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FRACTIONAL VERSIONS OF HADAMARD INEQUALITIES FOR STRONGLY (s,m)-CONVEX FUNCTIONS VIA CAPUTO FRACTIONAL DERIVATIVES

Ghulam Farid, Sidra Bibi, Laxmi Rathour*, Lakshmi Narayan Mishra, and Vishnu Narayan Mishra

Abstract. We aim in this article to establish variants of the Hadamard inequality for Caputo fractional derivatives. We present the Hadamard inequality for strongly (s,m)-convex functions which will provide refinements as well as generalizations of several such inequalities already exist in the literature. The error bounds of these inequalities are also given by applying some known identities. Moreover, various associated results are deduced.

1. Introduction

Fractional integral inequalities are in focus of several researchers in these days (see, [1, 21, 23] and references therein). In this paper we study Hadamard inequalities for Caputo fractional derivatives of strongly (s,m)-convex function. The Caputo fractional derivatives are defined as follows:

Definition 1. [15] Let $\psi \in AC^n[a,b]$ and $n = [\Re(\beta)] + 1$. Then Caputo fractional derivatives of order $\beta \in \mathbb{C}$, $\Re(\beta) > 0$ of ψ are defined as follows:

$$^{C}D^{\beta}_{a+}\psi(x) = \frac{1}{\Gamma(n-\beta)} \int_a^x \frac{\psi^{(n)}(z)}{(x-z)^{\beta-n+1}} dz, x > a, \tag{1.1}$$

$$^{C}D^{\beta}_{b-}\psi(x) = \frac{(-1)^n}{\Gamma(n-\beta)} \int_x^b \frac{\psi^{(n)}(z)}{(z-x)^{\beta-n+1}} dz, x < b. \tag{1.2}$$

If $\beta = n \in \{1,2,3,...\}$ and usual derivative of order n exists, then Caputo fractional derivative $(^{C}D^{\beta}_{a+}\psi)(x)$ coincides with $\psi^{(n)}(x)$, whereas $(^{C}D^{\beta}_{b-}\psi)(x)$ coincides with $\psi^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular we have

$$(^{C}D^{0}_{a+}\psi)(x) = (^{C}D^{0}_{b-}\psi)(x) = \psi(x) \tag{1.3}$$

where $n = 1$ and $\beta = 0$.

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∗ Corresponding author.

The definition of strongly convex function was introduced by Polyak in [22]. Strongly convexity is a refined notion of the notion of convexity, some properties of strongly convex functions are just stronger versions of known properties of convex functions. Strongly convex functions have been used for proving the convergence of a gradient type algorithm for minimizing a function. They play an important role in optimization theory and mathematical economics.

Definition 2. [22] Let D be a convex subset of $\mathbb{X}$, $(\mathbb{X}, ||.||)$ be a normed space. A function $\psi : D \subset \mathbb{X} \to \mathbb{R}$ is called strongly convex function with modulus C if it satisfies

$$\psi(az + (1-z)b) \leq z\psi(a) + (1-z)\psi(b) - Cz(1-z)||a-b||^2 \tag{1.4}$$

$\forall a, b \in D, z \in [0,1]$ and $C > 0$.

Many authors have been inventing the properties and applications of strongly convex functions, for more information see [4, 13, 16, 18–20, 25].

The concepts of m-convex function and strongly m-convex function were introduced in [24] and [17] respectively.

Definition 3. [24] A function $\psi : [0, b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have:

$$\psi(tx + m(1-t)y) \leq t\psi(x) + m(1-t)\psi(y).$$

Lara et. al introduced strongly m-convex functions as follows:

Definition 4. [17] A function $\psi : I \to \mathbb{R}$ is called strongly m-convex function with modulus $C \geq 0$ if

$$\psi(za + m(1-z)b) \leq z\psi(a) + m(1-z)\psi(b) - Cmz(1-z)|a-b|^2$$

for $a, b \in I$ and $z \in [0, 1]$.

Further generalizations of above functions in the form of (s, m)-convex function and strongly (s, m)-convex function are defined as follows:

Definition 5. [6] A function $\psi : [0, b] \to \mathbb{R}$ is said to be (s, m)-convex function, where $(s, m) \in [0, 1]^2$ and $b > 0$, if for every $x, y \in [0, b]$ and $z \in [0, 1]$ we have

$$\psi(zx + m(1-z)y) \leq z^s\psi(x) + (1-z)^s\psi(y).$$

Definition 6. [2] A function $\psi : [0, +\infty] \to \mathbb{R}$ is said to be strongly (s, m)-convex function with modulus $C \geq 0$, for $(s, m) \in [0, 1]^2$, if

$$\psi(zx + m(1-z)y) \leq z^s\psi(x) + (1-z)^s\psi(y) - Cmz(1-z)|y-x|^2$$

holds for all $x, y \in [0, +\infty]$ and $z \in [0, 1]$.

A well-known inequality named Hadamard inequality is another interpretation of convex function. It is stated as follows:

Definition 7. [5] Let $\psi : I \to \mathbb{R}$ be a convex function on interval $I \subset \mathbb{R}$ and $a, b \in I$ where $a < b$. Then the undermentioned inequality holds:

$$\psi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \psi(x)dx \leq \frac{\psi(a)+\psi(b)}{2}. \tag{1.5}$$

If order in (1.5) is reversed, then it holds for concave function.

The main goal of this article is to study the Hermite Hadamard type inequality for strongly (s,m)-convex functions via Caputo fractional derivatives which provide simultaneously refinements and generalizations of different fractional versions of such inequalities exist in the literature. Farid et. al [9] has proved the undermentioned Hadamard inequality for Caputo fractional derivatives:

THEOREM 1. *Let $\psi:[a,b]\to\mathbb{R}$ be the function with $\psi\in C^n[a,b]$ and $0\le a<b$. Also let $\psi^{(n)}$ be positive and convex function on $[a,b]$. Then the undermentioned inequality holds for Caputo fractional derivatives:*

$$\begin{aligned}\psi^{(n)}\left(\frac{a+b}{2}\right)&\le\frac{\Gamma(n-\beta+1)}{2(b-a)^{n-\beta}}\left[\left({}^C D^{\beta}_{a^+}\psi\right)(b)+(-1)^n\left({}^C D^{\beta}_{b^-}\psi\right)(a)\right]\\&\le\frac{[\psi^{(n)}(a)+\psi^{(n)}(b)]}{2}.\end{aligned}\tag{1.6}$$

They also established the undermentioned identity:

LEMMA 1. [9] *Let $\psi:[a,b]\to\mathbb{R}$, $0\le a<b$, be the function such that $\psi\in C^n[a,b]$. Then the undermentioned equality for Caputo fractional derivatives holds:*

$$\begin{aligned}&\frac{\psi^{(n)}(a)+\psi^{(n)}(b)}{2}-\frac{\Gamma(n-\beta+1)}{2(b-a)^{n-\beta}}\left[({}^C D^{\beta}_{a^+}\psi)(b)+(-1)^n({}^C D^{\beta}_{b^-}\psi)(a)\right]\\&=\frac{b-a}{2}\int_0^1\left[(1-z)^{n-\beta}-z^{n-\beta}\right]\psi^{(n+1)}\left(za+(1-z)b\right)dz.\end{aligned}\tag{1.7}$$

Farid et. al [9] also proved the undermentioned inequality for Caputo fractional derivatives:

THEOREM 2. *Let $\psi:[a,b]\to\mathbb{R}$, $0\le a<b$ be the function with $\psi\in C^{n+1}[a,b]$ and also let $|\psi^{(n+1)}|$ is convex on $[a,b]$. Then the undermentioned inequality for Caputo fractional derivatives holds:*

$$\begin{aligned}&\left|\frac{\psi^{(n)}(a)+\psi^{(n)}(b)}{2}-\frac{\Gamma(n-\beta+1)}{2(b-a)^{n-\beta}}\left[({}^C D^{\beta}_{a^+}\psi)(b)+(-1)^n({}^C D^{\beta}_{b^-}\psi)(a)\right]\right|\\&\le\frac{(b-a)}{2(n-\beta+1)}\left(1-\frac{1}{2^{n-\beta}}\right)\left[|\psi^{(n+1)}(a)|+|\psi^{(n+1)}(b)|\right].\end{aligned}\tag{1.8}$$

Kang et. al [14] proved the undermentioned version of the Hadamard inequality for Caputo fractional derivatives:

THEOREM 3. *Let $\psi:[a,b]\to\mathbb{R}$ be a positive function with $\psi\in C^n[a,b]$ and $0\le a<b$. If $\psi^{(n)}$ is convex function on $[a,b]$, then the undermentioned inequality for Caputo fractional derivatives holds:*

$$\begin{aligned}\psi^{(n)}\left(\frac{a+b}{2}\right)&\le\frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(b-a)^{n-\beta}}\left[\left({}^C D^{\beta}_{(\frac{a+b}{2})^+}\psi\right)(b)+(-1)^n\left({}^C D^{\beta}_{(\frac{a+b}{2})^-}\psi\right)(a)\right]\\&\le\frac{\psi^{(n)}(a)+\psi^{(n)}(b)}{2}.\end{aligned}\tag{1.9}$$

Farid et. al [10] established the following identity.

LEMMA 2. *Let $\psi : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $\psi \in C^{n+1}[a,b]$, then the undermentioned equality for Caputo fractional derivatives holds:*

(1.10)

$$\frac{2^{n-\frac{\beta}{k}-1} k\Gamma_k(n-\frac{\beta}{k}+k)}{(mb-a)^{n-\frac{\beta}{k}}} \left[({}^C D^{\beta,k}_{(\frac{a+bm}{2})^+}\psi)(mb) + m^{n-\frac{\beta}{k}+1}(-1)^n ({}^C D^{\beta,k}_{(\frac{a+mb}{2m})^-}\psi)\left(\frac{a}{m}\right)\right]$$
$$-\frac{1}{2}\left[\psi^{(n)}\left(\frac{a+mb}{2}\right) + m\psi^{(n)}\left(\frac{a+mb}{2m}\right)\right] = \frac{mb-a}{4}$$
$$\times \left[\int_0^1 z^{n-\frac{\beta}{k}}\psi^{(n+1)}\left(\frac{z}{2}a + m\left(\frac{2-z}{2}\right)b\right) - \int_0^1 z^{n-\frac{\beta}{k}}\psi^{(n+1)}\left(\frac{2-z}{2m}a + \frac{z}{2}b\right)dz\right],$$

with $\beta > 0$.

Kang et. al [14] also proved the undermentioned inequalities for Caputo fractional derivatives:

THEOREM 4. *Let $\psi : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $\psi \in C^{n+1}[a,b]$ and $a < b$. If $|\psi^{(n+1)}|^q$ is convex on $[a,b]$ for $q \geq 1$, then the undermentioned inequality for Caputo fractional derivatives holds:*

(1.11)

$$\left|\frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(b-a)^{n-\beta}}\left[({}^C D^{\beta}_{(\frac{a+b}{2})^+}\psi)(b) + (-1)^n({}^C D^{\beta}_{(\frac{a+b}{2})^-}\psi)(a)\right] - \psi^{(n)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4(n-\beta+1)^{\frac{1}{p}}}\left(\frac{1}{2(n-\beta+1)(n-\beta+2)}\right)^{\frac{1}{q}}$$
$$\left[\left((n-\beta+1)|\psi^{(n+1)}(a)|^q + (n-\beta+3)|\psi^{(n+1)}(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left((n-\beta+3)|\psi^{(n+1)}(a)|^q + (\beta+1)|\psi^{(n+1)}(b)|^q\right)^{\frac{1}{q}}\right].$$

THEOREM 5. [14] *Let $\psi : [a,b] \to \mathbb{R}$ be a function with $\psi \in C^{n+1}[a,b]$ and $a < b$. If $|\psi^{(n+1)}|^q$ is convex on $[a,b]$ for $q > 1$, then the undermentioned inequality for Caputo fractional derivatives holds:*

(1.12)

$$\left|\frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(b-a)^{n-\beta}}\left[({}^C D^{\beta}_{(\frac{a+b}{2})^+}\psi)(b) + (-1)^n({}^C D^{\beta}_{(\frac{a+b}{2})^-}\psi)(a)\right] - \psi^{(n)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4(np-\beta p+1)^{\frac{1}{p}}}\left[\left(\frac{|\psi^{(n+1)}(a)|^q + 3|\psi^{(n+1)}(b)|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{3|\psi^{(n+1)}(a)|^q + |\psi^{(n+1)}(b)|^q}{4}\right)^{\frac{1}{q}}\right] \leq \frac{b-a}{4}\left(\frac{4}{3(np-\beta p+1)}\right)^{\frac{1}{p}}$$
$$[|\psi^{(n+1)}(a)| + |\psi^{(n+1)}(b)|],$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

In Section 2, we will study the Hadamard inequality for Caputo fractional derivatives by using strongly (s,m)-convex functions. Two Caputo fractional versions are obtained from which some interesting consequences are derived and connected with already known results. In Section 3, by applying some established identities we will derive refinements of error estimations of Hadamard inequalities.

2. Main Results

The following result gives the Hadamard inequality for Caputo fractional derivatives of strongly (s,m)-convex functions.

THEOREM 6. *Let $s \in [0,1]$ and $\psi : [a,b] \to \mathbb{R}$ be a positive function with $\psi \in C^n[a,b]$, $m \in (0,1]$ and $0 \leq a < mb$. If $\psi^{(n)}$ is strongly (s,m)-convex function with modulus $C \geq 0$, then the undermentioned inequality for Caputo fractional derivatives holds:*

$$\begin{aligned}
&\psi^{(n)}\left(\frac{bm+a}{2}\right) + \frac{mC(n-\beta)}{4(n-\beta+2)}\left\{(b-a)^2 + \frac{2(b-a)(\frac{a}{m}-mb)}{(n-\beta+1)}\right.\\
&\left.+\frac{2}{(n-\beta)(n-\beta+1)}\right\} \leq \frac{\Gamma(n-\beta+1)}{2^s(bm-a)^{n-\beta}}\left[m^{n-\beta+1}(-1)^n\left({}^CD^{\beta}_{b-}\psi\right)\left(\frac{a}{m}\right)\right.\\
&\left.+\left({}^CD^{\beta}_{a+}\psi\right)(mb)\right] \leq \frac{(n-\beta)}{2^s}\left\{\left[m^2\psi^{(n)}\left(\frac{a}{m^2}\right)+m\psi^{(n)}(b)\right]\beta(s+1,n-\beta)\right.\\
&\left.+\frac{[m\psi^{(n)}(b)+\psi^{(n)}(a)]}{n-\beta+s} - \frac{Cm\left((b-a)^2+\left(b-\frac{a}{m^2}\right)^2\right)}{(n-\beta+1)(n-\beta+2)}\right\}
\end{aligned} \tag{2.1}$$

with $\beta > 0$.

Proof. Since $\psi^{(n)}$ is strongly (s,m)-convex function with modulus C, for $x, y \in [a,b]$, we have

$$\psi^{(n)}\left(\frac{mx+y}{2}\right) \leq \frac{m\psi^{(n)}(x)+\psi^{(n)}(y)}{2^s} - \frac{mC}{4}|x-y|^2. \tag{2.2}$$

Let $x = (1-z)\frac{a}{m} + zb \leq b$ and $y = m(1-z)b + za \geq a$, $z \in [0,1]$. Then we have

$$\begin{aligned}
\psi^{(n)}\left(\frac{bm+a}{2}\right) \leq &\frac{m\psi^{(n)}\left((1-z)\frac{a}{m}+zb\right)+\psi^{(n)}(m(1-z)b+za)}{2^s}\\
&-\frac{mC}{4}\left((1-z)\frac{a}{m}+zb-(m(1-z)b+za)\right)^2.
\end{aligned} \tag{2.3}$$

Multiplying (2.3) with $z^{n-\beta-1}$ on both sides and making integration over $[0,1]$ we get

$$\begin{aligned}
&\psi^{(n)}\left(\frac{bm+a}{2}\right)\int_0^1 z^{n-\beta-1}dz \leq \frac{1}{2^s}\left(\int_0^1 m\psi^{(n)}((1-z)\frac{a}{m}+zb)z^{n-\beta-1}dz\right.\\
&\left.+\int_0^1 \psi^{(n)}(m(1-z)b+za)z^{n-\beta-1}dz\right) - \frac{mC}{4}\int_0^1\left(z^{n-\beta-1}\right.\\
&\left.\times\left((1-z)\frac{a}{m}+zb-(m(1-z)b+za)\right)^2\right)dz.
\end{aligned} \tag{2.4}$$

By using change of the variables and computing the last integral, from (2.4) we get

(2.5)
$$\frac{1}{n-\beta}\psi^{(n)}\left(\frac{bm+a}{2}\right)$$
$$\leq \frac{1}{2^s(bm-a)}\left(\int_{\frac{a}{m}}^{b} m^2\psi^{(n)}(x)\left(\frac{mx-a}{bm-a}\right)^{n-\beta-1}dx+\int_a^{mb}\psi^{(n)}(y)\left(\frac{bm-y}{bm-a}\right)^{n-\beta-1}dy\right)$$
$$-\frac{mC}{4}\left\{\frac{(b-a)^2}{n-\beta+2}+\frac{2(b-a)(\frac{a}{m}-mb)}{(n-\beta+1)(n-\beta+2)}+\frac{2}{(n-\beta)(n-\beta+1)(n-\beta+2)}\right\}.$$

Further, it takes the following form

(2.6)
$$\psi^{(n)}\left(\frac{bm+a}{2}\right)\leq\frac{\Gamma(n-\beta+1)}{2^s(bm-a)^{n-\beta}}\left[m^{n-\beta+1}(-1)^n\left({}^{C}D^{\beta}_{b^-}\psi\right)\left(\frac{a}{m}\right)+\left({}^{C}D^{\beta}_{a^+}\psi\right)(mb)\right]$$
$$-\frac{mC(n-\beta)}{4}\left\{\frac{(b-a)^2}{n-\beta+2}+\frac{2(b-a)(\frac{a}{m}-mb)}{(n-\beta+1)(n-\beta+2)}+\frac{2}{(n-\beta)(n-\beta+1)(n-\beta+2)}\right\}.$$

Since $\psi^{(n)}$ is strongly (s,m)-convex function with modulus C, for $z\in[0,1]$, then one has

(2.7)
$$m\psi^{(n)}\left((1-z)\frac{a}{m}+zb\right)+\psi^{(n)}(m(1-z)b+za)\leq mz^s\psi^{(n)}(b)+m^2(1-z)^s\psi^{(n)}\left(\frac{a}{m^2}\right)$$
$$+z^s\psi^{(n)}(a)+m(1-z)^s\psi^{(n)}(b)-Cmz(1-z)(b-a)^2-Cm^2z(1-z)\left(b-\frac{a}{m^2}\right)^2.$$

Multiplying (2.7) with $z^{n-\beta-1}$ on both sides and making integration over $[0,1]$ we get

(2.8)
$$\int_0^1 m\psi^{(n)}\left((1-z)\frac{a}{m}+zb\right)z^{n-\beta-1}dz+\int_0^1\psi^{(n)}(m(1-z)b+za)z^{n-\beta-1}dz\leq$$
$$\int_0^1 m\psi^{(n)}(b)z^{n-\beta+s-1}dz+\int_0^1 m^2\psi^{(n)}\left(\frac{a}{m^2}\right)(1-z)^sz^{n-\beta-1}dz+\int_0^1\psi^{(n)}(a)z^{n-\beta+s-1}dz$$
$$+\int_0^1 m\psi^{(n)}(b)(1-z)^sz^{n-\beta}dz-Cm\left((b-a)^2+\left(b-\frac{a}{m^2}\right)\right)\int_0^1(1-z)z^{n-\beta}dz.$$

By using change of the variables and computing the last integral, from (2.8) we get

$$\frac{1}{bm-a}\left(\int_{\frac{a}{m}}^{b} m^2\psi^{(n)}(x)\left(\frac{mx-a}{bm-a}\right)^{n-\beta-1}dx+\int_a^{mb}\psi^{(n)}(y)\left(\frac{bm-y}{bm-a}\right)^{n-\beta-1}dy\right)\tag{2.9}$$
$$\leq\left[m^2\psi^{(n)}\left(\frac{a}{m^2}\right)+m\psi^{(n)}(b)\right]\beta(s+1,n-\beta)+\frac{m\psi^{(n)}(b)+\psi^{(n)}(a)}{n-\beta+s}$$
$$-\frac{Cm\left((b-a)^2+\left(b-\frac{a}{m^2}\right)^2\right)}{(n-\beta+1)(n-\beta+2)}.$$

Further, it takes the following form

$$
\begin{aligned}
&\frac{\Gamma(n-\beta+1)}{2^s(bm-a)^{n-\beta}}\left[m^{n-\beta+1}(-1)^n\left({}^C D_{b^-}^{\beta}\psi\right)\left(\frac{a}{m}\right)+\left({}^C D_{a^+}^{\beta}\psi\right)(mb)\right]\\
&\le\frac{(n-\beta)}{2^s}\left\{\left[m^2\psi^{(n)}\left(\frac{a}{m^2}\right)+m\psi^{(n)}(b)\right]\beta(s+1,n-\beta)\right.\\
&\left.+\frac{[m\psi^{(n)}(b)+\psi^{(n)}(a)]}{n-\beta+s}-\frac{Cm\left((b-a)^2+\left(b-\frac{a}{m^2}\right)^2\right)}{(n-\beta+1)(n-\beta+2)}\right\}.
\end{aligned}\tag{2.10}
$$

Inequalities (2.6) and (2.10) constitute the required inequality. □

The consequences of Theorem 6 are stated in the undermentioned corollaries and remark:

Corollary 1. *By setting $m=1$ in inequality (2.1), we will get the following inequality for strongly s-convex function via Caputo fractional derivatives*

$$
\begin{aligned}
&\psi^{(n)}\left(\frac{b+a}{2}\right)+\frac{C(n-\beta)}{4(n-\beta+2)}\left\{(b-a)^2-\frac{2(b-a)^2}{(n-\beta+1)}\right.\\
&\left.+\frac{2}{(n-\beta)(n-\beta+1)}\right\}\le\frac{\Gamma(n-\beta+1)}{2^s(b-a)^{n-\beta}}\left[(-1)^n\left({}^C D_{b^-}^{\beta}\psi\right)(a)\right.\\
&\left.+\left({}^C D_{a^+}^{\beta}\psi\right)(b)\right]\le\frac{(n-\beta)}{2^s}\left\{\left[\psi^{(n)}(a)+\psi^{(n)}(b)\right]\beta(s+1,n-\beta)\right.\\
&\left.+\frac{\left[\psi^{(n)}(a)+\psi^{(n)}(b)\right]}{n-\beta+s}-\frac{2C(b-a)^2}{(n-\beta+1)(n-\beta+2)}\right\}.
\end{aligned}
$$

Corollary 2. *By setting $C=0$ and $m=1$ in inequality (2.1), we will get the following inequality for s-convex function via Caputo fractional derivatives*

$$
\begin{aligned}
\psi^{(n)}\left(\frac{b+a}{2}\right)&\le\frac{\Gamma(n-\beta+1)}{2^s(b-a)^{n-\beta}}\left[(-1)^n\left({}^C D_{b^-}^{\beta}\psi\right)(a)+\left({}^C D_{a^+}^{\beta}\psi\right)(b)\right]\\
&\le\frac{(n-\beta)\left[\psi^{(n)}(a)+\psi^{(n)}(b)\right]}{2^s}\left\{\beta(s+1,n-\beta)+\frac{1}{n-\beta+s}\right\}.
\end{aligned}
$$

Remark 1.
(i) If $s=1$ in (2.1), then we will get the fractional Hadamard inequality for strongly m-convex function which is given in [8, Theorem 6].
(ii) If $s=1$ and $m=1$ in (2.1), then we will get the fractional Hadamard inequality for strongly convex function which is given in [7, Theorem 6].
(iii) If $C=0$, $m=1$ and $s=1$ in (2.1), then we will get the fractional Hadamard inequality stated in Theorem 1.
(iv) If $C=0$ in (2.1), then we will get the Caputo fractional Hadamard inequality for (s,m)-convex functions.

Remark 2. For $C>0$ all the above mentioned results provide refinements of corresponding inequalities for convex, m-convex, s-convex, (s,m)-convex functions.

The upcoming result is another version of the Hadamard inequality for Caputo fractional derivatives of strongly (s,m)-convex functions.

Theorem 7. *Let $s \in [0,1]$ and $\psi : [a,b] \to \mathbb{R}$ be a positive function with $\psi \in C^n[a,b]$, $m \in (0,1]$ and $0 \leq a < mb$. If $\psi^{(n)}$ is a strongly (s,m)-convex function with modulus $C \geq 0$, then the undermentioned inequality for Caputo fractional derivatives holds:*

(2.11)

$$\begin{aligned}
&\psi^{(n)}\left(\frac{bm+a}{2}\right) + \frac{mC(n-\beta)}{8(n-\beta+2)}\left\{\frac{(b-a)^2}{2} + \frac{(b-a)(\frac{a}{m}-mb)(n-\beta+3)}{(n-\beta+1)}\right.\\
&\left.+\frac{(\frac{a}{m}-mb)^2[(n-\beta)^2+5n-5\beta+8]}{2(n-\beta)(n-\beta+1)}\right\} \leq \frac{2^{n-\beta-s}\Gamma(n-\beta+1)}{(bm-a)^{n-\beta}}\\
&\left[m^{n-\beta+1}(-1)^n\left({}^{C}D^{\beta}_{(\frac{a+bm}{2m})^-}\psi\right)\left(\frac{a}{m}\right) + \left({}^{C}D^{\beta}_{(\frac{a+bm}{2})^+}\psi\right)(mb)\right] \leq \frac{(n-\beta)}{2^s}\\
&\left\{\frac{\Gamma(n-\beta+1)\left[m\psi^{(n)}(b)+m^2\psi^{(n)}\left(\frac{a}{m^2}\right)\right]\,{}^{C}D^{\alpha}_{1^+}(g(2))}{2^s(n-\beta)} + \frac{(m\psi^{(n)}(b)+\psi^{(n)}(a))}{2^s(n-\beta+s)}\right.\\
&\left.-\frac{mC(n-\beta+3)\left[(b-a)^2+m\left(b-\frac{a}{m^2}\right)^2\right]}{4(n-\beta+1)(n-\beta+2)}\right\}
\end{aligned}$$

with $\beta > 0$.

Proof. Let us consider $x = \frac{a}{m}\left(\frac{2-z}{2}\right) + b\frac{z}{2}$ and $y = a\frac{z}{2} + m\left(\frac{2-z}{2}\right)b$ for $z \in [0,1]$, in (2.2). Then we have

$$\begin{aligned}
\psi^{(n)}\left(\frac{bm+a}{2}\right) &\leq \frac{m\psi^{(n)}\left(\frac{a}{m}\left(\frac{2-z}{2}\right)+b\frac{z}{2}\right)+\psi^{(n)}\left(a\frac{z}{2}+m\left(\frac{2-z}{2}\right)b\right)}{2^s}\\
&-\frac{mC}{4}\left(\frac{a}{m}\left(\frac{2-z}{2}\right)+b\frac{z}{2}-a\frac{z}{2}-m\left(\frac{2-z}{2}\right)b\right)^2.
\end{aligned} \tag{2.12}$$

By multiplying (2.12) with $z^{n-\beta-1}$ on both sides and making integration over $[0,1]$, we get

(2.13)

$$\begin{aligned}
&\psi^{(n)}\left(\frac{bm+a}{2}\right)\int_0^1 z^{n-\beta-1}dz\\
&\leq \frac{1}{2^s}\int_0^1\left(m\psi^{(n)}\left(\frac{a}{m}\left(\frac{2-z}{2}\right)+b\frac{z}{2}\right)+\psi^{(n)}\left(a\frac{z}{2}+m\left(\frac{2-z}{2}\right)b\right)\right)z^{n-\beta-1}dz\\
&-\frac{mC}{4}\int_0^1\left(\frac{a}{m}\left(\frac{2-z}{2}\right)+b\frac{z}{2}-a\frac{z}{2}-m\left(\frac{2-z}{2}\right)b\right)^2 z^{n-\beta-1}dz.
\end{aligned}$$

By using change of variables and computing the last integral, from (2.13) we get

$$\begin{aligned}
&\frac{1}{n-\beta}\psi^{(n)}\left(\frac{bm+a}{2}\right) \le \frac{1}{2^s}\left(m\int_{\frac{a}{m}}^{(\frac{a+bm}{2m})}\left(\frac{2m(y-\frac{a}{m})}{bm-a}\right)^{n-\beta-1}\psi^{(n)}(y)\frac{2mdy}{bm-a}\right.\\
&\left.+\int_{(\frac{a+mb}{2})}^{mb}\left(\frac{2(mb-x)}{bm-a}\right)^{n-\beta-1}\psi^{(n)}(x)\frac{2dx}{bm-a}\right) - \frac{mC}{4}\left\{\frac{(b-a)^2}{4(n-\beta+2)}\right.\\
&\left.+\frac{(b-a)(\frac{a}{m}-mb)(n-\beta+3)}{2(n-\beta+1)(n-\beta+2)} + \frac{(\frac{a}{m}-mb)^2[(n-\beta)^2+5n-5\beta+8]}{4(n-\beta)(n-\beta+1)(n-\beta+2)}\right\}.
\end{aligned} \tag{2.14}$$

Further it takes the following form

$$\begin{aligned}
&\psi^{(n)}\left(\frac{bm+a}{2}\right) \le \frac{2^{n-\beta-s}\Gamma(n-\beta+1)}{(bm-a)^{n-\beta}}\left[m^{n-\beta+1}(-1)^n\left({}^C D^{\beta}_{(\frac{a+bm}{2m})^-}\psi\right)\left(\frac{a}{m}\right)\right.\\
&\left.+\left({}^C D^{\beta}_{(\frac{a+bm}{2})^+}\psi\right)(mb)\right] - \frac{mC(n-\beta)}{8}\left\{\frac{(b-a)^2}{2(n-\beta+2)}\right.\\
&\left.+\frac{(b-a)(\frac{a}{m}-mb)(n-\beta+3)}{(n-\beta+1)(n-\beta+2)} + \frac{(\frac{a}{m}-mb)^2[(n-\beta)^2+5n-5\beta+8]}{2(n-\beta)(n-\beta+1)(n-\beta+2)}\right\}.
\end{aligned} \tag{2.15}$$

Since $\psi^{(n)}$ is strongly (s,m)-convex function and $z\in[0,1]$, we have the following inequality:

$$\begin{aligned}
&m\psi^{(n)}\left(\frac{a}{m}\left(\frac{2-z}{2}\right)+b\frac{z}{2}\right) + \psi^{(n)}\left(a\frac{z}{2}+m\left(\frac{2-z}{2}\right)b\right) \le m^2\left(\frac{2-z}{2}\right)^s\\
&\psi^{(n)}\left(\frac{a}{m^2}\right) + m\left(\frac{z}{2}\right)^s\psi^{(n)}(b) + \left(\frac{z}{2}\right)^s\psi^{(n)}(a) + m\left(\frac{2-z}{2}\right)^s\psi^{(n)}(b)\\
&-m^2C\left(\frac{z(2-z)}{4}\right)\left(b-\frac{a}{m^2}\right)^2 - mC\left(\frac{z(2-z)}{4}\right)(b-a)^2.
\end{aligned} \tag{2.16}$$

By multiplying (2.16) with $z^{n-\beta-1}$ on both sides and making integration over $[0,1]$ we get

(2.17)

$$\begin{aligned}
&\int_0^1 m\psi^{(n)}\left(\frac{a}{m}\left(\frac{2-z}{2}\right)+b\frac{z}{2}\right)z^{n-\beta-1}dz + \int_0^1\psi^{(n)}\left(a\frac{z}{2}+m\left(\frac{2-z}{2}\right)b\right)z^{n-\beta-1}dz\\
&\le \int_0^1 m^2\left(\frac{2-z}{2}\right)^s\psi^{(n)}\left(\frac{a}{m^2}\right)z^{n-\beta-1}dz + \int_0^1 m\left(\frac{z}{2}\right)^s\psi^{(n)}(b)z^{n-\beta-1}dz\\
&+\int_0^1\left(\frac{z}{2}\right)^s\psi^{(n)}(a)z^{n-\beta-1}dz + \int_0^1 m\left(\frac{2-z}{2}\right)^s\psi^{(n)}(b)z^{n-\beta-1}dz\\
&-\int_0^1 mC\left(\frac{z(2-z)}{4}\right)\left[m\left(b-\frac{a}{m^2}\right)^2+(b-a)^2\right]z^{n-\beta-1}dz.
\end{aligned}$$

By using change of variables and computing the last integral, from (2.17) we get

(2.18)
$$\frac{2}{bm-a}\left\{\int_{\frac{a}{m}}^{\frac{a+bm}{2m}}\left(\frac{2m(y-\frac{a}{m})}{bm-a}\right)^{n-\beta-1} m^2\psi^{(n)}(y)dy+\int_{\frac{a+mb}{2}}^{mb}\left(\frac{2(mb-x)}{bm-a}\right)^{n-\beta-1}\psi^{(n)}(x)dx\right\}$$
$$\leq\left[m\psi^{(n)}(b)+m^2\psi^{(n)}\left(\frac{a}{m^2}\right)\right]\int_0^1\left(\frac{2-z}{2}\right)^s z^{n-\beta-1}dz+\frac{m\psi^{(n)}(b)+\psi^{(n)}(a)}{2^s(n-\beta+s)}$$
$$-\frac{mC(n-\beta+3)\left[(b-a)^2+m\left(b-\frac{a}{m^2}\right)^2\right]}{4(n-\beta+1)(n-\beta+2)}.$$

Further it takes the following form
(2.19)
$$\frac{2^{n-\beta-s}\Gamma(n-\beta+1)}{(bm-a)^{n-\beta}}\left[m^{n-\beta+1}(-1)^n\left({}^CD^{\beta}_{(\frac{a+bm}{2m})^-}\psi\right)\left(\frac{a}{m}\right)+\left({}^CD^{\beta}_{(\frac{a+bm}{2})^+}\psi\right)(mb)\right]$$
$$\leq\frac{(n-\beta)}{2^s}\left\{\left[m\psi^{(n)}(b)+m^2\psi^{(n)}\left(\frac{a}{m^2}\right)\right]\int_0^1\left(\frac{2-z}{2}\right)^s z^{n-\beta-1}+\frac{m\psi^{(n)}(b)+\psi^{(n)}(a)}{2^s(n-\beta+s)}\right.$$
$$\left.-\frac{mC(n-\beta+3)\left[(b-a)^2+m\left(b-\frac{a}{m^2}\right)^2\right]}{4(n-\beta+1)(n-\beta+2)}\right\}.$$

The integral in the right hand side is calculated as follows: By setting $u=2-z$, multiplying and dividing with $\Gamma(n-\beta)$, we have

$$\frac{1}{2^s}\int_0^1 z^{n-\alpha-1}(2-z)^s dz=\frac{\Gamma(n-\beta)}{2^s\Gamma(n-\beta)}\int_1^2(2-u)^{n-\beta-1}u^s du. \tag{2.20}$$

Let $g(x)=\frac{x^{s+n}}{(s+1)_n}$, where $(s+1)_n=(s+1)(s+2)\dots(s+n)$ is the Pochhammer symbol. Then $g^{(n)}(x)=x^s$. Now equation (2.20) becomes

$$\frac{1}{2^s}\int_0^1 z^{n-\beta-1}(2-z)^s dz=\frac{\Gamma(n-\beta)}{2^s\Gamma(n-\beta)}\int_1^2(2-u)^{n-\beta-1}\frac{d^n}{du^n}\left(\frac{u^{s+n}}{(s+1)_n}\right)du$$
$$=\frac{\Gamma(n-\beta)}{2^s}\,{}^CD^{\alpha}_{1^+}(g(2)).$$

By putting the above value in (2.19), we get
(2.21)
$$\frac{2^{n-\beta-s}\Gamma(n-\beta+1)}{(bm-a)^{n-\beta}}\left[m^{n-\beta+1}(-1)^n\left({}^CD^{\beta}_{(\frac{a+bm}{2m})^-}\psi\right)\left(\frac{a}{m}\right)+\left({}^CD^{\beta}_{(\frac{a+bm}{2})^+}\psi\right)(mb)\right]$$
$$\leq\frac{1}{2^s}\left\{\left[m\psi^{(n)}(b)+m^2\psi^{(n)}\left(\frac{a}{m^2}\right)\right]\frac{\Gamma(n-\beta+1)}{2^s}\,{}^CD^{\alpha}_{1^+}(g(2))\right.$$
$$\left.+\frac{(n-\beta)(m\psi^{(n)}(b)+\psi^{(n)}(a))}{2^s(n-\beta+s)}-\frac{mC(n-\beta)(n-\beta+3)\left[(b-a)^2+m\left(b-\frac{a}{m^2}\right)^2\right]}{4(n-\beta+1)(n-\beta+2)}\right\}.$$

From (2.15) and (2.19), (2.11) can be obtained. □

Corollary 3. *By setting $m = 1$ in inequality (2.11), we will get the following inequality for strongly s-convex function via Caputo fractional derivatives*

$$\psi^{(n)}\left(\frac{b+a}{2}\right)+\frac{C(n-\beta)}{8(n-\beta+2)}\left\{\frac{(b-a)^2}{2}-\frac{(b-a)^2(n-\beta+3)}{(n-\beta+1)}\right.$$
$$\left.+\frac{(a-b)^2[(n-\beta)^2+5n-5\beta+8]}{2(n-\beta)(n-\beta+1)}\right\}\leq\frac{2^{n-\beta-s}\Gamma(n-\beta+1)}{(b-a)^{n-\beta}}$$
$$\left[(-1)^n\left({}^CD^{\beta}_{(\frac{a+b}{2})^-}\psi\right)(a)+\left({}^CD^{\beta}_{(\frac{a+b}{2})^+}\psi\right)(b)\right]$$
$$\leq\frac{1}{2^s}\left\{\frac{\Gamma(n-\beta+1)\,{}^CD^{\alpha}_{1^+}(g(2))\left[\psi^{(n)}(b)+\psi^{(n)}(a)\right]}{2^s}\right.$$
$$\left.+\frac{(n-\beta)(\psi^{(n)}(b)+\psi^{(n)}(a))}{2^s(n-\beta+s)}-\frac{C(n-\beta)(n-\beta+3)(b-a)^2}{2(n-\beta+1)(n-\beta+2)}\right\}.$$

Corollary 4. *By setting $C = 0$ and $m = 1$ in inequality (2.11), we will get the following inequality for s-convex function via Caputo fractional derivatives*

$$\psi^{(n)}\left(\frac{b+a}{2}\right)\leq\frac{2^{n-\beta-s}\Gamma(n-\beta+1)}{(b-a)^{n-\beta}}\left[(-1)^n\left({}^CD^{\beta}_{(\frac{a+b}{2})^-}\psi\right)(a)+\left({}^CD^{\beta}_{(\frac{a+b}{2})^+}\psi\right)(b)\right]$$
$$\leq\frac{(\psi^{(n)}(a)+\psi^{(n)}(b))}{2^{2s}}\left\{\Gamma(n-\beta+1)\,{}^CD^{\alpha}_{1^+}(g(2))+\frac{(n-\beta)}{(n-\beta+s)}\right\}.$$

Remark 3.
- (i) If $s = 1$ in (2.11), then we will get the fractional Hadamard inequality for strongly m-convex function which is given in [8, Theorem 7].
- (ii) If $s = 1$ and $m = 1$ in (2.11), then we will get the fractional Hadamard inequality for strongly convex function which is given in [7, Theorem 7].
- (iii) If $C = 0$, $m = 1$ and $s = 1$ in (2.11), then we will get the fractional Hadamard inequality stated in Theorem 3.
- (iv) If $C = 0$ in (2.11), then we will get the Caputo fractional Hadamard inequality for (s, m)-convex functions.

Remark 4. For $C > 0$ all the above mentioned results provide refinements of corresponding inequalities for convex, m-convex, s-convex, (s, m)-convex functions.

3. Error bounds of fractional Hadamard inequalities

In this section we give refinements of the error bounds of fractional Hadamard inequalities for Caputo fractional derivatives:

Theorem 8. *Let $s \in [0, 1]$ and $\psi : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $\psi \in C^{n+1}[a, b]$, $m \in (0, 1]$ and $a < b$. If $|\psi^{(n+1)}|$ is a strongly (s, m)-convex function on $[a, b]$, then the undermentioned inequality for Caputo fractional* *derivatives holds:*

(3.1)
$$\left|\frac{\psi^{(n)}(a)+\psi^{(n)}(b)}{2}-\frac{\Gamma(n-\beta+1)}{2(b-a)^{n-\beta}}\left[({}^{C}D^{\beta}_{a^+}\psi)(b)+(-1)^n({}^{C}D^{\beta}_{b^-}\psi)(a)\right]\right|$$
$$\leq\frac{(b-a)\left[|\psi^{(n+1)}(a)|+|\psi^{(n+1)}(b)|\right]}{2}\left\{\frac{(2^{np-\beta p+1}-1)^{\frac{1}{p}}-(2^{qs+1}-1)^{\frac{1}{q}}}{(2^{np-\beta p+1}(np-\beta p+1))^{\frac{1}{p}}(2^{qs+1}(qs+1))^{\frac{1}{q}}}\right.$$
$$\left.-\frac{2^{n-\beta+s+1}-2}{2^{n-\beta+s+1}(n-\beta+s+1)}\right\}-\frac{Cm(b-a)\left(\frac{b}{m}-a\right)^2}{(n-\beta+2)(n-\beta+3)}\left(1-\frac{n-\beta+4}{2^{n-\beta+2}}\right),$$

with $\beta>0$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $|\psi'|$ is strongly (s,m)-convex function on $[a,b]$ and $z\in[a,b]$, we have

$$|\psi'(za+(1-z)b)|=\left|\psi'\left(za+m(1-z)\frac{b}{m}\right)\right|$$
$$\leq z^s|\psi'(a)|+m(1-z)^s\left|\psi'\left(\frac{b}{m}\right)\right|-Cmz(1-z)\left(\frac{b}{m}-a\right)^2.$$

By applying Lemma 1 and the strong (s,m)-convexity of $|\psi^{(n+1)}|$, we find

$$\left|\frac{\psi^{(n)}(a)+\psi^{(n)}(b)}{2}-\frac{\Gamma(n-\beta+1)}{2(b-a)^{n-\beta}}\left[({}^{C}D^{\beta}_{a^+}\psi)(b)+(-1)^n({}^{C}D^{\beta}_{b^-}\psi)(a)\right]\right| \tag{3.2}$$
$$\leq\frac{b-a}{2}\int_0^1\left|(1-z)^{n-\beta}-z^{n-\beta}\right|\left|\psi^{(n+1)}\left(za+m(1-z)\frac{b}{m}\right)\right|dz$$
$$\leq\frac{b-a}{2}\int_0^1\left|(1-z)^{n-\beta}-z^{n-\beta}\right|\left(z^s|\psi^{(n+1)}(a)|+m(1-z)^s\left|\psi^{(n+1)}\left(\frac{b}{m}\right)\right|\right.$$
$$\left.-Cmz(1-z)\left(\frac{b}{m}-a\right)^2\right)dz$$
$$\leq\frac{b-a}{2}\left[\int_0^{1/2}\left((1-z)^{n-\beta}-z^{n-\beta}\right)\left(z^s|\psi^{(n+1)}(a)|+m(1-z)^s\left|\psi^{(n+1)}\left(\frac{b}{m}\right)\right|\right.\right.$$
$$\left.-Cmz(1-z)\left(\frac{b}{m}-a\right)^2\right)dz+\int_{1/2}^1\left(z^{n-\beta}-(1-z)^{n-\beta}\right)$$
$$\left.\left(z^s|\psi^{(n+1)}(a)|+m(1-z)^s\left|\psi^{(n+1)}\left(\frac{b}{m}\right)\right|-Cmz(1-z)\left(\frac{b}{m}-a\right)^2\right)dz\right].$$

In the following we compute integrals appearing on the right side of inequality (3.2) by using Holder inequality:

(3.3)

$$\begin{aligned}
&\int_0^{1/2}\left((1-z)^{n-\beta}-z^{n-\beta}\right)\left(z^s|\psi^{(n+1)}(a)|+m(1-z)^s\left|\psi^{(n+1)}\left(\frac{b}{m}\right)\right|\right.\\
&\left.-Cmz(1-z)\left(\frac{b}{m}-a\right)^2\right)dz=|\psi^{(n+1)}(a)|\int_0^{1/2}\left(z^s(1-z)^{n-\beta}-z^{n-\beta+s}\right)dz\\
&+m\left|\psi^{(n+1)}\left(\frac{b}{m}\right)\right|\int_0^{1/2}\left((1-z)^{n-\beta+s}-(1-z)^s z^{n-\beta}\right)dz\\
&-Cm\left(\frac{b}{m}-a\right)^2\left(\int_0^{1/2}z(1-z)^{n-\beta+1}dz-\int_0^{1/2}z^{n-\beta+1}(1-z)dz\right)\\
&=|\psi^{(n+1)}(a)|\left(\left(\frac{1-(1/2)^{np-\beta p+1}}{np-\beta p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2^{qs+1}(qs+1)}\right)^{\frac{1}{q}}-\frac{(1/2)^{n-\beta+s+1}}{n-\beta+s+1}\right)\\
&+m\left|\psi^{(n+1)}\left(\frac{b}{m}\right)\right|\left(\frac{2^{n-\beta+s+1}-1}{2^{n-\beta+s+1}(n-\beta+s+1)}-\left(\frac{(1/2)^{np-\beta p+1}}{np-\beta p+1}\right)^{\frac{1}{p}}\left(\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right)^{\frac{1}{q}}\right)\\
&-\frac{Cm\left(\frac{b}{m}-a\right)^2}{(n-\beta+2)(n-\beta+3)}\left[1-\frac{(n-\beta+4)}{2^{n-\beta+2}}\right],
\end{aligned}$$

and

$$\begin{aligned}
(3.4)\quad &\int_{1/2}^{1}\left(z^{n-\beta}-(1-z)^{n-\beta}\right)\left(z|\psi^{(n+1)}(a)|+m(1-z)\left|\psi^{(n+1)}\left(\frac{b}{m}\right)\right|\right.\\
&\left.-Cmz(1-z)\left(\frac{b}{m}-a\right)^2\right)dz=|\psi^{(n+1)}(a)|\int_{1/2}^{1}\left(z^{n-\beta+s}-z^s(1-z)^{n-\beta}\right)dz\\
&+m\left|\psi^{(n+1)}\left(\frac{b}{m}\right)\right|\int_{1/2}^{1}\left((1-z)^s z^{n-\beta}-(1-z)^{n-\beta+s}\right)dz\\
&-Cm\left(\frac{b}{m}-a\right)^2\left(\int_{1/2}^{1}(1-z)z^{n-\beta+1}dz-\int_{1/2}^{1}z(1-z)^{n-\beta+1}dz\right)\\
&=|\psi^{(n+1)}(a)|\left(\frac{2^{n-\beta+s+1}-1}{2^{n-\beta+s+1}(n-\beta+s+1)}-\left(\frac{(1/2)^{np-\beta p+1}}{np-\beta p+1}\right)^{\frac{1}{p}}\left(\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right)^{\frac{1}{q}}\right)\\
&+m\left|\psi^{(n+1)}\left(\frac{b}{m}\right)\right|\left(\left(\frac{1-(1/2)^{np-\beta p+1}}{np-\beta p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2^{qs+1}(qs+1)}\right)^{\frac{1}{q}}-\frac{(1/2)^{n-\beta+s+1}}{n-\beta+s+1}\right)\\
&-\frac{Cm\left(\frac{b}{m}-a\right)^2}{(n-\beta+2)(n-\beta+3)}\left[1-\frac{n-\beta+4}{2^{n-\beta+2}}\right].
\end{aligned}$$

By putting the values of (3.3) and (3.4) in (3.2) we get (3.1). □

COROLLARY 5. *By setting $m = 1$ in inequality (3.1), we will get the following inequality for strongly s-convex function via Caputo fractional derivatives*

$$\begin{aligned}&\left|\frac{\psi^{(n)}(a)+\psi^{(n)}(b)}{2}-\frac{\Gamma(n-\beta+1)}{2(b-a)^{n-\beta}}\left[({}^C D^{\beta}_{a^+}\psi)(b)+(-1)^n({}^C D^{\beta}_{b^-}\psi)(a)\right]\right|\\&\leq\frac{(b-a)\left[|\psi^{(n+1)}(a)|+|\psi^{(n+1)}(b)|\right]}{2}\left\{\frac{(2^{np-\beta p+1}-1)^{\frac{1}{p}}-(2^{qs+1}-1)^{\frac{1}{q}}}{(2^{np-\beta p+1}(np-\beta p+1))^{\frac{1}{p}}(2^{qs+1}(qs+1))^{\frac{1}{q}}}\right.\\&\left.-\frac{2^{n-\beta+s+1}-2}{2^{n-\beta+s+1}(n-\beta+s+1)}\right\}-\frac{C\,(b-a)^3}{(n-\beta+2)(n-\beta+3)}\left(1-\frac{n-\beta+4}{2^{n-\beta+2}}\right).\end{aligned}$$

COROLLARY 6. *By setting $C = 0$ and $m = 1$ in inequality (3.1), we will get the following inequality for s-convex function via Caputo fractional derivatives*

$$\begin{aligned}&\left|\frac{\psi^{(n)}(a)+\psi^{(n)}(b)}{2}-\frac{\Gamma(n-\beta+1)}{2(b-a)^{n-\beta}}\left[({}^C D^{\beta}_{a^+}\psi)(b)+(-1)^n({}^C D^{\beta}_{b^-}\psi)(a)\right]\right|\\&\leq\frac{(b-a)\left[|\psi^{(n+1)}(a)|+|\psi^{(n+1)}(b)|\right]}{2}\left\{\frac{(2^{np-\beta p+1}-1)^{\frac{1}{p}}-(2^{qs+1}-1)^{\frac{1}{q}}}{(2^{np-\beta p+1}(np-\beta p+1))^{\frac{1}{p}}(2^{qs+1}(qs+1))^{\frac{1}{q}}}\right.\\&\left.-\frac{2^{n-\beta+s+1}-2}{2^{n-\beta+s+1}(n-\beta+s+1)}\right\}.\end{aligned}$$

REMARK 5. (i) If $s = 1$ in (3.1), then we will get the fractional Hadamard inequality for strongly m-convex function which is given in [8, Theorem 8].

(ii) If $s = 1$ and $m = 1$ in (3.1), then we will get the fractional Hadamard inequality for strongly convex function which is given in [7, Theorem 8].

(iii) If $C = 0$, $m = 1$ and $s = 1$ in (3.1), then we will get the fractional Hadamard inequality which is stated in Theorem 2.

By using Lemma 2 we give the following error bounds of Caputo fractional derivative inequality (2.11).

THEOREM 9. *Let $s \in [0, 1]$ and $\psi :\to \mathbb{R}$ be a differentiable mapping on (a, b) with $\psi \in C^{n+1}[a, b]$, $m \in (0, 1]$ and $a < b$. If $|\psi^{(n+1)}|^q$ is strongly (s, m)-convex function on $[a, b]$ for $q \geq 1$, then the undermentioned inequality for Caputo fractional derivatives holds:*

(3.5)

$$\begin{aligned}&\left|\frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(mb-a)^{n-\beta}}\left[({}^C D^{\beta}_{(\frac{a+bm}{2})^+}\psi)(mb)+m^{n-\beta+1}(-1)^n({}^C D^{\beta}_{(\frac{a+mb}{2m})^-}\psi)\left(\frac{a}{m}\right)\right]\right.\\&\left.-\frac{1}{2}\left[\psi^{(n)}\left(\frac{a+mb}{2}\right)+m\psi^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|\leq\frac{mb-a}{4(n-\beta+1)^{\frac{1}{p}}}\left[\left(\frac{|\psi^{(n+1)}(a)|^q}{2^s(n-\beta+s+1)}\right.\right.\\&\left.+\frac{m|\psi^{(n+1)}(b)|^q(2^{qs+1}-1)^{\frac{1}{q}}}{2^s(np-\beta p+1)^{\frac{1}{p}}(qs+1)^{\frac{1}{q}}}-\frac{Cm(b-a)^2(n-\beta+4)}{4(n-\beta+2)(n-\beta+3)}\right)^{\frac{1}{q}}\\&\left.+\left(\frac{m\left|\psi^{(n+1)}\left(\frac{a}{m^2}\right)\right|^q(2^{qs+1}-1)^{\frac{1}{q}}}{2^s(np-\beta p+1)^{\frac{1}{p}}(qs+1)^{\frac{1}{q}}}+\frac{|\psi^{(n+1)}(b)|^q}{2^s(np-\beta p+1)^{\frac{1}{p}}}-\frac{Cm\left(b-\frac{a}{m^2}\right)^2(n-\beta+4)}{4(n-\beta+2)(n-\beta+3)}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. By taking $k=1$ in Lemma 2 and with the help of modulus property, we have

$$\Bigg|\frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(mb-a)^{n-\beta}}\left[({}^{C}D^{\beta}_{(\frac{a+bm}{2})+}\psi)(mb)+m^{n-\beta+1}(-1)^{n}({}^{C}D^{\beta}_{(\frac{a+mb}{2m})-}\psi)\left(\frac{a}{m}\right)\right]$$
$$-\frac{1}{2}\left[\psi^{(n)}\left(\frac{a+mb}{2}\right)+m\psi^{(n)}\left(\frac{a+mb}{2m}\right)\right]\Bigg|$$
$$\leq\frac{mb-a}{4}\left[\int_0^1 z^{n-\beta}\left(\left|\psi^{(n+1)}\left(\frac{z}{2}a+m\left(\frac{2-z}{2}\right)b\right)\right|+\left|\psi^{(n+1)}\left(\frac{2-z}{2m}a+\frac{z}{2}b\right)\right|\right)dz\right].$$

Now, applying Lemma 2, using power mean inequality we have

$$\Bigg|\frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(mb-a)^{n-\beta}}\left[({}^{C}D^{\beta}_{(\frac{a+bm}{2})+}\psi)(mb)+m^{n-\beta+1}(-1)^{n}({}^{C}D^{\beta}_{(\frac{a+mb}{2m})-}\psi)\left(\frac{a}{m}\right)\right]$$
$$-\frac{1}{2}\left[\psi^{(n)}\left(\frac{a+mb}{2}\right)+m\psi^{(n)}\left(\frac{a+mb}{2m}\right)\right]\Bigg|$$
$$\leq\frac{mb-a}{4}\left[\int_0^1 z^{n-\beta}\left(\left|\psi^{(n+1)}\left(\frac{z}{2}a+m\left(\frac{2-z}{2}\right)b\right)\right|+\left|\psi^{(n+1)}\left(\frac{2-z}{2m}a+\frac{z}{2}b\right)\right|\right)dz\right]$$
$$\leq\frac{mb-a}{4}\left(\int_0^1 z^{n-\beta}dz\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 z^{n-\beta}\left|\psi^{(n+1)}\left(a\frac{z}{2}+m\left(\frac{2-z}{2}\right)b\right)\right|^q dz\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 z^{n-\beta}\left|\psi^{(n+1)}\left(a\left(\frac{2-z}{2m}\right)+b\frac{z}{2}\right)\right|^q dz\right)^{\frac{1}{q}}\right].$$

By using the strong (s,m)-convexity of $|\psi^{(n+1)}|^q$, we have

$$\leq\frac{mb-a}{4(n-\beta+1)^{\frac{1}{p}}}\left[\left(|\psi^{(n+1)}(a)|^q\int_0^1\frac{z^{n-\beta+s}}{2^s}dz+m|\psi^{(n+1)}(b)|^q\int_0^1\frac{z^{n-\beta}(2-z)^s}{2^s}dz\right.\right.$$
$$\left.-\frac{Cm(b-a)^2}{4}\int_0^1(2z^{n-\beta+1}-z^{n-\beta+2})dz\right)^{\frac{1}{q}}+\left(m\left|\psi^{(n+1)}\left(\frac{a}{m^2}\right)\right|^q\int_0^1\frac{z^{n-\beta}(2-z)^s}{2^s}dz\right.$$
$$\left.\left.+|\psi^{(n+1)}(b)|^q\int_0^1\frac{z^{n-\beta+s}}{2^s}dz-\frac{Cm\left(b-\frac{a}{m^2}\right)^2}{4}\int_0^1(2z^{n-\beta+1}-z^{n-\beta+2})dz\right)^{\frac{1}{q}}\right]$$
$$=\frac{mb-a}{4(n-\beta+1)^{\frac{1}{p}}}\left[\left(\frac{|\psi^{(n+1)}(a)|^q}{2^s(n-\beta+s+1)}+\frac{m|\psi^{(n+1)}(b)|^q(2^{qs+1}-1)^{\frac{1}{q}}}{2^s(np-\beta p+1)^{\frac{1}{p}}(qs+1)^{\frac{1}{q}}}\right.\right.$$
$$\left.-\frac{Cm(b-a)^2(n-\beta+4)}{4(n-\beta+2)(n-\beta+3)}\right)^{\frac{1}{q}}+\left(\frac{m\left|\psi^{(n+1)}\left(\frac{a}{m^2}\right)\right|^q(2^{qs+1}-1)^{\frac{1}{q}}}{2^s(np-\beta p+1)^{\frac{1}{p}}(qs+1)^{\frac{1}{q}}}\right.$$
$$\left.\left.+\frac{|\psi^{(n+1)}(b)|^q}{2^s(np-\beta p+1)^{\frac{1}{p}}}-\frac{Cm\left(b-\frac{a}{m^2}\right)^2(n-\beta+4)}{4(n-\beta+2)(n-\beta+3)}\right)^{\frac{1}{q}}\right],$$

inequality (3.5) is obtained. □

COROLLARY 7. *By setting $m = 1$ in inequality (3.5), we will get the following inequality for strongly s-convex function via Caputo fractional derivatives*

$$\left|\frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(b-a)^{n-\beta}}\left[({}^{C}D^{\beta}_{(\frac{a+b}{2})^{+}}\psi)(b)+(-1)^{n}({}^{C}D^{\beta}_{(\frac{a+b}{2})^{-}}\psi)(a)\right]-\psi^{(n)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4(n-\beta+1)^{\frac{1}{p}}}\left[\left(\frac{|\psi^{(n+1)}(a)|^{q}}{2^{s}(n-\beta+s+1)}+\frac{|\psi^{(n+1)}(b)|^{q}(2^{qs+1}-1)^{\frac{1}{q}}}{2^{s}(np-\beta p+1)^{\frac{1}{p}}(qs+1)^{\frac{1}{q}}}\right.\right.$$
$$\left.-\frac{C(b-a)^{2}(n-\beta+4)}{4(n-\beta+2)(n-\beta+3)}\right)^{\frac{1}{q}}+\left(\frac{\left|\psi^{(n+1)}(a)\right|^{q}(2^{qs+1}-1)^{\frac{1}{q}}}{2^{s}(np-\beta p+1)^{\frac{1}{p}}(qs+1)^{\frac{1}{q}}}\right.$$
$$\left.\left.+\frac{|\psi^{(n+1)}(b)|^{q}}{2^{s}(np-\beta p+1)^{\frac{1}{p}}}-\frac{C(b-a)^{2}(n-\beta+4)}{4(n-\beta+2)(n-\beta+3)}\right)^{\frac{1}{q}}\right].$$

COROLLARY 8. *By setting $C = 0$ and $m = 1$ in inequality (3.5), we will get the following inequality for s-convex function via Caputo fractional derivatives*

$$\left|\frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(b-a)^{n-\beta}}\left[({}^{C}D^{\beta}_{(\frac{a+b}{2})^{+}}\psi)(b)+(-1)^{n}({}^{C}D^{\beta}_{(\frac{a+b}{2})^{-}}\psi)(a)\right]-\psi^{(n)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4(n-\beta+1)^{\frac{1}{p}}}\left[\left(\frac{|\psi^{(n+1)}(a)|^{q}}{2^{s}(n-\beta+s+1)}+\frac{|\psi^{(n+1)}(b)|^{q}(2^{qs+1}-1)^{\frac{1}{q}}}{2^{s}(np-\beta p+1)^{\frac{1}{p}}(qs+1)^{\frac{1}{q}}}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left|\psi^{(n+1)}(a)\right|^{q}(2^{qs+1}-1)^{\frac{1}{q}}}{2^{s}(np-\beta p+1)^{\frac{1}{p}}(qs+1)^{\frac{1}{q}}}+\frac{|\psi^{(n+1)}(b)|^{q}}{2^{s}(np-\beta p+1)^{\frac{1}{p}}}\right)^{\frac{1}{q}}\right].$$

REMARK 6. (i) If $s = 1$ in (3.5), then we will get the fractional Hadamard inequality for strongly m-convex function which is given in [8, Theorem 9].

(ii) If $s = 1$ and $m = 1$ in (3.5), then we will get the fractional Hadamard inequality for strongly convex function which is given in [7, Theorem 9].

(iii) If $C = 0$, $m = 1$ and $s = 1$ in (3.5), then we will get the fractional Hadamard inequality which is stated in Theorem 4.

THEOREM 10. *Let $s \in [0,1]$ and $\psi : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $\psi \in C^{n+1}[a,b]$, $m \in (0,1]$ and $a < b$. If $|\psi^{(n+1)}|^{q}$ is a strongly (s,m)-convex function on $[a,b]$ for $q > 1$, then the undermentioned inequality for Caputo fractional derivatives holds:*

$$\left|\frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(mb-a)^{n-\beta}}\left[({}^{C}D^{\beta}_{(\frac{a+bm}{2})^{+}}\psi)(mb)+m^{n-\beta+1}(-1)^{n}({}^{C}D^{\beta}_{(\frac{a+mb}{2m})^{-}}\psi)\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[\psi^{(n)}\left(\frac{a+mb}{2}\right)+m\psi^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|\leq \frac{bm-a}{4(np-\beta p+1)^{\frac{1}{p}}}$$
$$\left[\left(\frac{\left(|\psi^{(n+1)}(a)|+m^{\frac{1}{q}}|\psi^{(n+1)}(b)|\right)^{q}}{2^{s}(s+1)}-\frac{Cm(b-a)^{2}}{6}\right)^{\frac{1}{q}}\right. \tag{3.6}$$
$$\left.+\left(\frac{\left(m^{\frac{1}{q}}\left|\psi^{(n+1)}\left(\frac{a}{m^{2}}\right)\right|+|\psi^{(n+1)}(b)|\right)^{q}}{2^{s}(s+1)}-\frac{Cm\left(b-\frac{a}{m^{2}}\right)^{2}}{6}\right)^{\frac{1}{q}}\right].$$

Proof. By taking $k=1$ in Lemma 2 and with the help of modulus property, we get

$$\left|\frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(mb-a)^{n-\beta}}\left[({}^{C}D^{\beta}_{(\frac{a+bm}{2})^{+}}\psi)(mb)+m^{n-\beta+1}(-1)^{n}({}^{C}D^{\beta}_{(\frac{a+mb}{2m})^{-}}\psi)\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[\psi^{(n)}\left(\frac{a+mb}{2}\right)+m\psi^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|\leq\frac{mb-a}{4}$$
$$\left[\int_0^1 z^{n-\beta}\left|\psi^{(n+1)}\left(\frac{z}{2}a+m\frac{(2-z)}{2}b\right)\right|dz+\int_0^1 z^{n-\beta}\left|\psi^{(n+1)}\left(\frac{2-z}{2m}a+\frac{z}{2}b\right)\right|dz\right].$$

Now applying Holder's inequality, we get

$$\left|\frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(mb-a)^{n-\beta}}\left[({}^{C}D^{\beta}_{(\frac{a+bm}{2})^{+}}\psi)(mb)+m^{n-\beta+1}(-1)^{n}({}^{C}D^{\beta}_{(\frac{a+mb}{2m})^{-}}\psi)\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[\psi^{(n)}\left(\frac{a+mb}{2}\right)+m\psi^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{4(np-\beta p+1)^{\frac{1}{p}}}\left[\left(\int_0^1\left|\psi^{(n+1)}\left(\frac{z}{2}a+m\frac{(2-z)b}{2}\right)\right|^q dz\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|\psi^{(n+1)}\left(\frac{2-z}{2m}a+\frac{z}{2}b\right)\right|^q dz\right)^{\frac{1}{q}}\right].$$

Using strong (s,m)-convexity of $|\psi^{(n+1)}|^q$, we get

$$\left|\frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(mb-a)^{n-\beta}}\left[({}^{C}D^{\beta}_{(\frac{a+bm}{2})^{+}}\psi)(mb)+m^{n-\beta+1}(-1)^{n}({}^{C}D^{\beta}_{(\frac{a+mb}{2m})^{-}}\psi)\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[\psi^{(n)}\left(\frac{a+mb}{2}\right)+m\psi^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{4(np-\beta p+1)^{\frac{1}{p}}}\left[\left(|\psi^{(n+1)}(a)|^q\int_0^1\left(\frac{z}{2}\right)^s dz+m|\psi^{(n+1)}(b)|^q\int_0^1\left(\frac{2-z}{2}\right)^s dz\right.\right.$$
$$\left.-\frac{Cm(b-a)^2}{4}\int_0^1(2z-z^2)dz\right)^{\frac{1}{q}}+\left(m\left|\psi^{(n+1)}\left(\frac{a}{m^2}\right)\right|^q\int_0^1\left(\frac{2-z}{2}\right)^s dz\right.$$
$$\left.\left.+|\psi^{(n+1)}(b)|^q\int_0^1\left(\frac{z}{2}\right)^s dz-\frac{Cm\left(b-\frac{a}{m^2}\right)^2}{4}\int_0^1(2z-z^2)dz\right)^{\frac{1}{q}}\right]$$

$$= \frac{mb-a}{4}\left(\frac{1}{np-\beta p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\psi^{(n+1)}(a)|^q + m|\psi^{(n+1)}(b)|^q}{2^s(s+1)} - \frac{2Cm(b-a)^2}{3}\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\frac{m\left|\psi^{(n+1)}\left(\frac{a}{m^2}\right)\right|^q + |\psi^{(n+1)}(b)|^q}{2^s(s+1)} - \frac{2Cm\left(b-\frac{a}{m^2}\right)^2}{3}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{bm-a}{4(np-\beta p+1)^{\frac{1}{p}}}\left[\left(\frac{\left(|\psi^{(n+1)}(a)| + m^{\frac{1}{q}}|\psi^{(n+1)}(b)|\right)^q}{2^s(s+1)} - \frac{Cm(b-a)^2}{6}\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\frac{\left(m^{\frac{1}{q}}\left|\psi^{(n+1)}\left(\frac{a}{m^2}\right)\right| + |\psi^{(n+1)}(b)|\right)^q}{2^s(s+1)} - \frac{Cm\left(b-\frac{a}{m^2}\right)^2}{6}\right)^{\frac{1}{q}}\right].$$

Here we have used the fact that $(a_1+b_1)^q \geq (a_1)^q + (b_1)^q$, where $q>1$, $a_1, b_1 \geq 0$. This completes the proof. □

COROLLARY 9. *By setting $m=1$ in inequality (3.6), we will get the following inequality for strongly s-convex function via Caputo fractional derivatives*

$$\left|\frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(b-a)^{n-\beta}}\left[({}^C D^{\beta}_{(\frac{a+b}{2})^+}\psi)(b) + (-1)^n({}^C D^{\beta}_{(\frac{a+b}{2})^-}\psi)(a)\right] - \psi^{(n)}\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{2(np-\beta p+1)^{\frac{1}{p}}}\left(\frac{\left(|\psi^{(n+1)}(a)| + |\psi^{(n+1)}(b)|\right)^q}{2^s(s+1)} - \frac{C(b-a)^2}{6}\right)^{\frac{1}{q}}.$$

COROLLARY 10. *By setting $C=0$ and $m=1$ in inequality (3.6), we will get the following inequality for s-convex function via Caputo fractional derivatives*

$$\left|\frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(b-a)^{n-\beta}}\left[({}^C D^{\beta}_{(\frac{a+b}{2})^+}\psi)(b) + (-1)^n({}^C D^{\beta}_{(\frac{a+b}{2})^-}\psi)(a)\right]\right.$$

$$\left. -\psi^{(n)}\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{2(np-\beta p+1)^{\frac{1}{p}}}\left(\frac{\left(|\psi^{(n+1)}(a)| + |\psi^{(n+1)}(b)|\right)^q}{2^s(s+1)}\right)^{\frac{1}{q}}.$$

REMARK 7. (i) If $s=1$ in (3.6), then we will get the fractional Hadamard inequality for strongly m-convex function which is given in [8, Theorem 10].

(ii) If $s=1$ and $m=1$ in (3.6), then we will get the fractional Hadamard inequality for strongly convex function which is given in [7, Theorem 10].

(iii) If $C=0$, $m=1$ and $s=1$ in (3.6), then we will get the fractional Hadamard inequality which is stated in Theorem 5.

Concluding remarks

This paper provides fractional derivative inequalities for strongly (s,m)-convex functions. There are many known inequalities are associated with the special cases of the results presented here. The refinements of many inequalities are concluded in the form of corollaries and remarks for convex, s-convex, m-convex, (s, m)-convex functions.

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Ghulam Farid
Department of Mathematics, COMSATS University Islamabad,
Attock Campus, Pakistan.
E-mail: faridphdsms@outlook.com, ghlmfarid@ciit-attock.edu.pk

Sidra Bibi
Govt. Girls Primary School, Kamra Khurd, Attock 43570, Pakistan.
E-mail: sidra.mpa2016@gmail.com

Laxmi Rathour
Ward number: 16, Bhagatbandh, Anuppur 484 224, Madhya Pradesh, India.
E-mail: laxmirathour817@gmail.com, rathourlaxmi562@gmail.com

Lakshmi Narayan Mishra
Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology, Vellore, Tamil Nadu 632 014, India
E-mail: lakshminarayanmishra04@gmail.com

Vishnu Narayan Mishra
Department of Mathematics, Indira Gandhi National Tribal University,
Lalpur, Amarkantak, Anuppur, Madhya Pradesh 484 887, India.
E-mail: vishnunarayanmishra@gmail.com