n x, d^n x, \cdots, d^n x)}{d^{2n}}}(t) \geq \rho_{d^n x, d^n x, \cdots, d^n x}(t).$$

Taking the limit as $n \to \infty$, we find that $R : X \to Y$ satisfies (3.1) for all $x, y \in X$. Since $Q : X \to Y$ is even, $R : X \to Y$ is even. So the mapping $R : X \to Y$ is quadratic. Letting the limit as $n \to \infty$ in (3.9), we get (3.3).

Next, we prove the uniqueness of the quadratic mapping $R : X \to Y$ subject to (3.3). Let us assume that there exists another quadratic mapping $L : X \to Y$ which satisfies (3.3). Since $R(d^n x) = d^{2n} R(x)$, $L(d^n x) = d^{2n} L(x)$ for all $x \in X$ and all $n \in \mathbb{N}$, from (3.8), it follows that

$$\begin{aligned}
(3.10) \quad \mu_{R(x)-L(x)}(2t) &= \mu_{R(d^n x)-L(d^n x)}(2 \cdot d^{2n} t) \\
&\geq T(\mu_{R(d^n x)-Q(d^n x)}(d^{2n} t), \mu_{Q(d^n x)-L(d^n x)}(d^{2n} t)) \\
&\geq T(T_{k=1}^{\infty} \left(\rho_{d^{n+(k-1)} x, d^{n+(k-1)} x, \cdots, d^{n+(k-1)} x}(d^2 - 1) d^{2n} t)\right), \\
&\quad (T_{k=1}^{\infty} \left(\rho_{d^{n+(k-1)} x, d^{n+(k-1)} x, \cdots, d^{n+(k-1)} x}((d^2 - 1) d^{2n} t)\right)
\end{aligned}$$

for all $x \in X$ and all $t > 0$. Letting $n \to \infty$ in (3.10), we conclude that $R = L$. □

THEOREM 3.2. *Let $Q : X \to Y$ be an even mapping for which there is a $\rho : X^n \to D^+$ ($\rho(x_1, x_2, \cdots, x_n)$ is denoted by $\rho_{x_1, x_2, \cdots, x_n}$) satisfying $Q(0) = 0$ and $0 < d < 1$. If*

$$(3.11) \qquad \mu_{DQ(x_1, x_2, \cdots, x_n)} \geq \rho_{x_1, x_2, \cdots, x_n}(t)$$

for all $x_1, x_2, \cdots, x_n \in X$ and all $t > 0$, and

$$(3.12) \qquad \lim_{n \to \infty} \rho_{\frac{x}{d^n}, \frac{x}{d^n}, \cdots, \frac{x}{d^n}}((1 - d^2) t) = 1$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique quadratic mapping $R : X \to Y$ such that

$$(3.13) \qquad \mu_{R(x)-Q(x)}(t) = T_{k=1}^{\infty} \left(\rho_{\frac{x}{d^k}, \frac{x}{d^k}, \cdots, \frac{x}{d^k}}((1 - d^2) t)\right)$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $x_1 = x_2 = \cdots = x_n = x$ in (3.11), we get

$$(3.14) \qquad \mu_{Q(dx)-d^2 Q(x)}(t) \geq \rho_{x, x, \cdots, x}(t)$$

Replacing x with $\frac{x}{d}$ in (3.14)

$$\mu_{Q(x)-d^2Q(\frac{x}{d})}(t) \geq \rho_{\frac{x}{d},\frac{x}{d},\cdots,\frac{x}{d}}(t) \tag{3.15}$$

for all $x \in X$ and all $t > 0$. Let

$$\psi_x(t) = \rho_{x,x,\cdots,x}(t)$$

Replacing x by $\frac{x}{d^n}$ and t with $\frac{t}{d^{2n}}$ in (3.15), we get

$$\begin{aligned} \mu_{Q(\frac{x}{d^n})-Q(\frac{x}{d^{n+1}})d^2}(\frac{t}{d^{2n}}) &\geq \psi_{\frac{x}{d^{n+1}}}(\frac{t}{d^{2n}}), \\ \mu_{Q(\frac{x}{d^n})d^{2n}-Q(\frac{x}{d^{n+1}})d^{2n+2}}(t) &\geq \psi_{\frac{x}{d^{n+1}}}(\frac{t}{d^{2n}}) \end{aligned} \tag{3.16}$$

for all $n \in N$, $x \in X$ and all $t > 0$. It follows from (3.16) and $1 \geq (\frac{1}{d^2} - 1)(d^2 + d^4 + \cdots + d^{2n})$ that

$$\begin{aligned} \mu_{Q(\frac{x}{d^n})d^{2n}-Q(x)}(t) &= \mu_{(Q(\frac{x}{d^n})d^{2n}-Q(\frac{x}{d^{(n-1)}})d^{2(n-1)}+\cdots+Q(\frac{x}{d})d^2-Q(x))}(t) \\ &\geq T_{k=1}^{n}\left(\mu_{(Q(\frac{x}{d^k})d^{2k}-Q(\frac{x}{d^{(k-1)}})d^{2(k-1)}}\left((\frac{1}{d^2}-1)d^{2k}t\right)\right) \\ &\geq T_{k=1}^{n}\left(\psi_{\frac{x}{d^k}}((1-d^2)t)\right) \\ &= T_{k=1}^{n}\left(\rho_{\frac{x}{d^k},\frac{x}{d^k},\cdots,\frac{x}{d^k}}((1-d^2)t)\right) \end{aligned} \tag{3.17}$$

for all $x \in X$ and all $n \in N$. Thus we have

$$\mu_{Q(\frac{x}{d^n})d^{2n}-Q(\frac{x}{d^m})d^{2m}}(t) \geq T_{k=m}^{n}\left(\rho_{\frac{x}{d^k},\frac{x}{d^k},\cdots,\frac{x}{d^k}}((1-d^2)t)\right). \tag{3.18}$$

Since the right hand side of the inequality (3.18) tends to 1 as m, n tend to infinity, the sequence $(Q(\frac{x}{d^n})d^{2n})$ is a Cauchy sequence. Thus we may define $R(x) = \lim_{n\to\infty} Q(\frac{x}{d^n})d^{2n}$ for all $x \in X$. Then

$$\mu_{R(x)-Q(x)}(t) = T_{k=1}^{\infty}\left(\rho_{\frac{x}{d^k},\frac{x}{d^k},\cdots,\frac{x}{d^k}}((1-d^2)t)\right) \tag{3.19}$$

Now we show that R is a quadratic mapping. Putting $x_1 = x_2 = \cdots = x_n = \frac{x}{d^n}$ in (3.11), we get

$$\mu_{DQ(\frac{x}{d^n},\frac{x}{d^n},\cdots,\frac{x}{d^n})d^{2n}}(t) \geq \rho_{\frac{x}{d^n},\frac{x}{d^n},\cdots,\frac{x}{d^n}}(t).$$

Taking the limit as $n \to \infty$, we find that $R : X \to Y$ satisfies (3.13) for all $x, y \in X$. So the mapping $R : X \to Y$ is quadratic. Letting the limit as $n \to \infty$ in (3.18), we get (3.13).

The rest of the proof is similar to the proof of Theorem 3.1. □

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