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ON LACUNARY Δ^m-STATISTICAL CONVERGENCE OF TRIPLE SEQUENCE IN INTUITIONISTIC FUZZY N-NORMED SPACE

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Abstract. In this article, we construct lacunary Δ^m-statistical convergence for triple sequences within the context of intuitionistic fuzzy n-normed spaces (IFnNS). For lacunary Δ^m-statistical convergence of triple sequence in IFnNS, we demonstrate numerous results. For this innovative notion of convergence, we further built lacunary Δ^m-statistical Cauchy sequences and offered the Cauchy convergence criterion.

1. Introduction

Fast [11] introduced the idea of statistical convergence, which has since been studied by a large number of authors. Active study on this topic was started after Fridy's publication [12, 13]. In a number of areas, including approximation theory [3], finitely additive set functions [2], sequence space [14, 15], and statistical convergence for fuzzy numbers [1, 21], mathematicians have studied the characteristics of convergence and statistical convergence.

Zadeh [28] gave the concept of fuzziness. It has been one of the most active areas of research in many branches of sciences, with a sizable number of research publications based on the concept of fuzzy sets/numbers appearing in the literature. Intuitionistic fuzzy normed space was first described by Saadati and Park in [22]. In a recent study, R. Antal et al. [1] explored the idea of double sequence Δ-statistical convergence in intuitionistic fuzzy normed space.

There have been numerous studies on difference sequence spaces and related generalisations published in the literature [7–10, 26, 27]. B.C.Tripathy et.al. studied a new type of generalized Difference Cesaro Sequence Spaces [26] and new type of difference sequence spaces [27]. A.Esi studied the generalized difference sequence spaces defined by Orlicz functions [7] and strongly generalized difference $[V^\lambda, \Delta^m, p]$-summable sequence spaces defined by a sequence of moduli [8]. Later on, saveral authors studied generalized Δ^m Statistical Convergence in Probabilistic Normed Space [9] and generalized Strongly difference convergent sequences associated with multiplier sequences [10], respectively.

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The function $X : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(C)$ can be used to define a triple sequence (real or complex), where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ stand for the sets of natural, real, and complex numbers, respectively. At the beginning, Sahiner et al. [23] introduced and studied the many conceptions of triple sequences and their statistical convergence. Triple sequence statistical convergence on probabilistic normed space was recently introduced by Savas and Esi [6], where as statistical convergence of triple sequences in topological groups was later introduced by Esi [5]. For further study on triple sequence spaces, we may refer to [1, 14–16, 18, 19].

Kizmaz [20] introduced the difference sequence space $Z(\Delta)$ as given below

$$Z(\Delta) = \{y = (y_k) : (\Delta y_k) \in Z\}$$

for $Z = \ell_\infty, c, c_0$ i.e. spaces of all bounded, convergent and null sequences respectively, where $\Delta_y = (\Delta y_k) = (y_k - y_{k+1})$. In particular, $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ are also Banach spaces, relative to a norm induced by $\|y\|_\Delta = |y_1| + \sup_k |\Delta y_k|$.

The generalized difference sequence spaces $Z(\Delta^m)$ was introduced by M.Et et.al. [4] as follows :

$$Z(\Delta^m) = \{y = (y_k) : (\Delta^m y_k) \in Z\}$$

for $Z = \ell_\infty, c, c_0$ where $\Delta^m(y) == (\Delta^m y_k) = (\Delta_{m-1} y_k - \Delta_{m-1} y_{k+1})$. So that $\Delta^m y_k = \sum_{r=0}^{p} (-1)^r \begin{pmatrix} m \\ r \end{pmatrix} x_{k+r}$.

The difference operator Δ on triple sequence x_{mnl} is defined as :

$\Delta_{x_{mnl}} = x_{mnl} - x_{(m+1)nl} - x_{m(n+1)l} - x_{mn(l+1)} = x_{(m+1)(n+1)l} + x_{(m+1)n(l+1)} + x_{m(n+1)(l+1)} - x_{(m+1)(n+1)(l+1)}$.

The generalized difference spaces for triple sequences can be approximated as:

$$Z(\Delta^m) = \{y = (y_{jkl}) : (\Delta^m y_{jkl}) \in Z\}$$

for $Z = \ell_\infty^3, c^3, c_0^3$ where $\Delta^m(y) == (\Delta^m y_{jkl}) = \big(\Delta_{m-1} y_{jkl} - \Delta_{m-1} y_{jk,(l+1)}\big)$. So that $\Delta^m y_k = \sum_{r=0}^{p} (-1)^{r+s+u} \begin{pmatrix} m \\ r \end{pmatrix} \begin{pmatrix} m \\ s \end{pmatrix} \begin{pmatrix} m \\ u \end{pmatrix} x_{j+r,k+s,l+u}$.

Here is a summary of the current endeavours. In Section 2, we go over the fundamental definitions of the intuitionistic fuzzy n-normed space. Lacunary Δ^m-statistical convergence in intuitionistic fuzzy n-normed space is presented in Section 3. Here, we established a number of results that show how generalised this convergence process is. For this innovative notion of convergence, we further built Lacunary Δ^m-statistical Cauchy sequences and provided the Cauchy convergence criterion.

2. Definitions and Preliminaries

Here we mention some basic definitions of intuitionistic fuzzy n-normed space and other preliminaries.

Definition 2.1. [24] A continuous t-norm is the mapping $\otimes : [0,1] \times [0,1] \to [0,1]$ such that

1. $\otimes$ is continuous, associative, commutative and with identity 1 ,
2. $a_1 \otimes b_1 \le a_2 \otimes b_2$ whenever $a_1 \le a_2$ and $b_1 \le b_2, \forall a_1, a_2, b_1, b_2 \in [0,1]$.

Definition 2.2. [24] A continuous -conorm is the mapping $\odot : [0,1] \times [0,1] \to [0,1]$ such that

1. $\odot$ is continuous, associative, commutative and with identity 0 ,
2. $a_1 \odot b_1 \leq a_2 \odot b_2$ whenever $a_1 \leq a_2$ and $b_1 \leq b_2, \forall a_1, a_2, b_1, b_2 \in [0,1]$.

DEFINITION 2.3. [22] An intuitionistic fuzzy normed space (IFNS) is referred to the 5tuple $(X, \varphi, \vartheta, \otimes, \odot)$ with vector space X, fuzzy sets φ, ϑ on $X \times (0, \infty)$, continuous t-norm $\otimes$ and continuous t-conorm $\odot$, if for each $y, z \in X$ and $s, t > 0$, we have

1. $\varphi(y,t) + \vartheta(y,t) \leq 1$
2. $\varphi(y,t) > 0$ and $\vartheta(y,t) \leq 1$,
3. $\varphi(y,t) = 1$ and $\vartheta(y,t) = 0 \Longleftrightarrow y = 0$,
4. $\varphi(\alpha y, t) = \varphi\left(y, \frac{t}{|\alpha|}\right)$ for $\alpha \neq 0$,
5. $\varphi(y,s) \otimes \varphi(z,t) \leq \varphi(y+z, s+t)$ and $\vartheta(y,s) \odot \vartheta(z,t) \leq \vartheta(y+z, s+t)$,
6. $\vartheta(y,o) : (0,\infty) \to [0,1]$ and $\vartheta(y,o) : (0,\infty) \to [0,1]$ are continuous,
7. $\lim_{t\to\infty} \varphi(y,t) = 1, \lim_{t\to 0} \varphi(y,t) = 0, \lim_{t\to\infty} \vartheta(y,t) = 1$ and $\lim_{t\to 0} \vartheta(y,t) = 0$.

Then (φ, ϑ) is known as intuitionistic fuzzy norm.

DEFINITION 2.4. [22] Let $(X, \|o\|)$ be any normed space. For every $t > 0$ and $y \in X$, take $\varphi = \frac{t}{t+\|y\|}, \vartheta = \frac{\|y\|}{t+\|y\|}$. Also, $a \otimes b = ab$ and $a \odot b = \min\{a+b, 1\} \forall a, b \in [0,1]$.

Then, a 5-tuple $(X, \varphi, \vartheta, \otimes, \odot)$ is an IFNS which satisfies the above mentioned conditions.

DEFINITION 2.5. [22] Let $(X, \varphi, \vartheta, \otimes, \odot)$ be an IFNS with norm (φ, ϑ). A sequence $y = (y_k)$ in X is called convergent to some $\xi \in X$ with respect to the intuitionic fuzzy norm (φ, ϑ) if there exists $k_0 \in \mathbb{N}$ for each $\epsilon > 0$ and $t > 0$ such that $\varphi(yk - \xi, t) > 1 - \epsilon$ and $\vartheta(yk - \xi, t) < \epsilon$ for all $k \geq k_0$. It is denoted by $(\varphi, \vartheta) - \lim_{k\to\infty} y_k = \xi$.

DEFINITION 2.6. [22] Let $(X, \varphi, \vartheta, \otimes, \odot)$ be an IFNS with norm (φ, ϑ). A sequence $y = (y_k)$ in X is called convergent to some $\xi \in X$ with respect to the intuitionic fuzzy norm (φ, ϑ) if there exists $k_0 \in \mathbb{N}$ for each $\epsilon > 0$ and $t > 0$

$$\delta\left(\{k \in \mathbb{N} : \varphi\left(y_k - \xi, t\right) \leq 1 - \epsilon \quad \text{or} \quad \vartheta\left(y_k - \xi, t\right) \geq \epsilon\}\right) = 0.$$

It is denoted by $S^{\varphi,\vartheta} - \lim_{k\to\infty} y_k = \xi$.

A subset E of the set $\mathbb{N}$ of natural numbers is said to have a "natural density" $\delta(E)$ if

$$\delta(E) = \lim_n \frac{1}{n} |\{k \leq n : k \in E\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

The number sequence $x = (x_k)$ is said to be statistically convergent to number l if for each $\epsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - l| \geq \epsilon\}| = 0,$$

and x is said to be statistically cauchy sequence if for every $\epsilon > 0$ there exists a number $N = N(\epsilon)$ such that

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - x_N| \geq \epsilon\}| = 0.$$

DEFINITION 2.7. [22] Let $(X, \varphi, \vartheta, \otimes, \odot)$ be an IFNS with norm (φ, ϑ). A double sequence $y = (y_{jk})$ in X is called statistically convergent to some $\xi \in X$ with respect to the intuitionic fuzzy norm (φ, ϑ) if there exists $k_0 \in \mathbb{N}$ for each $\epsilon > 0$ and $t > 0$

$$\delta\left(\{k \in \mathbb{N} : \varphi\left(y_{jk} - \xi, t\right) \leq 1 - \epsilon \text{ or } \vartheta\left(y_{jk} - \xi, t\right) \geq \epsilon\}\right) = 0.$$

It is denoted by $S^(\varphi, \vartheta) - \lim_{k\to\infty} y_{jk} = \xi$.

Definition 2.8. [6] The triple sequence $\theta_{j,k,l} = \{(j_r, k_s, l_t)\}$ is called the triple lacunary sequence if there exist three increasing sequences of integers such that

$$j_o = 0, h_r = j_r - j_{r-1} \to \infty \text{ as } r \to \infty,$$
$$k_o = 0, h_s = k_s - k_{s-1} \to \infty \text{ as } s \to \infty,$$

and

$$I_o = 0, h_t = I_t - I_{t-1} \to \infty \text{ as } t \to \infty$$

Let $k_{r,s,t} = j_r k_s I_t, h_{r,s,t} = h_r h_s h_t$ and $\theta_{j,k,l}$ is determined by

$$I_{r,s,t} = \{(j,k,l) : j_{r-1} < j \leq j_r, k_{s-1} < k \leq k_s \text{ and } I_{t-1} < I \leq I_t\}$$
$$q_r = \frac{j_r}{j_{r-1}}, q_s = \frac{k_s}{k_{s-1}}, q_t = \frac{l_t}{l_{t-1}} \text{ and } q_{r,s,t} = q_r q_s q_t.$$

Let $K \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. The number

$$\delta_3^\theta = \lim_{r,s,t} \frac{1}{h_{r,s,t}} \left|\{(j,k,l) \in I_{r,s,t} : (j,k,l) \in K\}\right|$$

is said to be the $\theta_{r,s,t}$-density of K, provided the limit exists.

Below is the definition of n-mored space:

Definition 2.9. Let $n \in \mathbb{N}$ and X be a real vector space of dimension $d \geq n$. (Here we allow d to be infinite). A real-valued function $||, ..., ||$ on X^n satisfying the following four properties:

(1) $||f_1, f_2, ..., f_n|| = 0$ if and only if $f_1, f_2, ..., f_n$ are linearly dependent;
(2) $||f_1, f_2, ..., f_n||$ is invariant under permutation;
(3) $||f_1, f_2, ..., f_{n-1}, \alpha f_n|| = |\alpha| ||f_1, f_2, ..., f_{n-1}, f_n||$ for any $\alpha \in R$;
(4) $||f_1, f_2, ..., f_{n-1}, y+z|| \leq ||f_1, f_2, ..., f_{n-1}, y+f_1, f_2, ..., f_{n-1}, z||$, is called an n-norm on X and the pair $(X, ||f_1, f_2, ..., f_n||)$ is called an n-normed space.

Definition 2.10. [25] An IFnNLS is the five-tuple $(X, \mu, v, *, \circ)$ where X is a linear space over a field $F, *$ is a continuous t-norm, $\circ$ is a continuous t-conorm, μ, v are fuzzy sets on $X^n \times (0, \infty), \mu$ denotes the degree of membership and v denotes the degree of nonmembership of $(x_1, x_2, \ldots, x_n, t) \in X^n \times (0, \infty)$ satisfying the following conditions for every $(x_1, x_2, \ldots, x_n) \in X^n$ and $s, t > 0$:

(i) $\mu\left(x_1, x_2, \ldots, x_n, t\right) + v\left(x_1, x_2, \ldots, x_n, t\right) \leq 1$,
(ii) $\mu\left(x_1, x_2, \ldots, x_n, t\right) > 0$,
(iii) $\mu\left(x_1, x_2, \ldots, x_n, t\right) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
(iv) $\mu\left(x_1, x_2, \ldots, x_n, t\right)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,
(v) $\mu\left(x_1, x_2, \ldots, cx_n, t\right) = \mu\left(x_1, x_2, \ldots, x_n, \frac{t}{|c|}\right)$ for all $c \neq 0, c \in F$,
(vi) $\mu\left(x_1, x_2, \ldots, x_n, s\right) * \mu\left(x_1, x_2, \ldots, x_n', t\right) \leq \mu\left(x_1, x_2, \ldots, x_n + x_n', s+t\right)$,
(vii) $\mu\left(x_1, x_2, \ldots, x_n, t\right) : (0, \infty) \to [0, 1]$ is continuous in t,
(viii) $\lim_{t\to\infty} \mu\left(x_1, x_2, \ldots, x_n, t\right) = 1$ and $\lim_{t\to 0} \mu\left(x_1, x_2, \ldots, x_n, t\right) = 0$,
(ix) $v\left(x_1, x_2, \ldots, x_n, t\right) < 1$

(x) $v(x_1, x_2, \ldots, x_n, t) = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
(xi) $v(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,
(xii) $v(x_1, x_2, \ldots, cx_n, t) = v\left(x_1, x_2, \ldots, x_n, \frac{t}{|c|}\right)$ for all $c \neq 0, c \in F$,
(xiii) $v(x_1, x_2, \ldots, x_n, s) \circ v(x_1, x_2, \ldots, x'_n, t) \geq v(x_1, x_2, \ldots, x_n + x'_n, s + t)$
(xiv) $v(x_1, x_2, \ldots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t,
(xv) $\lim_{t\to\infty} v(x_1, x_2, \ldots, x_n, t) = 0$ and $\lim_{t\to 0} v(x_1, x_2, \ldots, x_n, t) = 1$.

Example 2.11. [25] Let $(X, \|\bullet, \ldots, \bullet\|)$ be an n-normed linear space. Also let $a * b = ab$ and $a \circ b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$

$$\mu(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|} \quad \text{and} \quad v(x_1, x_2, \ldots, x_n, t) = \frac{\|x_1, x_2, \ldots, x_n\|}{t + \|x_1, x_2, \ldots, x_n\|}.$$

Then $(X, \mu, v, *, 0)$ is an IFnNLS.

3. Triple Lacunary Δ^m-statistical convergence in IFnNS.

In the context of intuitionistic fuzzy normed spaces for triple sequences, we define Lacunary Δ^m-statistical convergence and establish certain results.

Definition 3.1. Let $(X, \varphi, \vartheta, \otimes, \odot)$ be a IFnNS with norm $(\varphi, \vartheta)^n$ and $\theta_{j,k,l}$ be a triple lacunary sequence. A triple sequence $y = (y_{jkl})$ in X is called lacunary Δ^m-statistically convergent to some $\xi \in X$ with respect to the intuitionistic fuzzy norm (φ, ϑ) if for each $\epsilon > 0$, $t > 0$ and $f_1, f_2, ..., f_{n-1} \in X$.

$$\begin{aligned} &\delta_3^\theta(\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t) \leq 1 - \epsilon \\ &\qquad \text{or} \quad \vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t) \geq \epsilon\}) = 0 \end{aligned} \tag{1}$$

or equivalently

$$\begin{aligned} &\delta_3^\theta(\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t) > 1 - \epsilon \\ &\qquad \text{or} \quad \vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t) < \epsilon\}) = 1. \end{aligned} \tag{1*}$$

In this case, we write $S_{\theta^{(\varphi,\vartheta)^n}_{j,k,l}} - \lim_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$ or $X_{jkl} \stackrel{(\varphi,\vartheta)^n}{\longrightarrow} \xi \left(S_{\theta_{j,k,l}}\right)$ and denote the set of all $S_{\theta_{j,k,l}}$-convergent triple sequences in the intuitionistic fuzzy normed space by $S^{(\varphi,\vartheta)^n}_{\theta_{j,k,l}}$.

Definition 3.2. Let $(X, \varphi, \vartheta, \otimes, \odot)$ be a IFnNS with norm $(\varphi, \vartheta)^n$ and $\theta_{j,k,l}$ be a triple lacunary sequence. A triple sequence $y = (y_{jkl})$ in X is called lacunary Δ^m-statistically Cauchy with respect to the intuitionistic fuzzy norm (φ, ϑ) if there exists $j_0, k_0, l_o \in \mathbb{N}$ for each $\epsilon > 0$ and $t > 0$ such that for all $j, r \geq j_0, k, s \geq k_0$, $l, u \geq l_0$ and $f_1, f_2, ..., f_{n-1} \in X$, we have

$$\begin{aligned} \delta_3^\theta(\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \Delta^m y_{rsu}, t) \leq 1 - \epsilon \ \text{ or} \\ \vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \Delta^m y_{rsu}, t) \geq \epsilon\}) = 0. \end{aligned}$$

It is denoted by $S^{(\varphi,\vartheta)^n}_{jkl} - \lim_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$.

From (1) and (1*), we have the following lemma.

Lemma 3.1. *Let* $(X, \varphi, \vartheta, \otimes, \odot)$ *be an* IFNS *with norm* $(\varphi, \vartheta)^n$ *and* $\theta_{j,k,l}$ *be a triple lacunary sequence. Then the following statements are equivalent for triple sequence* $y = (y_{jkl})$ *in* X *whenever* $\epsilon > 0$ *and* $t > 0$,

(i) $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$,

(ii) $\delta_3^\theta \left(\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) > 1-\epsilon\}\right)$
$= \delta_3^\theta(\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} :\ \vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) < \epsilon\}) = 1$,

(iii) $\delta_3^\theta \left(\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) \le 1-\epsilon\}\right)$
$= \delta_3^\theta(\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} :\ \vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) \ge \epsilon\}) = 0$,

(iv) $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim\limits_{j,k,l\to\infty} \varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) = 1$
and $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim_{j,k,l\to\infty} \vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) = 0$.

Theorem 3.3. *Let* $(X, \varphi, \vartheta, \otimes, \odot)$ *be a* IFNS *with norm* $(\varphi, \vartheta)^n$ *and* $\theta_{j,k,l}$ *be a triple lacunary sequence. If* $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$, *then* ξ *is unique.*

Proof. If possible, let $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi_1$ and $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi_2$.

For given $\epsilon \in (0,1)$ and $t > 0$, take $\alpha > 0$ such that $(1-\alpha)\otimes(1-\alpha) > 1-\epsilon$ and $\alpha \odot \alpha < \epsilon$.

Consider

$$K_{1,\varphi}(\alpha,t) = \left\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi_1, t/2\right) \le 1-\alpha\right\},$$
$$K_{2,\varphi}(\alpha,t) = \left\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi_2, t/2\right) \le 1-\alpha\right\},$$
$$K_{3,\vartheta}(\alpha,t) = \left\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi_1, t/2\right) \ge \alpha\right\},$$
$$K_{4,\vartheta}(\alpha,t) = \left\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi_2, t/2\right) \ge \alpha\right\}.$$

Using lemma 3.1, we have

$$\delta_3^\theta\left(K_1, \varphi(\alpha,t)\right) = \delta_3^\theta\left(K_3, \vartheta(\alpha,t)\right) = 0.$$
$$\delta_3^\theta\left(K_2, \varphi(\alpha,t)\right) = \delta_3^\theta\left(K_4, \vartheta(\alpha,t)\right) = 0.$$

Let $K_{\varphi,\vartheta}(\alpha,t) = [K_{1,\varphi}(\alpha,t) \bigcup K_{2,\varphi}(\alpha,t)] \bigcap [K_{3,\vartheta}(\alpha,t) \bigcup K_{4,\vartheta}(\alpha,t)]$. Clearly,

$$\delta_3^\theta K_{\varphi,\vartheta}(\alpha,t) = 0.$$

Whenever $(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} - K_{\varphi,\vartheta}(\alpha,t)$, we have two possibilities, either $(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} - [K_{1,\varphi}(\alpha,t) \bigcup K_{2,\varphi}(\alpha,t)]$ or $(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} - [K_{3,\vartheta}(\alpha,t) \bigcup K_{4,\vartheta}(\alpha,t)]$.

First, we consider $(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} - [K_{1,\varphi}(\alpha,t) \bigcup K_{2,\varphi}(\alpha,t)]$. Then

$$\begin{aligned}\varphi\left(\xi_1 - \xi_2, t\right) &\ge \varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi_1, t/2\right) \otimes \varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi_2, t/2\right)\\ &> (1-\alpha)\otimes(1-\alpha)\\ &> 1-\epsilon.\end{aligned}$$

As given $\epsilon \in (0,1)$ was arbitrary, then $\varphi\left(\xi_1 - \xi_2, t\right) = 1$ for all $t > 0$, then $\xi_1 = \xi_2$.

Similarly, if $(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} - [K_{3,\vartheta}(\alpha,t) \bigcup K_{4,\vartheta}(\alpha,t)]$

$$\begin{aligned}\vartheta\left(\xi_1 - \xi_2, t\right) &\le \vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi_1, t/2\right) \odot \vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi_2, t/2\right)\\ &< \alpha \odot \alpha\\ &< \epsilon.\end{aligned}$$

Since $\epsilon \in (0,1)$ was arbitrary, then $\varphi(\xi_1, \xi_2, t) = 0$ for all $t > 0$, i.e., $\xi_1 = \xi_2$. Therefore $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$ exists uniquely. $\square$

THEOREM 3.4. *Let $(X, \varphi, \vartheta, \otimes, \odot)$ be a IFNS with norm $(\varphi, \vartheta)^n$ and $\theta_{j,k,l}$ be a triple lacunary sequence. If $(\varphi, \vartheta)^n - \lim_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$, then $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$. But converse may not be true.*

Proof. Let $(\varphi, \vartheta)^n - \lim_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$. Then, there exists j_0, k_0 and $l_0 \in \mathbb{N}$ for given $\epsilon > 0$ and any $t > 0$ such that for all $j \geq j_0, k \geq k_0$ and $l \geq l_0$,
we have $\varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t) > 1 - \epsilon$ and $\vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t) < \epsilon$.

Further, the set $A(\epsilon, t) = \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t) \leq 1 - \epsilon$ or
$\vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t) \geq \epsilon\}$, contains only finite number of elements. We know that natural density of any finite set is always zero. Therefore, $\delta_3^\theta(A(\epsilon, t)) = 0$ i.e. $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$. $\square$

But converse is not true, this can be justified with the example.

EXAMPLE 3.5. Let $(\mathbb{R}, ||)$ be the real normed space under the usual norm. Define $a \otimes b = ab$ and $a \odot b = \min\{a + b, 1\}$ $\forall\, a, b \in [0,1]$. Also for every $t > 0$ and all $y \in \mathbb{R}$, consider $\varphi(y,t) = \frac{t}{t+|y|}$ and $\vartheta(y,t) = \frac{|y|}{t+|y|}$. Then, clearly $(\mathbb{R}, \varphi, \vartheta, \otimes, \odot)$ is an IFNS. Define the sequence

$$\Delta^m x_{jkl} = \begin{cases} \qquad jkl, \quad \text{for } j_r - [|\sqrt{h_r}|] + 1 \leq j \leq j_r \\ k_s - [|\sqrt{h_s}|] + 1 \leq k \leq k_s \\ \qquad \text{and } l_t - [|\sqrt{h_t}|] + 1 \leq l \leq l_t \\ \qquad \xi, \qquad \text{otherwise.} \end{cases}$$

By given $\epsilon > 0$ and $t > 0$, we obtain the below set for $\xi = 0$.

$$\begin{aligned} K(\epsilon, t) &= \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl}, t) \leq 1 - \epsilon \\ &\quad \text{or} \quad \vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl}, t) \geq \epsilon\} \\ &= \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |\Delta^m y_{jkl}| \geq \frac{\epsilon t}{1-\epsilon} > 0 \\ &= \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |\Delta^m y_{jkl}| = jkl \\ &= \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |\Delta^m y_{jkl}| \geq \frac{\epsilon t}{1-\epsilon} > 0 \\ &= \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : j_r - [|\sqrt{h_r}|] + 1 \leq j \leq j, \\ &\qquad\qquad k_s - [|\sqrt{h_s}|] + 1 \leq k \leq k_s \\ &\quad \text{and} \qquad\quad l_t - [|\sqrt{h_t}|] + 1 \leq l \leq l_t\} \end{aligned}$$

and so, we get

$$\lim_{r,s,t} \frac{1}{h_r,h_s,h_t} \mid \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : j_r - [|\sqrt{h_r}|] + 1 \leq j \leq j_r$$

$$k_s - [|\sqrt{h_s}|] + 1 \leq k \leq k_s$$

$$\text{and } \; l_t - [|\sqrt{h_t}|] + 1 \leq l \leq l_t\}$$

$$\leq \lim_{r,s,t} \frac{\sqrt{h_r}\sqrt{h_s}\sqrt{h_t}}{h_r h_s h_t} = 0.$$

Hence $S^{(\varphi,\vartheta)^n}_{\theta_{j,k,l}} - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = 0$. By the above defined sequence $(\Delta^m y_{jkl})$, we get

$$\varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m x_{jkl}, t\right) = \begin{cases} \frac{t}{t+|jkl|}, & \text{for } _r - [|\sqrt{h_r}|] + 1 \leq j \leq j_r \\ & k_s - [|\sqrt{h_s}|] + 1 \leq k \leq k_s \\ & \text{and } l_t - [|\sqrt{h_t}|] + 1 \leq l \leq l_t \\ 0, & \text{otherwise.} \end{cases}$$

i.e $\varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m x_{jkl}, t\right) \leq 1, \quad \forall j,k,l$. And

$$\vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m x_{jkl}, t\right) = \begin{cases} \frac{|jkl|}{t+|jkl|}, & \text{for } j_r - [|\sqrt{h_r}|] + 1 \leq j \leq j_r, \\ & k_s - [|\sqrt{h_s}|] + 1 \leq k \leq k_s \\ & \text{and } l_t - \left[\left|\sqrt{h_t}\,\right|\right] + 1 \leq l \leq l_t \\ 0, & \text{otherwise.} \end{cases}$$

i.e $\vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m x_{jkl}, t\right) \geq 0, \quad \forall j,k,l$.

This shows that $(\varphi,\vartheta)^n - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} \neq 0$.

THEOREM 3.6. *Let $(X, \varphi, \vartheta, \otimes, \odot)$ be a IFnNS with norm $(\varphi,\vartheta)^n$ and $\theta_{j,k,l}$ be a triple lacunary sequence. Then $S_{\theta_{j,k,l},\vartheta} - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi \iff$ there exists a set $P = \{(j_a, k_b, l_c) : a,b,c = 1,2,3,\ldots\} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta(P) = 1$ and $(\varphi,\vartheta)^n - \lim\limits_{j_a,k_b,l_c\to\infty} \Delta^m y_{j_a k_b l_c} = \xi$.*

Proof. Assume that $S^{(\varphi,\vartheta)^n}_{\theta_{j,k,l}} - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$. For $t > 0$ and $\alpha \in \mathbb{N}$, we take $M(\alpha,t) = \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t > 1 - 1/\alpha$ and $\vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t) < 1/\alpha\}$, and

$$K(\alpha,t) = \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) \leq 1 - 1/\alpha$$

or

$$\vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) \geq 1/\alpha\},$$

as $S^{(\varphi,\vartheta)^n}_{\theta_{j,k,l}} - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$, then $\delta_3(K(\alpha,t)) = 0$. Also, for any $t > 0$ and $\alpha \in \mathbb{N}$, evidently we get $M(\alpha,t) \supset M(\alpha+1,t)$, and

$$\delta_3(M(\alpha,t)) = 1, \tag{3.1}$$

For $(j,k,l) \in M(\alpha,t)$, we prove $(\varphi,\vartheta)^n - \lim\limits_{j_a,k_b,l_c\to\infty} \Delta^m y_{j_a k_b l_c} = \xi$.

On the contrary, suppose that triple sequence $y = (y_{jkl})$ is not Δ^m-convergent to ξ for all $(j,k,l) \in M(\alpha,t)$. So, there exists some $\alpha > 0$ and $k_0 \in \mathbb{N}$ such that

$\varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t) \leq 1 - \rho$
or $\vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t) \geq \rho\}$ for all $j,k,l \geq k_0$
$\Longrightarrow \varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t) \geq 1 - \rho$ and $\vartheta(\Delta^m y_{jkl} - \xi, t) \leq \rho\}$ for all $j,k,l \geq k_0$

Therefore, $\delta_3\left(\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) \geq 1 - \rho \text{ and } \vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) \leq \rho\}\}\right) = 0$
i.e. $\delta_3(M(\rho,t)) = 0$. Since $\rho > 1/\alpha$, then $\delta_3(M(\alpha,t)) = 0$ as $M(\alpha,t) \subset M(\rho,t)$, which is a contradiction to (3.1). This shows that there exists a set $M(\alpha,t)$ for which $\delta_3(M(\alpha,t)) = 1$ and the triple sequence $y = (y_{jkl})$ is statistically Δ^m-convergent to ξ.

Converesely, suppose there exists a subset $P = \{(j_a, k_b, l_c) : a,b,c = 1,2,3,\ldots\} \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ with $\delta_3(P) = 1$
and $(\varphi,\vartheta)^n - \lim\limits_{j_a,k_b,l_c\to\infty} \Delta^m y_{j_a k_b l_c} = \xi$. i.e. for given $\rho > 0$ and any $t > 0$ we have $N_0 \in \mathbb{N}$, which gives

$$\varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) > 1 - \rho$$

and

$$\vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) < \rho \text{ for all } j,k,l \geq N_0.$$

Now, let $K(\rho,t) = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) \leq 1 - \rho$ or $\vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) \geq \rho\}\}$. Then,

$$K(\rho,t) \subseteq \mathbb{N} - \left\{\left(j_{N_0+1}, k_{N_0+1}, l_{N_0+1}\right), \ldots\right\}. \text{As } \delta_3(P) = 1 \Longrightarrow \delta_3(K(\alpha,t)) \leq 0.$$

Hence, $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$.

□

Theorem 3.7. *Let $(X, \varphi, \vartheta, \otimes, \odot)$ be a IFNS with norm $(\varphi,\vartheta)^n$ and $\theta_{j,k,l}$ be a triple lacunary sequence. Let $y = (y_{jkl})$ be any triple sequence. Then $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi \Longleftrightarrow$ there is a triple sequence $x = (x_{jkl})$ such that $(\varphi,\vartheta)^n - \lim_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$ and $\delta_3\left(\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \Delta^m y_{jkl} = \Delta^m x_{jkl}\}\right) = 1$*

Proof. Assume that $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$. By Theorem (3.3), we set

$$P = \{(j_a, k_b, l_c) : a,b,c = 1,2,3,\ldots\} \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}$$

with $\delta_3(P) = 1$ and $(\varphi,\vartheta)^n - \lim\limits_{j,k,l\to\infty} \Delta^m y_{j_a k_b l_c} = \xi$. Consider the sequence

$$\Delta^m x_{jkl} = \begin{cases} \Delta^m y_{jkl}, & (j,k,l) \in P \\ \xi, & \text{otherwise,} \end{cases}$$

which gives the required result.

Converesely, consider $x = (x_{jkl})$ and $z = (z_{jkl})$ in X with $(\varphi,\vartheta)^n - \lim_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$ and $\delta_3\left(\left\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \Delta^m_{yjkl} = \Delta^m x_{jkl}\right\}\right) = 1$. Then for each $\epsilon > 0$ and $t > 0$,

$$\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) \leq 1 - \epsilon$$

or

$$\vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) \geq \epsilon\} \subseteq A \cup B$$

where

$$A = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) \leq 1-\epsilon$$

or

$$\vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) \geq \epsilon\}$$

and

$$B = \left\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : (\Delta^m y_{jkl} \neq \Delta^m x_{jkl})\right\}.$$

Since $(\varphi,\vartheta)^n - \lim\limits_{j,k,l\to\infty} \Delta^m y_{j_a k_b l_c} = \xi$ then the set A contains at most finitely many terms. Also $\delta_3(B) = 0$ as $\delta_3\left(B^c\right) = 1$ where $B^c = \{(j,k,l)\mathbb{N}\times\mathbb{N}\times\mathbb{N} : \Delta^m y_{jkl} = \Delta^m x_{jkl}\}$.

Therefore

$$\delta_3\left\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \varphi\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) \leq 1-\epsilon\right.$$

or

$$\left.\vartheta\left(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t\right) \geq \epsilon\right\}.$$

We get $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$. □

Theorem 3.8. *Let $(X,\varphi,\vartheta,\otimes,\odot)$ be a* IFNS *with norm $(\varphi,\vartheta)^n$ and $\theta_{j,k,l}$ be a triple lacunary sequence. Let $y = (y_{jkl})$ be any triple sequence. Then $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi \iff$ there exists two triple sequence $z = (z_{jkl})$ and $x = (x_{jkl})$ in X such that $\Delta^m y_{jkl} = \Delta^m z_{jkl} + \Delta^m x_{jkl}$ for all $(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ where $(\varphi,\vartheta)^n - \lim\limits_{j,k,l\to\infty} \Delta^m y_{j_a k_b l_c} = \xi$ and $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$.*

Proof. Assume that $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$. By Theorem (3.5), we set $P = \{(j_a, k_b, l_c) : a,b,c = 1,2,3,\ldots\} \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ wit $\delta_3(P) = 1$ and $(\varphi,\vartheta)^n - \lim\limits_{j,k,l\to\infty} \Delta^m y_{ja} k_b l_c = \xi$.

Consider two triple sequences $z = (z_{jkl})$ and $x = (x_{jkl})$, then

$$\Delta^m z_{jkl} = \begin{cases} \Delta^m y_{jkl}, & (j,k,l) \in P \\ \xi, & \text{otherwise.} \end{cases}$$

and

$$\Delta^m x_{jkl} = \begin{cases} 0, & (j,k,l) \in P \\ \Delta^m y_{jkl} - \xi, & \text{otherwise,} \end{cases}$$

which gives the required result.

Converesely, consider $x = (x_{jkl})$ and $z = (z_{jkl})$ in X with $\Delta^m y_{jkl} = \Delta^m z_{jkl} + \Delta^m x_{jkl}$ for all $(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ where $(\varphi,\vartheta)^n - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$ and $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim\limits_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$. Then we get result using Theorem (3.6) and Theorem (3.7). □

Theorem 3.9. *A triple sequence $y = (y_{jkl})$ in* IFNS$(X,\varphi,\vartheta,\otimes,\odot)$ *is lacurnary $\Delta^m\_$ statistically convergent with respect to $(\varphi,\vartheta)^n$ if and only if it is lacunary Δ^m-statistically Cauchy with respect to $(\varphi,\vartheta)^n$.*

Proof. Let $S_{\theta_{j,k,l}}^{(\varphi,\vartheta)^n} - \lim_{j,k,l\to\infty} \Delta^m y_{jkl} = \xi$. Then, for each $\epsilon > 0$ and $t > 0$, take $\alpha > 0$ such that $(1-\alpha) \otimes (1-\alpha) > 1-\epsilon$ and $\alpha \odot \alpha < \epsilon$.

Let $K(\alpha, t) = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t/2) \leq 1-\alpha$ or $\vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t/2) \geq \alpha\}$, therefore $\delta_3(K(\alpha,t)) = 0$ and $\delta_3([K(\alpha,t)]^c) = 1$.

Let $M(\epsilon, t) = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \Delta^m y_{rsu}, t) \leq 1-\epsilon$ or $\vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \Delta^m y_{rsu}, t) \geq \epsilon\}$.

Now, we prove $M(\epsilon,t) = K(\epsilon,t)$, for this if $(j,k,l) \in M(\epsilon,t) = K(\epsilon,t)$. Then we get $\varphi(f_1, f_2, ..., f_{n-1}, \Delta_p y_{jkl} - \xi, t/2) \leq 1-\alpha$ or $\vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t/2) \geq \alpha\}$.

Also

$$\begin{aligned}
&1-\epsilon \\
&\geq \varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \Delta^m y_{rsu}, t) \\
&\geq \varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t/2) \otimes \vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t/2) \\
&> (1-\alpha) \otimes (1-\alpha) \\
&> 1-\epsilon
\end{aligned}$$

and

$$\begin{aligned}
\epsilon &\geq \vartheta(f_1, f_2, ..., f_{n-1}, \Delta_p y_{jkl} - \Delta_p y_{rsu}, t) \\
&\leq \vartheta(f_1, f_2, ..., f_{n-1}, \Delta_p y_{jkl} - \xi, t/2) \odot \varphi(f_1, f_2, ..., f_{n-1}, \Delta_p y_{jkl} - \xi, t/2) \\
&< \alpha \odot \alpha \\
&< \epsilon.
\end{aligned}$$

which is not possible. Therefore $M(\epsilon,t) \subset K(\alpha,t)$ and $\delta_3(M(\epsilon,t)) = 0$ i.e. $y = (y_{jkl})$ is Δ^m-statistically convergent with respect to (φ, ϑ).

Coversely, assume that $y = (y_{jkl})$ is Δ^m-statiscally Cauchy with respect to $(\varphi,\vartheta)^n$ but not Δ^m-statiscally convergent with respect to $(\varphi,\vartheta)^n$. Thus for $\epsilon > 0$ and $t > 0$, $\delta_3(M(\epsilon,t)) = 0$, where

$$M(\epsilon,t) = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \Delta^m y_{j_0k_0l_0}, t) \leq 1-\epsilon$$

or

$$\vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \Delta^m y_{j_0k_0l_0}, t) \geq \epsilon\}.$$

Take $\alpha > 0$ such that $(1-\alpha) \otimes (1-\alpha) > 1-\epsilon$ and $\alpha \odot \alpha < \epsilon$. Also, $\delta_3(K(\alpha,t)) = 0$, where $K(\alpha,t) = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t/2) \geq 1-\epsilon$ or $\vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t/2) < \epsilon\}$. Now

$$\begin{aligned}
&\varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \Delta^m y_{j_0k_0l_0}, t) \\
&\geq \varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t/2) \otimes \vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{j_0k_0l_0} - \xi, t/2) \\
&> (1-\alpha) \otimes (1-\alpha) \\
&> 1-\epsilon
\end{aligned}$$

and

$$\begin{aligned}
&\vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \Delta^m y_{j_0k_0l_0}, t) \\
&\leq \vartheta(f_1, f_2, ..., f_{n-1}, \Delta^m y_{jkl} - \xi, t/2) \odot \varphi(f_1, f_2, ..., f_{n-1}, \Delta^m y_{j_0k_0l_0} - \xi, t/2) \\
&< \alpha \odot \alpha \\
&< \epsilon.
\end{aligned}$$

Therefore, $\delta_3\left([M(\epsilon,t)]^c\right)=0$ i.e. $\delta_3(M(\epsilon,t))=1$, which is a contradiction as $y=(y_{jkl})$ is Δ^m-statistically cauchy. Hence, $y=(y_{jkl})$ is Δ^m-statiscally convergent with respect to $(\varphi,\vartheta)^n$. □

4. Conclusion.

This work establishes certain conclusions and defines Lacunary Δ^m-statistical convergence on intuitionistic fuzzy n-normed space. Since any regular norm implies an intuitionistic fuzzy norm, the findings are more pervasive than in analogous normed spaces.

5. Declaration

Conflicts of interests: There is no conflict of interest.

Availibility of data and materials: This paper has no associated data.

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