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SOME RESULTS CONCERNING FIXED POINT IN VECTOR SPACES

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Abstract. In this paper, we study the generalization of the Banach contraction principle in the vector space, involving four rational square terms in the inequality, by using the notation of bilinear functional. We also present an extension of Selberg's inequality to vector space.

1. Inroduction and Basic Concepts

The study of properties and application of fixed points of various types of contractive mapping in Hilbert spaces were obtained, among others, by Browder and Petryshyn [1], Hicks and Huffman, [3], Huffman [4], Koparde and Waghmode [5]. We refer to [8, 10] for more examples and properties of fixed point theorems in Hilbert space.

The object of the present note is to present a fixed point theorem in vector space by using the quadratic form. In [9], the authors studied the generalization of the Banach contraction principle in the Hilbert space, involving four rational square terms in the inequality. This paper obtains similar results for bilinear functionals in vector space. Meanwhile, we present the extension of some well-known results in Hilbert space to vector space using the notion of bilinear functional (see also [6, 7] and the references therein).

This section gives some definitions and preliminary results, which will be used in our paper.

Definition 1.1. ([2], Definition 4.3.1) By a bilinear functional φ on a complex vector space E, we mean a mapping $\varphi : E \times E \to \mathbb{C}$ satisfying the following two conditions:

1. $\varphi(\alpha x_1 + \beta x_2, y) = \alpha\varphi(x_1, y) + \beta\varphi(x_2, y)$,
2. $\varphi(x, \alpha y_1 + \beta y_2) = \overline{\alpha}\varphi(x, y_1) + \overline{\beta}\varphi(x, y_2)$,

for any scalars α and β and any $x,\ x_1,\ x_2,\ y,\ y_1,\ y_2 \in E$.

Definition 1.2. ([2], Definition 4.3.6) Let φ be a bilinear functional on a vector space E. The function $\Phi : E \to \mathbb{C}$ defined by $\Phi(x) = \varphi(x, x)$ is called the quadratic

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form associated with φ. A quadratic form Φ on a normed space E is called bounded if there exists a constant $K > 0$ such that

$$|\Phi(x)| \leq K\|x\|^2$$

for all $x \in E$.

Definition 1.3. Let φ be a bilinear functional, then $x \to \varphi(x, x)$ is a continuous bilinear functional if its components are continuous.

2. Some New Results for Bilinear Functional

This section is dedicated to studying new results around bilinear functionals. We start with the following theorem, which can be considered an extension of parallelogram law in vector space.

Theorem 2.1. *For any two elements x and y of vector space E, we have*

$$\Phi(x+y) + \Phi(x-y) = 2(\Phi(x) + \Phi(y)). \tag{1}$$

Proof. We have

$$\varphi(x+y, x+y) = \varphi(x,x) + \varphi(x,y) + \varphi(y,x) + \varphi(y,y). \tag{2}$$

Moreover

$$\varphi(x-y, x-y) = \varphi(x,x) - \varphi(x,y) - \varphi(y,x) + \varphi(y,y). \tag{3}$$

By adding (2) and (3), we obtain the desired result, since

$$2\varphi(x,x) + 2\varphi(y,y) = 2(\Phi(x) + \Phi(y)).$$

□

Theorem 2.2. *If Y is a closed convex subset of a complete vector space E, and $x_0 \in E$, there is a unique element y_0 of Y such that*

$$\sqrt{\Phi(x_0 - y_0)} \leq \sqrt{\Phi(x_0 - y)}, \quad (y \in Y). \tag{4}$$

Moreover

$$\operatorname{Re}\varphi(y_0, x_0 - y_0) \geq \operatorname{Re}\varphi(y, x_0 - y_0), \ (y \in Y). \tag{5}$$

Proof. With

$$d = \inf\left\{\sqrt{\Phi(x_0 - y)} : \ y \in Y\right\}$$

there is a sequence $\{y_n\}$ of a elements of Y such that $\sqrt{\Phi(x_0 - y_n)} \to d$. By Theorem 2.1

$$2\Phi(x_0 - y_m) + 2\Phi(x_0 - y_n) = \Phi(2x_0 - y_m - y_n) + \Phi(y_n - y_m)$$

for all positive integer m and n. Since $\frac{1}{2}(y_m + y_n) \in Y$, we have

$$\sqrt{\Phi(2x_0 - y_m - y_n)} = 2\sqrt{\Phi\left(x_0 - \frac{1}{2}(y_m + y_n)\right)} \geq 2d$$

and therefore

$$\begin{aligned}\Phi(y_n - y_m) &= 2\Phi(x_0 - y_m) + 2\Phi(x_0 - y_n) - \Phi(2x_0 - y_m - y_n) \\ &\leq 2\Phi(x_0 - y_m) + 2\Phi(x_0 - y_n) - 4d^2 \to 0\end{aligned}$$

as min $(m,n) \to \infty$. Hence $\{y_n\}$ is a Cauchy sequence and so converges to an element y_0 of E. Moreover, $y_0 \in Y$ since Y is closed, and y_0 satisfies (4), since

$$\sqrt{\Phi(x_0 - y_0)} = \lim_{n\to\infty} \sqrt{\Phi(x_0 - y_n)} = d = \inf\left\{\sqrt{\Phi(x_0 - y)} : \ y \in Y\right\}.$$

If $y_0^{'}$ is another element of Y that satisfies (4), then $\sqrt{\Phi\left(x_0 - y_0^{'}\right)} = \sqrt{\Phi(x_0 - y_0)} = d$. We can apply the preceding reasoning, with y_0 and $y_0^{'}$ in place of y_m and y_n, to obtain

$$\begin{aligned}\Phi\left(y_0^{'} - y_0\right) &= 2\Phi(x_0 - y_0) + 2\Phi\left(x_0 - y_0^{'}\right) - 4\Phi\left(x_0 - \frac{1}{2}\left(y_0 + y_0^{'}\right)\right) \\ &\leq 2d^2 + 2d^2 - 4d^2 = 0.\end{aligned}$$

Hence $y_0^{'} = y_0$, and y_0 is uniquely determined by (4). For each y in Y and t in $(0,1)$, $y_0 + t(y - y_0) \in Y$, and (4) gives

$$\begin{aligned}\Phi(x_0 - y_0) &\leq \Phi(x_0 - y_0 - t(y - y_0)) \\ &= \Phi(x_0 - y_0) - 2t \operatorname{Re}\varphi(y - y_0, x_0 - y_0) + t^2\Phi(y - y_0).\end{aligned}$$

Hence

$$-2\operatorname{Re}\varphi(y - y_0, x_0 - y_0) + t\Phi(y - y_0) \geq 0, \quad (0 < t < 1)$$

and this gives (5) when $t \to 0$. □

Theorem 2.3. *If Y is a closed subspace of a complete vector space E, each element x_0 of E can be expressed uniquely in the form $y_0 + z_0$, with y_0 in Y and z_0 in $Y^{\perp}$. Moreover, y_0 is the unique point in Y that is closest to x_0.*

Proof. Since Y is a closed convex subset of E, we can choose y_0 as in Theorem 2.2, and define $z_0 = x_0 - y_0$. From (4) and (5), y_0 is the (unique) point in Y that is closest to x_0, and $\operatorname{Re}\varphi(y, z_0) \leq \operatorname{Re}\varphi(y_0, z_0)$ for each y in Y. By writing ay in place of y, we obtain

$$\operatorname{Re}\ a\varphi(y, z_0) \leq \operatorname{Re}\varphi(y_0, z_0)$$

for $y \in Y$ and $a \in \mathbb{C}$. Hence $\varphi(y, z_0) = 0$ for each y in Y, and $z_0 \in Y^{\perp}$. This proves the existence of a decomposition $x_0 = y_0 + z_0$, with y_0 in Y and z_0 in $Y^{\perp}$. If, also $x_0 = y_1 + z_1$, with y_1 in Y and z_1 in $Y^{\perp}$, then

$$y_0 + z_0 = y_1 + z_1$$

and

$$y_0 - y_1 = z_1 - z_0 \in Y \cap Y^{\perp} = \{0\}$$

therefore $y_0 = y_1$, and $z_0 = z_1$. □

Theorem 2.4. *If Y is a closed subspace of a complete vector space E and $X \subseteq E$, then*

$$\left(Y^{\perp}\right)^{\perp} = Y.$$

Proof. If $y \in Y$, then y is orthogonal to each element of $Y^{\perp}$, and so $y \in \left(Y^{\perp}\right)^{\perp}$. This shows that $Y \subseteq \left(Y^{\perp}\right)^{\perp}$, and we have to prove the reverse inclusion. With x_0 in $\left(Y^{\perp}\right)^{\perp}$, we can choose y_0 in Y and z_0 in $Y^{\perp}$ so that $x_0 = y_0 + z_0$, by Theorem 2.3. Then $x_0 \in \left(Y^{\perp}\right)^{\perp}$, $y_0 \in Y \subseteq \left(Y^{\perp}\right)^{\perp}$, and therefore $z_0 = x_0 - y_0 \in \left(Y^{\perp}\right)^{\perp}$. Hence

$$z_0 \in Y^{\perp} \cap \left(Y^{\perp}\right)^{\perp} = \{0\}$$

and $x_0 = y_0 \in Y$. This gives the required inclusion $\left(Y^{\perp}\right)^{\perp} \subseteq Y$, so $\left(Y^{\perp}\right)^{\perp} = Y$. If $Y = E$, then $Y^{\perp} = E^{\perp} = \{0\}$; convesely, if $Y^{\perp} = \{0\}$, then $Y = \left(Y^{\perp}\right)^{\perp} = \{0\}^{\perp} = E$. □

3. An Interesting Inequality Involving Bilinear Functional

Selberg's inequality is an interesting inequality that is also a generalization of the Cauchy-Schwarz and Bessel inequality. In this section, we present a Selberg-type inequality involving bilinear functionals.

THEOREM 3.1. *In a vector space E*

$$\sum_{j=1}^{n} \frac{\varphi\left(x, y_j\right)}{\sum\limits_{k=1}^{n} \varphi\left(y_j, y_k\right)} \leq \varphi\left(x, x\right) \tag{6}$$

for all $x \in E$ and $y_j \neq 0$ $(y_j \in E)$. The equality (6) *holds if and only if*

$$x = \sum_{j=1}^{n} \alpha_j y_j, \quad (\alpha_j \in \mathbb{C}) \tag{7}$$

and for each pair (j, k), $j \neq k$,

$$\varphi\left(y_j, y_k\right) = 0 \tag{8}$$

or

$$|\alpha_j| = |\alpha_k| \ \text{ and } \varphi\left(\alpha_j y_j, \alpha_k y_k\right) \geq 0. \tag{9}$$

Proof. For any $\alpha_j \in \mathbb{C}$, we have

$$\begin{aligned}
0 &\leq \Phi\left(x - \sum_{j=1}^{n} \alpha_j y_j\right) \\
&= \varphi\left(x - \sum_{j=1}^{n} \alpha_j y_j, x - \sum_{j=1}^{n} \alpha_j y_j\right) \\
&= \varphi\left(x, x\right) - \varphi\left(x, \sum_{j=1}^{n} \alpha_j y_j\right) - \varphi\left(\sum_{j=1}^{n} \alpha_j y_j, x\right) + \varphi\left(\sum_{j=1}^{n} \alpha_j y_j, \sum_{j=1}^{n} \alpha_j y_j\right) \\
&= \varphi\left(x, x\right) - \sum_{j=1}^{n} \alpha_j \varphi\left(y_j, x\right) - \sum_{j=1}^{n} \overline{\alpha_j} \varphi\left(x, y_j\right) + \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \overline{\alpha_k} \varphi\left(y_j, y_k\right).
\end{aligned}$$

From $0 \leq \left(|\alpha_j| - |\alpha_k|\right)^2$, we have $|\alpha_j \overline{\alpha_k}| \leq \frac{1}{2}|\alpha_j|^2 + \frac{1}{2}|\alpha_k|^2$, so the last quantity is equal to or less than the following

$$\varphi\left(x, x\right) - \sum_{j=1}^{n} \alpha_j \overline{\varphi\left(x, y_j\right)} - \sum_{j=1}^{n} \overline{\alpha_j} \varphi\left(x, y_j\right) + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} |\alpha_j|^2 \varphi\left(y_j, y_k\right) + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} |\alpha_k|^2 \varphi\left(y_j, y_k\right).$$

We can choose $\alpha_j = \frac{\varphi(x,y_j)}{\sum\limits_{k=1}^{n}\varphi(y_j,y_k)}$. Thus, the above quantity will be equal to

$$\varphi(x,x) - \sum_{j=1}^{n} \frac{\varphi(x,y_j)\overline{\varphi(x,y_j)}}{\sum\limits_{k=1}^{n}\varphi(y_j,y_k)} - \sum_{j=1}^{n} \frac{\overline{\varphi(x,y_j)}\varphi(x,y_j)}{\sum\limits_{k=1}^{n}\varphi(y_j,y_k)} + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{(\varphi(x,y_j))^2\varphi(y_j,y_k)}{\left(\sum\limits_{k=1}^{n}\varphi(y_j,y_k)\right)^2}$$

$$+ \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{(\varphi(x,y_k))^2\varphi(y_j,y_k)}{\left(\sum\limits_{k=1}^{n}\varphi(y_k,y_j)\right)^2},$$

which is, after a simple calculation, equal to

$$\varphi(x,x) - \sum_{j=1}^{n} \frac{(\varphi(x,y_j))^2}{\sum\limits_{k=1}^{n}\varphi(y_j,y_k)} - \sum_{j=1}^{n} \frac{(\varphi(x,y_j))^2}{\sum\limits_{k=1}^{n}\varphi(y_j,y_k)} + \frac{1}{2}\sum_{j=1}^{n}(\varphi(x,y_j))^2 \frac{\sum\limits_{k=1}^{n}\varphi(y_j,y_k)}{\left(\sum\limits_{k=1}^{n}\varphi(y_j,y_k)\right)^2}$$

$$+ \frac{1}{2}\sum_{k=1}^{n}(\varphi(x,y_j))^2 \frac{\sum\limits_{j=1}^{n}\varphi(y_j,y_k)}{\left(\sum\limits_{j=1}^{n}\varphi(y_j,y_k)\right)^2}$$

and, of course, it is equivalent to

$$\varphi(x,x) - 2\sum_{j=1}^{n} \frac{(\varphi(x,y_j))^2}{\sum\limits_{k=1}^{n}\varphi(y_j,y_k)} + \frac{1}{2}\sum_{j=1}^{n} \frac{(\varphi(x,y_j))^2}{\sum\limits_{k=1}^{n}\varphi(y_j,y_k)} + \frac{1}{2}\sum_{j=1}^{n} \frac{(\varphi(x,y_k))^2}{\sum\limits_{j=1}^{n}\varphi(y_j,y_k)}$$

$$= \varphi(x,x) - 2\sum_{j=1}^{n} \frac{(\varphi(x,y_j))^2}{\sum\limits_{k=1}^{n}\varphi(y_j,y_k)} + \frac{1}{2}\sum_{j=1}^{n} \frac{(\varphi(x,y_j))^2}{\sum\limits_{k=1}^{n}\varphi(y_j,y_k)} + \frac{1}{2}\sum_{j=1}^{n} \frac{(\varphi(x,y_j))^2}{\sum\limits_{k=1}^{n}\varphi(y_j,y_k)}$$

and $\sum\limits_{j=1}^{n} \frac{(\varphi(x,y_j))^2}{\sum\limits_{k=1}^{n}\varphi(y_j,y_k)} \le \varphi(x,x)$ follows. We will show that, if $x = \sum\limits_{j=1}^{n}\alpha_j y_j$, $\alpha_j \in \mathbb{C}$, and for each pair (j,k), $j \ne k$, then

$$\sum_{j=1}^{n} \frac{(\varphi(x,y_j))^2}{\sum\limits_{k=1}^{n}\varphi(y_j,y_k)} = \varphi(x,x)$$

so

$$x = \sum_{j=1}^{n}\alpha_j y_j \;\wedge\; 2\alpha_j\overline{\alpha_k}\varphi(y_j,y_k) = |\alpha_j|^2\varphi(y_j,y_k) + |\alpha_k|^2\varphi(y_j,y_k). \tag{10}$$

Let $x = \sum\limits_{j=1}^{n}\alpha_j y_j$, $\alpha_j \in \mathbb{C}$. Then for each (j,k), $j \ne k$ where $\varphi(y_j,y_k) = 0$, we have

$$\varphi(\alpha_k y_k, \alpha_j y_j) = |\alpha_j|^2\varphi(y_k,y_j) \tag{11}$$

and for each (j,k), $j \neq k$ where (9) is true, we have (11). Moreover,

$$\begin{aligned}
\sum_{j=1}^{n} \frac{\left(\varphi\left(\sum\limits_{k=1}^{n} \alpha_k y_k, y_j\right)\right)^2}{\sum\limits_{k=1}^{n} \varphi\left(y_j, y_k\right)} &= \sum_{j=1}^{n} \frac{\left(\sum\limits_{k=1}^{n} \alpha_k \varphi\left(y_k, y_j\right)\right)^2}{\sum\limits_{k=1}^{n} \varphi\left(y_j, y_k\right)} \\
&= \sum_{j=1}^{n} \frac{\left(\sum\limits_{k=1}^{n} \alpha_k \varphi\left(y_k, y_j\right)\right)^2 |\alpha_j|^2}{\sum\limits_{k=1}^{n} \varphi\left(y_j, y_k\right)|\alpha_j|^2} \\
&= \sum_{j=1}^{n} \frac{\left(\sum\limits_{k=1}^{n} \alpha_k \varphi\left(y_k, y_j\right)\right) \sum\limits_{k=1}^{n} \overline{\alpha_k} \varphi\left(y_j, y_k\right) \alpha_j \overline{\alpha_j}}{\sum\limits_{k=1}^{n} \varphi\left(y_j, y_k\right)|\alpha_j|^2} \\
&= \sum_{j=1}^{n} \frac{\left(\sum\limits_{k=1}^{n} \varphi\left(\alpha_k y_k, \alpha_j y_j\right)\right) \sum\limits_{k=1}^{n} \alpha_j \overline{\alpha_k} \varphi\left(y_j, y_k\right)}{\sum\limits_{k=1}^{n} \varphi\left(y_k, y_j\right)|\alpha_j|^2}.
\end{aligned}$$

We use (11), and we have

$$\sum_{j=1}^{n} \frac{\left(\sum\limits_{k=1}^{n} \varphi\left(\alpha_k y_k, \alpha_j y_j\right)\right) \sum\limits_{k=1}^{n} \alpha_j \overline{\alpha_k} \varphi\left(y_j, y_k\right)}{\sum\limits_{k=1}^{n} \varphi\left(\alpha_k y_k, \alpha_j y_j\right)} = \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \overline{\alpha_k} \varphi\left(y_j, y_k\right)$$

and

$$\varphi(x,x) = \varphi\left(\sum_{j=1}^{n} \alpha_j y_j, \sum_{k=1}^{n} \alpha_k y_k\right) = \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \overline{\alpha_k} \varphi\left(y_j, y_k\right).$$

Hence (7) implies

$$\sum_{j=1}^{n} \frac{\left(\varphi\left(x, y_j\right)\right)^2}{\sum\limits_{k=1}^{n} \varphi\left(y_j, y_k\right)} = \varphi(x,x). \tag{12}$$

If (12) hold, then choose $\alpha_j = \frac{\varphi(x,y_j)}{\sum_{k=1}^{n} \varphi(y_j,y_k)}$. From the proof of the inequality (6), we have that equality (6) holds when

$$0 = \Phi\left(x - \sum_{j=1}^{n} \alpha_j y_j\right)$$

and

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \overline{\alpha_k} \varphi\left(y_j, y_k\right) = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} |\alpha_j|^2 \varphi\left(y_j, y_k\right) + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} |\alpha_k|^2 \varphi\left(y_j, y_k\right).$$

For each pair (j,k), $j \neq k$ we have

$$\frac{1}{2}|\alpha_j|^2\varphi(y_j,y_k)+\frac{1}{2}|\alpha_k|^2\varphi(y_j,y_k)\geq 0$$

and

$$\alpha_j\overline{\alpha_k}\varphi(y_j,y_k)\leq\frac{1}{2}|\alpha_j|^2\varphi(y_j,y_k)+\frac{1}{2}|\alpha_k|^2\varphi(y_j,y_k)$$

Hence $\sum\limits_{j=1}^{n}\frac{(\varphi(x,y_j))^2}{\sum\limits_{k=1}^{n}\varphi(y_j,y_k)}=\varphi(x,x)$ implies (10). If (10), then for each pair (j,k), $j\neq k$, assume that (8) is not true. Then

$$\varphi(\alpha_j y_j,\alpha_k y_k)\geq 0$$

and

$$\frac{2\alpha_j\overline{\alpha_k}\varphi(y_j,y_k)}{\varphi(y_j,y_k)}=|\alpha_j|^2+|\alpha_k|^2$$

so

$$\frac{|2\alpha_j\overline{\alpha_k}\varphi(y_j,y_k)|}{\varphi(y_j,y_k)}=|\alpha_j|^2+|\alpha_k|^2$$

hence

$$2|\alpha_j||\alpha_k|=|\alpha_j|^2+|\alpha_k|^2$$

therefore

$$|\alpha_j|=|\alpha_k|.$$

It means (10) implies (7). □

4. A Fixed Point Theorem Involving Bilinear Functional

In this section, we study the generalization of the Banach contraction principle in the vector space, involving four rational square terms in the inequality, by using the notation of bilinear functional.

THEOREM 4.1. *Suppose Φ is a bounded and continuous quadratic form. Let X be a complete closed subset of a vector space and $T: X\to X$ be a self-mapping satisfying the following condition*

(13)

$$\Phi(Tx-Ty)\leq a_1\frac{\Phi(y-Ty)(1+\Phi(x-Tx))}{1+\Phi(x-y)}+a_2\frac{\Phi(x-Tx)(1+\Phi(y-Ty))}{1+\Phi(x-y)}$$
$$+a_3\frac{\Phi(x-Ty)(1+\Phi(y-Tx))}{1+\Phi(x-y)}+a_4\frac{\Phi(y-Tx)(1+\Phi(x-Ty))}{1+\Phi(x-y)}+a_5\Phi(x-y)$$

for each $x,y\in X$ and $x\neq y$, where a_1, a_2, a_3, a_4, a_5 are non-negative reals with

(14) $$0\leq a_1+a_2+a_3+a_4+a_5<1.$$

Then T has a unique fixed point in X.

Proof. For some $x_0\in X$, we define a sequence $\{x_n\}$ of iterator of T as follows

(15) $$x_1=Tx_0,\ x_2=Tx_1,\ x_3=Tx_2,\dots,x_{n+1}=Tx_n$$

for $n=1,2,3,\dots$.

Existence: We show that $\{x_n\}$ is a Cauchy sequence in X. For this, according to (15), we define

$$\Phi\left(x_{n+1}-x_n\right)=\Phi\left(Tx_n-Tx_{n-1}\right)$$

then by using (13), we have

(16)

$$\Phi\left(x_{n+1}-x_n\right)\leq a_1\frac{\Phi\left(x_{n-1}-Tx_{n-1}\right)\left(1+\Phi\left(x_n-Tx_n\right)\right)}{1+\Phi\left(x_n-x_{n-1}\right)}+a_2\frac{\Phi\left(x_n-Tx_n\right)\left(1+\Phi\left(x_{n-1}-Tx_{n-1}\right)\right)}{1+\Phi\left(x_n-x_{n-1}\right)}$$
$$a_3\frac{\Phi\left(x_n-Tx_{n-1}\right)\left(1+\Phi\left(x_{n-1}-Tx_n\right)\right)}{1+\Phi\left(x_n-x_{n-1}\right)}+a_4\frac{\Phi\left(x_{n-1}-Tx_n\right)\left(1+\Phi\left(x_n-Tx_{n-1}\right)\right)}{1+\Phi\left(x_n-x_{n-1}\right)}+a_5\Phi\left(x_n-x_{n-1}\right)$$

so (16) implies that

$$\begin{aligned}&\left(1-a_2-2a_4\right)\Phi\left(x_{n+1}-x_n\right)+\left(1-a_1-a_2\right)\Phi\left(x_{n+1}-x_n\right)\Phi\left(x_n-x_{n-1}\right)\\&\leq\left(\left(a_1+2a_4+a_5\right)+a_5\Phi\left(x_n-x_{n-1}\right)\right)\Phi\left(x_n-x_{n-1}\right)\end{aligned}$$

therefore

$$\Phi\left(x_{n+1}-x_n\right)\leq p\left(n\right)\Phi\left(x_n-x_{n-1}\right) \tag{17}$$

where

$$p\left(n\right)=\frac{a_1+2a_4+a_5+a_5\Phi\left(x_n-x_{n-1}\right)}{\left(1-a_2-2a_4\right)+\left(1-a_1-a_2\right)\Phi\left(x_n-x_{n-1}\right)} \tag{18}$$

for $n=1,2,3,...$. Clearly $p\left(n\right)<1$, for all n as $0\leq a_1+a_2+a_3+4a_4+a_5<1$.

Repeating the same argument, we find some $S<1$, such that

$$\Phi\left(x_{n+1}-x_n\right)\leq\lambda^n\Phi\left(x_1-x_0\right)$$

where $\lambda=S^2$. Letting $n\to\infty$, we obtain $\Phi\left(x_{n+1}-x_n\right)\to 0$. It follows that $\{x_n\}$ is a Cauchy sequence in X. So by completeness of X there exist a point $\mu\in X$ such that $x_n\to\mu$ as $n\to\infty$. Also $\{x_{n+1}\}=\{Tx_n\}$ is a subsequence of $\{x_n\}$ converges to the same limit μ. Since T is continuous, we obtain

$$T\left(\mu\right)=T\left(\lim_{n\to\infty}x_n\right)=\lim_{n\to\infty}Tx_n=\lim_{n\to\infty}x_{n+1}=\mu$$

Hence μ is a fixed point of T in X.

Uniquence: Now, we show the uniqueness of μ. If T has another fixed point γ and $\gamma\neq\mu$, then

(19)

$$\begin{aligned}&\Phi\left(\mu-\gamma\right)=\Phi\left(T\mu-T\gamma\right)\\&\leq a_1\frac{\Phi\left(\gamma-T\gamma\right)\left(1+\Phi\left(\mu-T\mu\right)\right)}{1+\Phi\left(\mu-\gamma\right)}+a_2\frac{\Phi\left(\mu-T\mu\right)\left(1+\Phi\left(\gamma-T\gamma\right)\right)}{1+\Phi\left(\mu-\gamma\right)}\\&a_3\frac{\Phi\left(\mu-T\gamma\right)\left(1+\Phi\left(\gamma-T\mu\right)\right)}{1+\Phi\left(\mu-\gamma\right)}+a_4\frac{\Phi\left(\gamma-T\mu\right)\left(1+\Phi\left(\mu-T\gamma\right)\right)}{1+\Phi\left(\mu-\gamma\right)}+a_5\Phi\left(\mu-\gamma\right)\end{aligned}$$

hence (19) implies that

$$\Phi\left(\mu-\gamma\right)\leq\left(a_3+a_4+a_5\right)\Phi\left(\mu-\gamma\right).$$

This gives a contradiction for $a_3+a_4+a_5<1$. Thus μ is a unique fixed point of T in X. □

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