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A SIMPLE PROOF FOR JI-KIM-OH'S THEOREM

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Abstract. In 1911, Dubouis determined all positive integers represented by sums of k nonvanishing squares for all $k \geq 4$. As a generalization, Y.-S. Ji, M.-H. Kim and B.-K. Oh determined all positive definite binary quadratic forms represented by sums of k nonvanishing squares for all $k \geq 5$. In this article, we give a simple proof for Ji-Kim-Oh's theorem for all $k \geq 10$.

1. Introduction

A classical result of the Lagrange Four Square Theorem states that the sum of four squares of integers represents all positive integers. Before Lagrange, Descartes proposed interesting variants [1]. That is, finding all positive integers represented by exactly four nonvanishing squares of integers. His conjecture was proved by Dubouis [2, Chapter 6] in 1911. Dubouis determined the set $\mathbf{E}_k$ of all positive integers not represented by a sum of k nonvanishing squares for all $k \geq 4$. It is still an open conjecture to determine $\mathbf{E}_3$, although $\mathbf{E}_1$ and $\mathbf{E}_2$ are simple to be determined. Recently, there have been some generalizations of Dubouis' results to real quadratic fields [4–6].

In 1930, Mordell generalized the Four Square Theorem in another direction. That is, the sum of five squares of integral linear forms can represent all positive definite integral binary quadratic forms. Recently, Y.-S. Ji, M.-H. Kim, and B.-K. Oh [3] determined all positive definite integral binary quadratic forms that are not represented by a sum of nonvanishing k squares for all $k \geq 5$. It is also an interesting generalization of Dubouis' results. The critical part of Ji-Kim-Oh's theorem is making tables of all positive definite binary quadratic forms that are not represented by a sum of five nonvanishing squares.

In this article, we give a simple proof of Ji-Kim-Oh's Theorem [3, Theorem 5.5] under the assumption $k \geq 10$ without using the tables mentioned above.

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2. Notation and preliminaries

We begin by setting some notions and terminologies. Here, let

$$f(x,y) = ax^2 + 2bxy + cy^2 \in \mathbb{Z}[x,y]$$

be a positive definite binary quadratic form of discriminant $\mathrm{disc}(f) = ac - b^2$. We say f is *Minkowski reduced* if $0 \le 2b \le a \le c$. Note that every positive definite binary quadratic form is equivalent to a unique Minkowski reduced binary quadratic form. For abbreviation, a binary form $f(x,y)$ will always mean a positive definite Minkowski reduced binary quadratic form throughout the remainder of this article.

If there exist linear forms $a_ix + b_iy \in \mathbb{Z}[x,y]$ such that

$$f(x,y) = (a_1x + b_1y)^2 + (a_2x + b_2y)^2 + \cdots + (a_kx + b_ky)^2, \tag{1}$$

then we say that f *is represented by a sum of k squares*, and if there exist nonzero $a_ix + b_iy \in \mathbb{Z}[x,y]$ satisfying above (1), then we say that f *is represented by a sum of nonvanishing k squares.*

For Minkowski reduced binary form f, we define

$$\mathrm{Min}(f) = a \quad \text{and} \quad \mathrm{Sos}(f) = a + c - b. \tag{2}$$

Note that the number $\mathrm{Sos}(f)$ is actually the maximum number k of squares when f is represented by a sum of nonvanishing k squares (see [3]). Also, from the Minkowski reduced property of f, we know

$$2c \ge \mathrm{Sos}(f) = a + c - b \ge \frac{3}{2}a. \tag{3}$$

We introduce some important previous works on representation by a sum of (nonvanishing) squares:

Theorem 2.1 (Dubouis). *Every positive integer n is represented by a sum of nonvanishing k squares for all $k \ge 4$ except*

$$n = \begin{cases} 1,2,3,5,6,8,9,11,14,17,29,41 \text{ and } 2\cdot 4^m, 6\cdot 4^m, 14\cdot 4^m \text{ for } m \ge 1 & \text{if } k = 4, \\ 1,2,3,4,6,7,9,10,12,15,18,33 & \text{if } k = 5, \\ 1,2,\ldots,k-1,k+1,k+2,k+4,k+5,k+7,k+10,k+13 & \text{if } k \ge 6. \end{cases}$$

Theorem 2.2 (Mordell). *Every binary form is represented by a sum of five squares.*

Lemma 2.3 (Ji-Kim-Oh). *Let $k \ge 8$. If a binary form f satisfies*

$$\mathrm{Min}(f) \ge 5 \quad \text{and} \quad \mathrm{Sos}(f) \ge k+2,$$

then f is represented by a sum of nonvanishing k squares.

Note that the above Lemma is [3, Lemma 5.4] and the following Theorem 2.4 is a part of the result of [3, Theorem 5.5]. We omit the full version of their result since it is too complicated to state here. The purpose of this paper is to provide concise proof of the following.

Theorem 2.4 (Ji-Kim-Oh). *Let $k \ge 10$. A binary form f satisfies*

$$\mathrm{Min}(f) \ge 4 \quad \text{and either} \quad \mathrm{Sos}(f) = k \quad \text{or} \quad \mathrm{Sos}(f) \ge k+2$$

if and only if f is represented by a sum of nonvanishing k squares.

We will use the following well-known facts to prove our main theorem.

Remark 2.5. A binary form f is not represented by a sum of four squares if and only if $\operatorname{disc}(f) \equiv 7 \cdot 4^k \pmod{8 \cdot 4^k}$ for some integer $k \geq 0$ (see [7]).

Remark 2.6. As a consequence of Mordell's Theorem and Remark 2.5, a binary form $f(x,y) = ax^2 + 2bxy + cy^2$ is represented by a sum of nonvanishing five squares if $\operatorname{disc}(f) = ac - b^2 \equiv 7 \pmod 8$.

3. Proof of the Main Theorem

In this section we determine all binary forms represented by a sum of nonvanishing k squares. Throughout this section, we let

$$f(x,y) = ax^2 + 2bxy + cy^2$$

be a positive definite Minkowski reduced quadratic form of the $\operatorname{disc}(f) = ac - b^2$.

Lemma 3.1. *Let $k \geq 5$ and f satisfies the following conditions:*

(a) $\operatorname{Min}(f) = a \geq 4$,
(b) $a \equiv c \equiv 0 \pmod 2$, $b \equiv 1 \pmod 2$ *and* $ac \equiv 0 \pmod 8$,
(c) $\operatorname{Sos}(f) \geq k + 2$.

Then f is represented by a sum of nonvanishing k squares.

Proof. The proof is given by induction on $\operatorname{Sos}(f) \geq 7$. Note that $\operatorname{Sos}(f) = a + c - b$ is always odd by the given condition (b).

First, suppose $\operatorname{Sos}(f) = 7$, then $k = 5$. Since f is Minkowski reduced, $7 = \operatorname{Sos}(f) = a + c - b \geq \frac{3}{2}a$, so $a \leq 4$. Since b is odd and $0 \leq 2b \leq a$, we have $b = 1$. Since $ac \equiv 0 \pmod 8$, we get $a = 4$ and $c = 4$. Thus

$$\begin{aligned} f(x,y) &= 4x^2 + 2xy + 4y^2 \\ &= (x+y)^2 + (x+y)^2 + (x-y)^2 + x^2 + y^2. \end{aligned}$$

Therefore f is represented by a sum of nonvanishing 5 squares.

Next, suppose the lemma is true for $7 \leq \operatorname{Sos}(f) \leq m$. That is, f is represented by a sum of nonvanishing k squares when $5 \leq k \leq \operatorname{Sos}(f) - 2 \leq m - 2$. Let

$$g(x,y) = px^2 + 2qxy + ry^2$$

be a positive definite Minkowski reduced quadratic form which satisfies (a), (b) and $\operatorname{Sos}(g) = p + q - r = m + 2$. By Remark 2.6 g is represented by a sum of nonvanishing 5 squares. We need to show that g is represented by a sum of nonvanishing k squares for all $6 \leq k \leq m$. Note that $r \geq 6$, since r is even and $2r \geq \operatorname{Sos}(g) = p + r - q = m + 2 \geq 9$.

Consider a quadratic form

$$g_1(x,y) = g(x,y) - 4y^2 = px^2 + 2qxy + (r-4)y^2.$$

Since $p > 0$, $r \geq 6$ and $\operatorname{disc}(g_1) = p(r-4) - q^2 \geq pr - 4p - \frac{pr}{4} = \frac{3}{4}p(r - \frac{16}{3}) > 0$, g_1 is positive definite. Since $\operatorname{disc}(g_1) \equiv 7 \pmod 8$, g_1 is represented by a sum of nonvanishing 5 squares by Remark 2.6. Thus $g = g_1(x,y) + (2y)^2$ is represented by a sum of nonvanishing 6 squares.

Consider quadratic forms g_i $(i = 2, 3, 4)$ where

$$\begin{aligned} g_2(x,y) &= g(x,y) - 2(x+y)^2 \\ &= (p-2)x^2 + 2(q-2)xy + (r-2)y^2 && \text{if } p \geq 6 \text{ and } q \geq 3, \\ g_3(x,y) &= g(x,y) - 2x^2 - 2y^2 \\ &= (p-2)x^2 + 2xy + (r-2)y^2 && \text{if } p \geq 6 \text{ and } q = 1, \\ g_4(x,y) &= g(x,y) - 2y^2 \\ &= 4x^2 + 2xy + (r-2)y^2 && \text{if } p = 4 \text{ and } q = 1. \end{aligned}$$

We can easily check that g_2, g_3 and g_4 are positive definite Minkowski reduced and satisfy the conditions (a) and (b). Also, we know $\operatorname{Sos}(g_2) = m$, $\operatorname{Sos}(g_3) = m-2$ and $\operatorname{Sos}(g_4) = m$. By the induction hypothesis, g_2 and g_4 are represented by a sum of nonvanishing k squares for all $5 \leq k \leq m-2$ and g_3 is represented by a sum of nonvanishing k squares for all $5 \leq k \leq m-4$. Therefore, g is represented by a sum of nonvanishing k squares for all $5 \leq k \leq m$. $\square$

THEOREM 3.2. *Let $k \geq 10$ and f satisfies*

$$\operatorname{Min}(f) = a \geq 4 \quad \textit{and} \quad \operatorname{Sos}(f) \geq k+2,$$

then f is represented by a sum of nonvanishing k squares.

Proof. We will show it by dividing several cases. The basic idea is to construct quadratic forms

$$\begin{aligned} g_i(x,y) &= f(x,y) - a_i(x+y)^2 - b_i x^2 - c_i y^2 \\ &= p_i x^2 + 2q_i xy + r_i y^2 \end{aligned}$$

satisfying the following conditions:

(a) g_i is positive definite, and either $g_i(x,y)$ or $g_i(y,x)$ is Minkowski reduced with $\operatorname{Min}(g_i) \geq 4$,
(b) $p_i \equiv 0 \pmod 2$, $r_i \equiv 0 \pmod 4$, $q_i \equiv 1 \pmod 2$,
(c) $f - g_i = a_i(x+y)^2 + b_i x^2 + c_i y^2$ is represented by a sum of nonvanishing h squares for some h, where $k-h \geq 5$ and $\operatorname{Sos}(g_i) \geq k-h+2$.

By Lemma 3.1, the above conditions imply that g_i is represented by a sum of nonvanishing $k-h$ squares. It follows $f = g_i + (f - g_i)$ is represented by a sum of nonvanishing k squares.

Case (1) $c - 4 \geq 2b$ and $b \geq 1$: We define the numbers a_1, b_1, and c_1 as followings:

$$a_1 = \begin{cases} 0 & \text{if } b \text{ is odd,} \\ 1 & \text{if } b \text{ is even,} \end{cases} \qquad b_1 = \begin{cases} 0 & \text{if } a - a_1 \text{ is even,} \\ 1 & \text{if } a - a_1 \text{ is odd,} \end{cases}$$

c_1 is the residue of $c - a_1$ modulo 4.

Let

$$\begin{aligned} g_1(x,y) &= f(x,y) - a_1(x+y)^2 - b_1 x^2 - c_1 y^2 \\ &= (a - a_1 - b_1)x^2 + 2(b - a_1)xy + (c - a_1 - c_1)y^2. \end{aligned}$$

From the definition of a_1, b_1, c_1, we can see that g_1 satisfies (b) and (c) with $h = a_1 + b_1 + c_1 \leq 5$, $k - h \geq 5$. Since $a - a_1 - b_1 \equiv 0 \pmod 2$, $a - a_1 - b_1 \geq 2(b - a_1) \geq 0$. Also,

$c - a_1 - c_1 \geq c - 4 \geq 2(b - a_1) \geq 0$. Hence, either $g_1(x,y)$ or $g_1(y,x)$ is positive definite Minkowski reduced except when $a = 4$ and $b = 2$. Therefore, if $f(x,y) \neq 4x^2 + 4xy + cy^2$, then $g_1(x,y)$ satisfies (a) and hence f is represented by a sum of nonvanishing k squares.

Suppose $f(x,y) = 4x^2 + 4xy + cy^2$. Since $\mathrm{Sos}(f) = c + 2 \geq k + 2$, we have $c \geq k$. Consider the following identities

$$f(x,y) = (2x+y)^2 + (c-1)y^2 \tag{4}$$

$$= 2x^2 + 2(x+y)^2 + (c-2)y^2 \tag{5}$$

$$= x^2 + (x+y)^2 + (x-y)^2 + (x+2y)^2 + (c-6)y^2. \tag{6}$$

Let $k_1 = k - 1$ and $k_2 = k - 4$. If $c = k + r$ with $r = 0, 3, 6, 8, 9, 11, 12$ or $r \geq 14$, then by Dubouis' Theorem, $c - 1$ is represented by a sum of nonvanishing $k_1 = k - 1$ squares. By the identity (4), f is represented by a sum of nonvanishing k squares. If $c = k + r$ with $r = 1, 4, 7, 10, 13$, then $c - 2$ is represented by a sum of nonvanishing $k_2 = k - 4$ squares. By the identity (5), f is represented by a sum of nonvanishing k squares. If $c = k + r$ with $r = 2, 5$, then $c - 6$ is represented by a sum of nonvanishing $k_2 = k - 4$ squares. By the identity (6), f is represented by a sum of nonvanishing k squares.

Case (2) $c - 4 \geq 2b$ and $b = 0$: In this case $\mathrm{Sos}(f) = a + c \geq k + 2$. Let a_1, b_1 and c_1 be the same as Case (1). Since $b = 0$, we have $a_1 = 1$. Let

$$\begin{aligned} g_2(x,y) &= f(x,y) - (x-y)^2 - b_1x^2 - c_1y^2 \\ &= (a - 1 - b_1)x^2 + 2xy + (c - 1 - c_1)y^2. \end{aligned}$$

If $a \geq 5$, then $g_2(x,y)$ satisfies the conditions (a) and (b). Also if $k \leq a + c - 4$, then $\mathrm{Sos}(g_2) = a + c - 3 - b_1 - c_1 \geq k + 2 - (3 + b_1 + c_1) \geq k - 5 \geq 5$. So g_2 satisfies (c) with $h = 3 + b_1 + c_1$. Thus f is represented by a sum of nonvanishing k squares. If $k = a + c - 2$ or $a + c - 3$, then from the identities

$$\begin{aligned} f(x,y) &= (x+y)^2 + (x-y)^2 + (a-2)x^2 + (c-2)y^2 \\ &= (2x)^2 + (a-4)x^2 + cy^2 \end{aligned}$$

we have that f is represented by a sum of nonvanishing k squares.

If $a = 4$, then by using a similar method used in Case (1) with

$$\begin{aligned} f(x,y) &= 4x^2 + cy^2 \\ &= (2x)^2 + cy^2 \\ &= 2x^2 + (x+y)^2 + (x-y)^2 + (c-2)y^2, \end{aligned}$$

we can show that f is represented by a sum of nonvanishing k squares.

Case (3) $c - 4 < 2b$ and $a \geq 7$: We define the numbers a_3, b_3 and c_3 as followings:

$$a_3 = \begin{cases} 2 & \text{if } b \text{ is odd,} \\ 3 & \text{if } b \text{ is even,} \end{cases} \qquad b_3 = \begin{cases} 0 & \text{if } a - a_3 \text{ is even,} \\ 1 & \text{if } a - a_3 \text{ is odd,} \end{cases}$$

c_3 is the residue of $c - a_3$ modulo 4.

Since $2b > c-4 \geq 3$, we have $b \geq 2$. Let

$$\begin{aligned} g_3(x,y) &= f(x,y) - a_3(x+y)^2 - b_3x^2 - c_3y^2 \\ &= (a-a_3-b_3)x^2 + 2(b-a_3)xy + (c-a_3-c_3)y^2. \end{aligned}$$

Then g_3 satisfies the conditions (a), (b) and $2 \leq a_3+b_3+c_3 \leq 7$. If $2 \leq a_3+b_3+c_3 \leq 5$, then g_3 also satisfies the condition (c) with $h = a_3+b_3+c_3 \leq 5$, since $\mathrm{Sos}(g_3) = \mathrm{Sos}(f) - a_3 - b_3 - c_3 \geq k-h+2$. Therefore f is represented by a sum of nonvanishing k squares. If $a_3+b_3+c_3 = 6, 7$, then we let

$$\begin{aligned} g_4(x,y) &= f(x,y) - a_4(x+y)^2 - b_4x^2 - c_4y^2 \\ &= (a-a_4-b_4)x^2 + 2(b-a_4)xy + (c-a_4-c_4)y^2, \end{aligned}$$

where $a_4 = a_3$, $b_4 = b_3 - 2$ and $c_4 = c_3$. Then g_4 satisfies condition (a), (b) and (c) with $h = a_4+b_4+c_4 = 4$ or 5. Therefore f is represented by a sum of nonvanishing k squares.

Case (4) $c-4 < 2b$ and $a \leq 6$: Since f is Minkowski reduced and $\mathrm{Sos}(f) = a+c-b \geq 12$, we have $6 \geq a \geq 2b > c-4 \geq 12-a+b-4 \geq b+2$ and hence $b=3$, $a=6$ and $c < 10$. Let c_5 be the residue of c modulo 4 and let

$$g_5(x,y) = f(x,y) - c_5y^2.$$

Then g_5 satisfies (a), (b) and (c) with $h = c_5 \leq 3$. Therefore f is represented by a sum of nonvanishing k squares. $\square$

For any binary quadratic form $g(x,y) = px^2 + 2qxy + ry^2$ which need not to be Minkowski reduced, we define

$$\mathcal{S}(g) = p + r - q.$$

We can easily see the following properties:

(a) for all binary quadratic forms f and g,

$$\mathcal{S}(f+g) = \mathcal{S}(f) + \mathcal{S}(g).$$

(b) $\mathcal{S}(x^2) = 1$ and $\mathcal{S}(x^2-2xy+y^2) = 3$ although x^2 and $x^2-2xy+y^2 = (x-y)^2$ are isometric. More generally, if $g(x,y) = px^2+2qxy+ry^2$ is isometric to a Minkowski reduced binary quadratic form $f(x,y) = ax^2+2bxy+cy^2$, then

$$a+c-b = \mathcal{S}(f) \leq \mathcal{S}(g) = p+r-q.$$

Moreover, if f is Minkowski reduced then $\mathcal{S}(f) = \mathrm{Sos}(f)$.

(c) For any binary linear form $\ell = ax+by$, we can see that

$$\mathcal{S}(\ell^2) = a^2+b^2-ab = \frac{1}{2}(a^2+b^2+(a-b)^2). \tag{7}$$

Since $\mathcal{S}(\ell^2) = a^2+b^2-ab \geq 2|ab| - ab \geq |ab|$, we get

$$\begin{cases} \mathcal{S}(\ell^2) = 1 & \text{iff } \ell^2 = x^2,\ y^2,\ (x+y)^2, \\ \mathcal{S}(\ell^2) \geq 3 & \text{otherwise.} \end{cases} \tag{8}$$

Although the statement of Theorem 3.3 has been known [3, Lemma 5.1], we here provide a new simple proof. In the proof, we use properties of $\mathcal{S}$, which are mentioned above.

THEOREM 3.3. *If a binary form f is represented by a sum of nonvanishing k squares, then*

$$\mathrm{Sos}(f) = k \quad or \quad \mathrm{Sos}(f) \geq k+2.$$

Proof. If f is represented by a sum of nonvanishing k squares, then there are nonzero linear forms $\ell_i(x,y) = a_i x + b_i y \in \mathbb{Z}[x,y]$ satisfying

$$f(x,y) = \sum_{1\leq i\leq k} [\ \ell_i(x,y)\ ]^2.$$

Since f is Minkowski reduced, we get

$$\mathrm{Sos}(f) = \mathcal{S}(f) = \mathcal{S}(\sum_{1\leq i\leq k} \ell_i^2\ ) = \sum_{1\leq i\leq k} \mathcal{S}(\ell_i^2) \geq k.$$

Suppose $\mathrm{Sos}(f) = k+1$. First, if $\mathcal{S}(\ell_i^2) = 1$ for all $1 \leq i \leq k$, then

$$k+1 = \mathrm{Sos}(f) = \mathcal{S}(f) = \mathcal{S}(\sum_{1\leq i\leq k} \ell_i^2\ ) = k,$$

which is impossible. Next, if $\mathcal{S}(\ell_j^2) \geq 3$ for some $1 \leq j \leq k$, then

$$k+1 = \mathrm{Sos}(f) = \mathcal{S}(f) = \mathcal{S}(\ell_j^2) + \mathcal{S}(\sum_{1\leq i\leq k,\ i\neq j} \ell_i^2\ ) \geq 3 + (k-1) = k+2,$$

which is also impossible. $\square$

By combining the results of Theorem 3.2 and Theorem 3.3, we get the followings: Assume $k \geq 10$ and let f be a binary form with $\mathrm{Min}(f) \geq 4$. Then f is represented by a sum of nonvanishing k squares if and only if

$$\text{either} \quad \mathrm{Sos}(f) = k \quad \text{or} \quad \mathrm{Sos}(f) \geq k+2.$$

When a binary form f satisfies $\mathrm{Min}(f) \leq 3$, we have the following theorem.

THEOREM 3.4. *Let $k \geq 10$ and let f be a binary form with $\mathrm{Min}(f) \leq 3$. Then f is represented by a sum of nonvanishing k squares if and only if*

$$\mathrm{Sos}(f) \geq k \quad and \quad \mathrm{Sos}(f) \neq k+t$$

where

$$t = \begin{cases} 1,2,4,5,7,10,13 & \text{if } f(x,y) = x^2 + cy^2, \\ 1,4,7 & \text{if } f(x,y) = 2x^2 + cy^2 \text{ or } f(x,y) = 3x^2 + cy^2, \\ 1,2,5 & \text{if } f(x,y) = 2x^2 + 2xy + cy^2, \\ 1 & \text{if } f(x,y) = 3x^2 + 2xy + cy^2. \end{cases}$$

Proof. Consider the following identities:

$$\begin{array}{lcl} 2x^2 + cy^2 & = & (x+y)^2 + (x-y)^2 + (c-2)y^2, \\ 2x^2 + 2xy + cy^2 & = & x^2 + (x+y)^2 + (c-1)y^2 \\ & = & (x+2y)^2 + (x-y)^2 + (c-5)y^2, \\ 3x^2 + cy^2 & = & (x+y)^2 + (x-y)^2 + (c-2)y^2, \\ 3x^2 + 2xy + cy^2 & = & 2x^2 + (x+y)^2 + (c-1)y^2 \\ & = & x^2 + (x+2y)^2 + (x-y)^2 + (c-5)y^2 \\ & = & 2(x+y)^2 + (x-y)^2 + (c-3)y^2. \end{array}$$

We obtain the necessity if we apply Dubouis' Theorem and the same method which is used in the proof of Theorem 3.2 Case (1)'s last part.

From the identity (8), we can list out all ℓ^2 for $\ell = ax + by$ such that $\mathcal{S}(\ell^2) \le 8$:

$$\begin{array}{ll} \mathcal{S}(\ell^2) = 1: & x^2, y^2, (x+y)^2, \\ \mathcal{S}(\ell^2) = 3: & (x-y)^2, (x+2y)^2, (2x+y)^2, \\ \mathcal{S}(\ell^2) = 4: & (2x)^2, (2y)^2, (2x+2y)^2, \\ \mathcal{S}(\ell^2) = 7: & (x-2y)^2, (2x-y)^2, (x+3y)^2, (3x+y)^2, (2x+3y)^2, (3x+2y)^2. \end{array}$$

The sufficiency is a consequence of identity $\mathcal{S}(f) = \sum_{i=1}^{k} \mathcal{S}((a_ix+b_iy)^2)$ when $f(x,y) = ax^2 + 2bxy + cy^2 = \sum_{i=1}^{k} (a_ix + b_iy)^2$ and the above list. □

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