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SOME RESULTS FOR THE CLASS OF ANALYTIC FUNCTIONS CONCERNED WITH SYMMETRIC POINTS

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Abstract. This paper's objectives are to present the $\mathcal{H}$ class of analytical functions and explore the many characteristics of the functions that belong to this class. Some inequalities regarding the angular derivative have been discovered for the functions in this class. In addition, the symmetry points on the unit disc are used for the obtained inequalities.

1. Introduction

Let g be an analytic function in the unit disc $\mathbb{D} = \{z : |z| < 1\}$, $g(0) = 0$ and $g : \mathbb{D} \to \mathbb{D}$. In accordance with the classical Schwarz lemma, for any point z in the unit disc $\mathbb{D}$, we have $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|g'(0)| \leq 1$. In addition, if the equality $|g(z)| = |z|$ holds for any $z \neq 0$, or $|g'(0)| = 1$, then g is a rotation; that is $g(z) = ze^{i\theta}$, θ real ([5], p.329). Schwarz lemma has important applications in engineering [15,16]. We will need to remember the following lemma in order to prove our study results [6].

Lemma 1.1 (Jack's lemma). *Let $g(z)$ be a non-constant anaytic function in $\mathbb{D}$ with $g(0) = 0$. If*

$$|g(z_0)| = \max\{|g(z)| : |z| \leq |z_0|\},$$

then there exists a real number $k \geq 1$ such that

$$\frac{z_0 g'(z_0)}{g(z_0)} = k.$$

A related boundary behavior of analytic functions is considered also in [12]. Also, some applications of Jack-Fukui-Sakaguchi Lemma have been given in [11].

In this study, the Schwarz lemma will be obtained for the following class $\mathcal{H}$ which will be given. Let $\mathcal{A}$ denote the class of functions $f(z) = z + a_2z^2 + a_3z^3 + ...$ that are analytic in $\mathbb{D}$. Also, let $\mathcal{H}$ be the subclass of $\mathcal{A}$ consisting of all functions f satisfying

$$\left|\frac{(zf(z))''}{f'(z)} - 2z\frac{(f(z)-f(-z))'}{f(z)-f(-z)}\right| < \frac{1}{2},\ z \in \mathbb{D}. \tag{1.1}$$

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The aim of this paper is to examine some properties of the function f which belongs to the class of $\mathcal{H}$ by employing Jack's Lemma.

Let $f \in \mathcal{H}$ and consider the following function

$$\Phi(z) = \frac{4z^2 f'(z)}{\left(f(z) - f(-z)\right)^2} - 1. \tag{1.2}$$

It is an analytic function in $\mathbb{D}$ and $\Phi(0) = 0$. Now, let us show that $|\Phi(z)| < 1$ in $\mathbb{D}$. From the logarithmic differentiations, we have

$$\frac{(zf(z))''}{f'(z)} - 2z\frac{(f(z) - f(-z))'}{f(z) - f(-z)} = \frac{z\Phi'(z)}{1 + \Phi(z)}.$$

We assume that there exists a $z_0 \in \mathbb{D}$ such that

$$\max_{|z| \leq |z_0|} |\Phi(z)| = |\Phi(z_0)| = 1.$$

From Jack's lemma, we obtain

$$\Phi(z_0) = e^{i\theta} \text{ and } \frac{z_0\Phi'(z_0)}{\Phi(z_0)} = k.$$

Thus, we have that

$$\begin{aligned} \left|\frac{(z_0f(z_0))''}{f'(z_0)} - 2z_0\frac{(f(z_0) - f(-z_0))'}{f(z_0) - f(-z_0)}\right| &= \left|\frac{z_0\Phi'(z_0)}{1 + \Phi(z_0)}\right| = \left|\frac{k\Phi(z_0)}{1 + \Phi(z_0)}\right| \\ &= \left|\frac{ke^{i\theta}}{1 + e^{i\theta}}\right| \end{aligned}$$

Since $\left|1 + e^{i\theta}\right| \leq 1 + \left|e^{i\theta}\right| = 2$ and $k \geq 1$, we obtain

$$\left|\frac{(z_0f(z_0))''}{f'(z_0)} - 2z_0\frac{(f(z_0) - f(-z_0))'}{f(z_0) - f(-z_0)}\right| \geq \frac{1}{2}.$$

This contradicts the $f \in \mathcal{H}$. This means that there is no point $z_0 \in \mathbb{D}$ such that $\max_{|z| \leq |z_0|} |\Phi(z)| = |\Phi(z_0)| = 1$. Hence, we take $|\Phi(z)| < 1$ in $\mathbb{D}$. From the Schwarz lemma, we obtain

$$\begin{aligned} \Phi(z) &= \frac{4z^2 f'(z)}{\left(f(z) - f(-z)\right)^2} - 1 = \frac{4z^2\left(1 + 2a_2z + 3a_3z^2 + ...\right)}{4z^2\left(1 + a_3z^2 + a_5z^4 + ...\right)^2} - 1 \\ &= \frac{1 + 2a_2z + 3a_3z^2 + ...}{\left(1 + a_3z^2 + a_5z^4 + ...\right)^2} - 1 \\ &= \frac{1 + 2a_2z + 3a_3z^2 + ...}{1 + 2\left(a_3z^2 + a_5z^4 + ...\right) + \left(a_3z^2 + a_5z^4 + ...\right)^2 + ...} - 1 \\ &= \frac{2a_2z + a_3z^2 + ...}{1 + 2\left(a_3z^2 + a_5z^4 + ...\right) + \left(a_3z^2 + a_5z^4 + ...\right)^2 + ...}, \end{aligned}$$

$$\frac{\Phi(z)}{z} = \frac{2a_2 + a_3z + ...}{1 + 2\left(a_3z^2 + a_5z^4 + ...\right) + \left(a_3z^2 + a_5z^4 + ...\right)^2 + ...}$$

$$|\Phi'(0)| \leq |2a_2| \leq 1$$

and

$$|a_2| \leq \frac{1}{2}.$$

We thus obtain the following lemma.

Lemma 1.2. *If $f \in \mathcal{H}$, then we have the inequality*

$$|f''(0)| \leq 1. \tag{1.3}$$

There are numerous research on the Schwarz Lemma because of how broadly applicable it is. Some of these studies, referred to as the boundary variant of the Schwarz Lemma, deal with estimating the derivative of the function from below the modulus at a particular boundary point of the unit disc. The boundary version of Schwarz Lemma is given as follows [13, 18]:

Lemma 1.3. *If $g(z)$ extends continuously to some boundary point $\varsigma \in \partial\mathbb{D} = \{z : |z| = 1\}$ with $|\varsigma| = 1$, $|g(z)| < 1$ for $z \in \mathbb{D}$, $g(0) = 0$ and if $|g(\varsigma)| = 1$ and $g'(\varsigma)$ exists, then*

$$|g'(\varsigma)| \geq \frac{2}{1 + |g'(0)|} \tag{1.4}$$

and

$$|g'(\varsigma)| \geq 1. \tag{1.5}$$

The geometric theory of functions greatly benefits from inequality (1.5) and its generalizations, which are still hot subjects in the mathematics literature [1–4, 7–10, 13, 14].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [17]).

Lemma 1.4 (Julia-Wolff lemma). *Let g be an analytic function in $\mathbb{D}$, $g(0) = 0$ and $g(\mathbb{D}) \subset \mathbb{D}$. If, in addition, the function g has an angular limit $g(\varsigma)$ at $\varsigma \in \partial\mathbb{D}$, $|g(\varsigma)| = 1$, then the angular derivative $g'(\varsigma)$ exists and $1 \leq |g'(\varsigma)| \leq \infty$.*

Corollary 1.5. *The analytic function g has a finite angular derivative $g'(\varsigma)$ if and only if g' has the finite angular limit $g'(\varsigma)$ at $\varsigma \in \partial\mathbb{D}$.*

2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma for $\mathcal{H}$ class. Also, in a class of analytic functions on the unit disc, assuming the existence of angular limit on the boundary point, the estimations below of the modulus of angular derivative have been obtained. In addition, the symmetry points on the unit disc are used for the obtained inequalities.

Theorem 2.1. *Let $f \in \mathcal{H}$. Assume that, for $1, -1 \in \partial\mathbb{D}$, f has an angular limit $f(1)$ and $f(-1)$ at the points 1 and -1, respectively, $f'(1) = 0$. Then we have the inequality*

$$|f''(1)| \geq \frac{|f(1) - f(-1)|^2}{4}. \tag{2.1}$$

Proof. If $f(1) = f(-1)$, the inequality is clear. Suppose $f(1) \neq f(-1)$. Consider the function

$$\Phi(z) = \frac{4z^2 f'(z)}{(f(z) - f(-z))^2} - 1.$$

Also, since $f'(1)=0$, we have $|\Phi(1)|=1$. Furthermore, since $f(z)$ has an angular limit at 1, then $f'(z)$ has an angular limit at 1, $\Phi(z)$ has an angular limit an from the Julia Wolff lemma the function $\Phi(z)$ has an angular derivative at 1. Therefore, from (1.5), we obtain

$$\begin{aligned} 1 &\leq |\Phi'(1)| = 4\left|\frac{(2f'(1)+f''(1))\,(f(1)-f(-1))^2}{(f(1)-f(-1))^4}\right. \\ &\quad \left. -\frac{2\,(f(1)-f(-1))\,(f'(1)+f'(-1))\,f'(1)}{(f(1)-f(-1))^4}\right| \\ &= 4\frac{|f''(1)|}{|f(1)-f(-1)|^2} \end{aligned}$$

and

$$|f''(1)| \geq \frac{|f(1)-f(-1)|^2}{4}.$$

□

The inequality (2.1) can be strengthened from below by taking into account, $a_2 = \frac{f''(0)}{2}$, the first coefficient of the expansion of the function $f(z)=z+a_2z^2+a_3z^3+\ldots$.

Theorem 2.2. *Using the same presumptions as in Theorem 2.1, we obtain*

$$|f''(1)| \geq \frac{|f(1)-f(-1)|^2}{2\,(1+2\,|c_2|)}. \tag{2.2}$$

Proof. Let the function Φ be the same given by (1.2). So, from (1.4), we obtain

$$\frac{2}{1+|\Phi'(0)|} \leq |\Phi'(1)| = 4\frac{|f''(1)|}{|f(1)-f(-1)|^2},$$

Since

$$|\Phi'(0)| = 2\,|a_2|\,,$$

we take

$$\frac{1}{1+2\,|a_2|} \leq 2\frac{|f''(1)|}{|f(1)-f(-1)|^2}$$

and

$$|f''(1)| \geq \frac{1}{2}\frac{|f(1)-f(-1)|^2}{1+2\,|a_2|}.$$

□

The inequality (2.2) can be strengthened as below by taking into account $a_3 = \frac{f'''(0)}{3!}$ which is the coefficient in the expansion of the function $f(z)=z+a_2z^2+a_3z^3+\ldots$.

Theorem 2.3. *Let $f\in\mathcal{H}$. Assume that, for $1,-1\in\partial\mathbb{D}$, f has an angular limit $f(1)$ and $f(-1)$ at the points 1 and -1, respectively, $f'(1)=0$. Then we have the inequality*

$$|f''(1)| \geq \frac{|f(1)-f(-1)|^2}{4}\left(1+\frac{2\,(1-2\,|c_2|)^2}{1-4\,|c_2|^2+|c_3|}\right). \tag{2.3}$$

Proof. Let Φ be the same as in the proof of Theorem 2.1 and $u(z) = z$. By the maximum principle, for each $z \in \mathbb{D}$, we have the inequality $|\Phi(z)| \le |u(z)|$. So,

$$\begin{aligned} p(z) &= \frac{\Phi(z)}{u(z)} = \frac{1}{z}\left(\frac{4z^2 f'(z)}{(f(z)-f(-z))^2} - 1\right) \\ &= \frac{1}{z}\frac{2a_2 z + a_3 z^2 + \dots}{1 + 2\left(a_3 z^2 + a_5 z^4 + \dots\right) + \left(a_3 z^2 + a_5 z^4 + \dots\right)^2 + \dots} \\ &= \frac{2a_2 + a_3 z + \dots}{1 + 2\left(a_3 z^2 + a_5 z^4 + \dots\right) + \left(a_3 z^2 + a_5 z^4 + \dots\right)^2 + \dots} \end{aligned}$$

is analytic function in $\mathbb{D}$ and $|p(z)| \le 1$ for $z \in \mathbb{D}$. In particular, we have

$$|p(0)| = |2a_2| \le 1 \tag{2.4}$$

and

$$|p'(0)| = |a_3|$$

Furthermore, the geometric meaning of the derivative and the inequality $|\Phi(z)| \le |u(z)|$ imply the inequality

$$\frac{\Phi'(1)}{\Phi(1)} = |\Phi'(1)| \ge |u'(1)| = \frac{u'(1)}{u(1)}.$$

The auxiliary function

$$\vartheta(z) = \frac{p(z) - p(0)}{1 - \overline{p(0)}p(z)}$$

is analytic in $\mathbb{D}$, $\vartheta(0) = 0$, $|\vartheta(z)| < 1$ for $|z| < 1$ and $|\vartheta(1)| = 1$ for $1 \in \partial\mathbb{D}$. From (1.4), we obtain

$$\begin{aligned} \frac{2}{1 + |\vartheta'(0)|} &\le |\vartheta'(1)| = \frac{1 - |p(0)|^2}{\left|1 - \overline{p(0)}p(1)\right|^2} |p'(1)| \\ &\le \frac{1 + |p(0)|}{1 - |p(0)|}\left\{|\Phi'(1)| - |u'(1)|\right\} \\ &= \frac{1 + |2a_2|}{1 - |2a_2|}\left(4\frac{|f''(1)|}{|f(1) - f(-1)|^2} - 1\right). \end{aligned}$$

Since

$$\vartheta'(z) = \frac{1 - |p(0)|^2}{\left(1 - \overline{p(0)}p(z)\right)^2} p'(z)$$

and

$$|\vartheta'(0)| = \frac{|p'(0)|}{1 - |p(0)|^2} = \frac{|a_3|}{1 - (|2a_2|)^2},$$

we obtain

$$\frac{2}{1 + \frac{|a_3|}{1 - (|2a_2|)^2}} \le \frac{1 + |2a_2|}{1 - |2a_2|}\left(4\frac{|f''(1)|}{|f(1) - f(-1)|^2} - 1\right),$$

$$\frac{2\left(1 - (|2a_2|)^2\right)}{1 - (|2a_2|)^2 + |a_3|}\frac{1 - |2a_2|}{1 + |2a_2|} \le 4\frac{|f''(1)|}{|f(1) - f(-1)|^2} - 1$$

and

$$|f''(1)| \geq \frac{|f(1)-f(-1)|^2}{4}\left(1+\frac{2\left(1-2\left|a_2\right|\right)^2}{1-4\left|a_2\right|^2+\left|a_3\right|}\right).$$

□

If $f(z)-z$ a have zeros different from $z=0$, taking into account these zeros, the inequality (2.3) can be strengthened in another way. This is given by the following Theorem.

THEOREM 2.4. *Let $f \in \mathcal{H}$ and $a_i, i=1,..,n$ be zeros of the function $f(z)-z$ in $\mathbb{D}$ that are different from zero. Assume that, for $1,-1 \in \partial\mathbb{D}$, f has an angular limit $f(1)$ and $f(-1)$ at the points 1 and -1, respectively, $f'(1)=0$. Then we have the inequality*

$$|f''(1)| \geq \frac{|f(1)-f(-1)|^2}{4}\left(1+\sum_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2}\right. \tag{2.5}$$
$$\left.+\frac{2\left(\prod_{i=1}^{n}|a_i|-2|a_2|\right)^2}{\left(\prod_{i=1}^{n}|a_i|\right)^2-4|a_2|^2+\prod_{i=1}^{n}|a_i|\left|a_3+2a_2\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}\right).$$

Proof. Let Φ be as in (1.2) and $a_i, i=1,..,n$ be zeros of the function $f(z)-z$ in $\mathbb{D}$ that are different from zero. Also, consider the function

$$B(z)=z\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}.$$

Here, B is analytic in $\mathbb{D}$ and $|B(z)|<1$ for $|z|<1$. By the maximum principle for each $z\in\mathbb{D}$, we have

$$|\Phi(z)| \leq |B(z)|.$$

Consider the function

$$\begin{aligned}
w(z) &= \frac{\Phi(z)}{B(z)}=\left(\frac{4z^2f'(z)}{\left(f(z)-f(-z)\right)^2}-1\right)\frac{1}{z\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}} \\
&= \frac{2a_2z+a_3z^2+...}{1+2\left(a_3z^2+a_5z^4+...\right)+\left(a_3z^2+a_5z^4+...\right)^2+...}\frac{1}{z\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}} \\
&= \frac{2a_2+a_3z+...}{1+2\left(a_3z^2+a_5z^4+...\right)+\left(a_3z^2+a_5z^4+...\right)^2+...}\frac{1}{\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}}.
\end{aligned}$$

w is analytic in $\mathbb{D}$ and $|w(z)|<1$ for $|z|<1$. In particular, we have

$$|w(0)|=\frac{2\left|a_2\right|}{\prod_{i=1}^{n}|a_i|}$$

and

$$|w'(0)|=\frac{\left|a_3+2a_2\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{\prod_{i=1}^{n}|a_i|}.$$

The auxiliary function

$$\varphi(z)=\frac{w(z)-w(0)}{1-\overline{w(0)}w(z)}$$

is analytic in $\mathbb{D}$, $|\varphi(z)|<1$ for $|z|<1$ and $\varphi(0)=0$. For $1\in\partial\mathbb{D}$ and $f'(1)=0$, we take $|\varphi(1)|=1$.

From (1.4), we obtain

$$\begin{aligned}\frac{2}{1+|\varphi'(0)|} &\leq |\varphi'(1)|=\frac{1-|w(0)|^2}{\left|1-\overline{w(0)}w(1)\right|}|w'(1)| \\ &\leq \frac{1+|w(0)|}{1-|w(0)|}\left(|\Phi'(1)|-|B'(1)|\right).\end{aligned}$$

It can be seen that

$$|\varphi'(0)|=\frac{|w'(0)|}{1-|w(0)|^2}$$

and

$$|\varphi'(0)|=\frac{\frac{\left|a_3+2a_2\sum\limits_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{\prod\limits_{i=1}^{n}|a_i|}}{1-\left(\frac{2|a_2|}{\prod\limits_{i=1}^{n}|a_i|}\right)^2}=\prod_{i=1}^{n}|z_i|\frac{\left|a_3+2a_2\sum\limits_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{\left(\prod\limits_{i=1}^{n}|a_i|\right)^2-4|a_2|^2}.$$

Also,we have

$$|B'(1)|=1+\sum_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2},\ 1\in\partial\mathbb{D}.$$

Therefore, we obtain

$$\frac{2}{1+\prod\limits_{i=1}^{n}|a_i|\frac{\left|a_3+2a_2\sum\limits_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{\left(\prod\limits_{i=1}^{n}|a_i|\right)^2-4|a_2|^2}}\leq\frac{\prod\limits_{i=1}^{n}|a_i|+2|a_2|}{\prod\limits_{i=1}^{n}|a_i|-2|a_2|}\left(4\frac{|f''(1)|}{|f(1)-f(-1)|^2}-1-\sum_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2}\right),$$

$$\frac{2\left(\left(\prod\limits_{i=1}^{n}|a_i|\right)^2-4|a_2|^2\right)}{\left(\prod\limits_{i=1}^{n}|a_i|\right)^2-4|a_2|^2+\prod\limits_{i=1}^{n}|a_i|\left|a_3+2a_2\sum\limits_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}$$

$$\leq\frac{\prod\limits_{i=1}^{n}|a_i|+2|a_2|}{\prod\limits_{i=1}^{n}|a_i|-2|a_2|}\left(4\frac{|f''(1)|}{|f(1)-f(-1)|^2}-1-\sum_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2}\right),$$

$$\frac{2\left(\prod_{i=1}^{n}|a_i|-2|a_2|\right)^2}{\left(\prod_{i=1}^{n}|a_i|\right)^2-4|a_2|^2+\prod_{i=1}^{n}|a_i|\left|a_3+2a_2\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}\le 4\frac{|f''(1)|}{|f(1)-f(-1)|^2}-1-\sum_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2}$$

and so, we get inequality (2.1). □

If $f(z)-z$ has no zeros different from $z=0$ in Theorem 2.3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

Theorem 2.5. *Let $f\in\mathcal{H}$, $f(z)-z$ has no zeros in $\mathbb{D}$ except $z=0$ and $a_2>0$. Assume that, for $1,-1\in\partial\mathbb{D}$, f has an angular limit $f(1)$ and $f(-1)$ at the points 1 and -1, respectively, $f'(1)=0$. Then we have the inequality*

$$|f''(1)|\ge\frac{|f(1)-f(-1)|^2}{4}\left(1-\frac{4a_2\ln^2(2a_2)}{4a_2\ln(2a_2)-|a_3|}\right). \tag{2.6}$$

Proof. Let $a_2>0$ in the expression of the function $f(z)$. Having in mind the inequality (2.4) and the function $f(z)-z$ has no zeros in $\mathbb{D}$ except $z=0$, we denote by $\ln p(z)$ the analytic branch of the logarithm normed by the condition

$$\ln p(0)=\ln(2a_2)<0.$$

The auxiliary function

$$\Theta(z)=\frac{\ln p(z)-\ln p(0)}{\ln p(z)+\ln p(0)}$$

is analytic in the unit disc $\mathbb{D}$, $|\Theta(z)|<1$, $\Theta(0)=0$ and $|\Theta(1)|=1$ for $1\in\partial\mathbb{D}$.

From (1.4), we obtain

$$\begin{aligned}\frac{2}{1+|\Theta'(0)|} &\le |\Theta'(1)|=\frac{|2\ln p(0)|}{|\ln p(1)+\ln p(0)|^2}\left|\frac{p'(1)}{p(1)}\right|\\ &= \frac{-2\ln p(0)}{\ln^2 p(0)+\arg^2 p(1)}\left\{|\Phi'(1)|-1\right\}.\end{aligned}$$

Replacing $\arg^2 p(1)$ by zero, then

$$\frac{1}{1-\frac{1}{2\ln(2a_2)}\frac{|a_3|}{2a_2}}\le\frac{-1}{\ln(2a_2)}\left\{4\frac{|f''(1)|}{|f(1)-f(-1)|^2}-1\right\}$$

and

$$1-\frac{4a_2\ln^2(2a_2)}{4a_2\ln(2a_2)-|a_3|}\le 4\frac{|f''(1)|}{|f(1)-f(-1)|^2}.$$

Thus, we obtain the inequality (2.6). □

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