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NEW QUANTUM VARIANTS OF SIMPSON-NEWTON TYPE INEQUALITIES VIA (α, m)-CONVEXITY

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Abstract. In this article, we will utilize (α, m)-convexity to create a new form of Simpson-Newton inequalities in quantum calculus by using q_{ϱ_1}-integral and q_{ϱ_1}-derivative. Newly discovered inequalities can be transformed into quantum Newton and quantum Simpson for generalized convexity. Additionally, this article demonstrates how some recently created inequalities are simply the extensions of some previously existing inequalities. The main findings are generalizations of numerous results that already exist in the literature, and some fundamental inequalities, such as Hölder's and Power mean, have been used to acquire new bounds.

1. Introduction

The field of mathematical inequalities has grown rapidly over the last several centuries and viewed as a classical field of research. From classical to contemporary applications, inequalities have been used in many areas of science and technology. Information theory, engineering and many other fields benefited from their use [1, 2]. Some basic inequalities like Hermite-Hadamard, Hölder, Ostrowski, Jensen, Hardy and Cauchy-Schwarz played an important part in the theory of classical calculus and q-calculus because their significance was well established in the past [6, 7].
The connection between inequalities and convex function has been found to be extremely strong. The study of convex function always presents magnificent sight of the beauty in advanced mathematics. Convexity is a growing area of research because mathematicians always put potential in this direction that has applications in complex analysis, number theory and many other fields. Convexity also has a significant impact on people's lives with numerous applications [1, 8] and convex function is defined as:

Definition 1.1. [8] Let $\Psi : [\varrho_1, \varrho_2] \subseteq \Re \to \Re$ is convex, if for every $\varkappa, \mathrm{y} \in [\varrho_1, \varrho_2]$ and every $\zeta \in [0, 1]$, we have

$$\Psi(\zeta\ \mathrm{y} + (1-\zeta)\varkappa) \ \leq \ \zeta\ \Psi(\mathrm{y}) + (1-\zeta)\ \Psi(\varkappa).$$

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DEFINITION 1.2. [9] Let $\Psi : [0, \varrho_2) \to \Re$ is called (α, m)-convex, if the identity

$$\Psi(\zeta\varkappa + m(1-\zeta)\mathrm{y}) \leq \zeta^{\alpha}\Psi(\varkappa) + m(1-\zeta^{\alpha})\Psi(\mathrm{y}),$$

for all $\varkappa, \mathrm{y} \in [0, \varrho_2), \zeta \in [0, 1], (\alpha, m) \in (0, 1]^2$.

Simpson inequality is important in a variety of mathematical disciplines. Classical Simpson inequality for four times continuously differentiable functions can be expressed as follows:

THEOREM 1.3. [10] *Let a mapping* $\Psi : [\varrho_1, \varrho_2] \to \Re$ *is four times continuously differentiable on* (ϱ_1, ϱ_2) *and suppose* $||\Psi^{(4)}||_\infty = sup_{\varkappa \in (\varrho_1, \varrho_2)}|\Psi^{(4)}(\varkappa)| < \infty$*. Then, we obtain following identity*

$$\left| \frac{1}{3}\left[\frac{\Psi(\varrho_1) + \Psi(\varrho_2)}{2} + 2\Psi\left(\frac{\varrho_1 + \varrho_2}{2}\right)\right] - \frac{1}{\varrho_2 - \varrho_1}\int_{\varrho_1}^{\varrho_2} \Psi(\varkappa)\, d\varkappa \right|$$
$$\leq \frac{1}{2880}||\Psi^{(4)}||_\infty(\varrho_2 - \varrho_1)^4.$$

Simpson developed crucial techniques for numerical integration and estimation of definite integrals, which he named Simpson's law. Many years ago, related estimation was introduced by J. Kepler. That's why it is called Kepler's law and stated as follows:
(1) Simpson's 1/3 rule:

$$\int_{\varrho_1}^{\varrho_2} \Psi(\varkappa)\, d\varkappa \approx \frac{\varrho_2 - \varrho_1}{6}\left[\Psi(\varrho_1) + 4\Psi\left(\frac{\varrho_1 + \varrho_2}{2}\right) + \Psi(\varrho_2)\right].$$

(2) Simpson's 3/8 rule:

$$\int_{\varrho_1}^{\varrho_2} \Psi(\varkappa)\, d\varkappa \approx \frac{\varrho_2 - \varrho_1}{8}\left[\Psi(\varrho_1) + 3\Psi\left(\frac{2\varrho_1 + \varrho_2}{3}\right) + 3\Psi\left(\frac{\varrho_1 + 2\varrho_2}{3}\right) + \Psi(\varrho_2)\right].$$

When there is no limit in calculus, it is referred as q-calculus. Euler is the creator of q-calculus and the inventor of the q-parameter. Jackson began his work in a symmetrical manner in the nineteenth century and presented q-definite integrals. Q-calculus is used in a wide range of subjects, including mathematics, number theory, hyper geometry and physics. One can see in [11–14] and references therein. In q-calculus, we substitutes conventional derivative with difference operator, allowing you to work with sets of non differentiable functions. Quantum difference operators are of tremendous importance because of their applications in a variety of mathematical disciplines, including basic hypergeometric functions, orthogonal polynomials, relativity, mechanics and combinatorics. Many essential concepts of quantum calculus are covered in Kac and Cheung's [15] book. These ideas help us to develop new inequalities, which can be useful in the discovery of new boundaries.
Tariboon and Ntouyas described the q-derivative and q-integral of ongoing work at intervals and confirmed some of its features in 2013 [16]. Many well-known inequalities such as Hölder, Hermite-Hadamard, trapezoid, Ostrowski, Cauchy-Bunyakovsky-Schwarz, Grüss and Grüss–Čebyšev inequalities have been investigated on the assumption that q-calculus, see [17] for more details. In 2020, Bermudo et al. [18] explained new q-derivative and q-integral for continuous work on a regular basis cost q^{ϱ_2}-calculus, while the definition of J. Tariboon and S. K. Ntouyas is called q_{ϱ_1}-calculus. Moreover, in their paper, they proved the inequality of Hermite-Hadamard for this new definition by utilizing convex and h-convex mappings. In [19], Alp et al. proved some

midpoint type inequalities for q_{ϱ_1}-integrals. Noor et al. established some inequalities of trapezoid type for q_{ϱ_1}-integrals in [20]. On the other hand, Budak present several midpoint and trapezoid type inequalities for q^{ϱ_2}-integrals in [21]. In [22], Budak et al proved some Simpson-Newton type inequalities by using the concept of quantum integrals. There are many papers devoted to quantum Simpson-Newton type inequalities [23–27].
The following are some of the fundamental definitions of quantum calculus.

2. Preliminaries

In this section, we will define certain derivative and integral definitions. Following notion will be used frequently:

$$[n]_q = \frac{1-q^n}{1-q}, \; q \in (0,1).$$

Jackson [12] defined the following q-Jackson integral for a function Ψ from 0 to ϱ_2 :

$$\int_0^{\varrho_2} \Psi(\varkappa)\, d_q\varkappa = (1-q)\,\varrho_2 \sum_{n=0}^{\infty} q^n \Psi(\varrho_2 q^n) \tag{1}$$

and in [12], he also defined another q-integral for a function f over $[\varrho_1, \varrho_2]$ as follows:

$$\int_{\varrho_1}^{\varrho_2} \Psi(\varkappa)\, d_q\varkappa = \int_0^{\varrho_2} \Psi(\varkappa)\, d_q\varkappa - \int_0^{\varrho_1} \Psi(\varkappa)\, d_q\varkappa. \tag{2}$$

Definition 2.1. [16] Let $\Psi : [\varrho_1, \varrho_2] \underset{cont.}{\longrightarrow} \Re$. Then q_{ϱ_1}-derivative of function Ψ at $\varkappa \in [\varrho_1, \varrho_2]$ is defined as:

$$_{\varrho_1}D_q\Psi(\varkappa) = \begin{cases} \dfrac{\Psi(\varkappa) - \Psi(q\varkappa + (1-q)\varrho_1)}{(1-q)(\varkappa - \varrho_1)}, & \text{if } \varkappa \neq \varrho_1 \\ \lim\limits_{\varkappa \to \varrho_1} {}_{\varrho_1}D_q\Psi(\varkappa), & \text{if } \varkappa = \varrho_1. \end{cases} \tag{3}$$

If $\varrho_1 = 0$ and $_0D_q\Psi(\varkappa) = D_q\Psi(\varkappa)$, then (3) reduces to

$$D_q\Psi(\varkappa) = \begin{cases} \dfrac{\Psi(\varkappa) - \Psi(q\varkappa)}{(1-q)\varkappa}, & \text{if} \varkappa \neq 0; \\ \lim\limits_{\varkappa \to 0} D_q\Psi(\varkappa), & \text{if } \varkappa = 0. \end{cases}$$

which is called q-Jackson derivative.

Theorem 2.2. [16] *If $\Psi, \Upsilon : [\varrho_1, \varrho_2] \to \Re$ are q-differentiable functions, then the following identities hold:*

(i) The product $\Psi\,\Upsilon : [\varrho_1, \varrho_2] \to \Re$ is q-differentiable on $[\varrho_1, \varrho_2]$ with

$$\begin{aligned} {}_{\varrho_1}D_q(\Psi\Upsilon)(\varkappa) &= \Psi(\varkappa)\,{}_{\varrho_1}D_q\Upsilon(\varkappa) + \Upsilon(q\varkappa + (1-q)\varrho_1)\,{}_{\varrho_1}D_q(\varkappa) \\ &= \Upsilon(\varkappa)\,{}_{\varrho_1}D_q\Psi(\varkappa) + \Psi(q\varkappa + (1-q)\varrho_1)\,{}_{\varrho_1}D_q\Upsilon(\varkappa) \end{aligned}$$

(ii) If $\Psi(\varkappa)\Upsilon(q\varkappa + (1-q)\varrho_1) \neq 0$, then Ψ/Υ is q-differentiable on $[\varrho_1, \varrho_2]$ with

$$_{\varrho_1}D_q\left(\frac{\Psi}{\Upsilon}\right)(\varkappa) = \frac{\Upsilon(\varkappa)\,{}_{\varrho_1}D_q\Psi(\varkappa) - \Psi(\varkappa)\,{}_{\varrho_1}D_q\Upsilon(\varkappa)}{\Upsilon(\varkappa)\Upsilon(q\varkappa + (1-q)\varrho_1)}$$

Definition 2.3. [16] If $\Psi : [\varrho_1, \varrho_2] \underset{cont.}{\longrightarrow} \Re$ and $z \in [\varrho_1, \varrho_2]$, then following identities hold:

$$\int_{\varrho_1}^{z} \Psi(\varkappa)\ {}_{\varrho_1}d_q\varkappa = (1-q)(z-\varrho_1)\sum_{k=0}^{\infty} q^n \Psi(q^n z + (1-q^n)\varrho_1). \tag{4}$$

If $\varrho_1 = 0$, then (4) reduces to

$$\int_{0}^{z} \Psi(\varkappa)_0 d_q\varkappa = \int_{0}^{z} \Psi(\varkappa) d_q\varkappa = (1-q)z\sum_{n=0}^{\infty} q^n \Psi(q^n z),$$

that is called q-Jackson integral.

Theorem 2.4. [16] *If $\Psi : [\varrho_1, \varrho_2] \underset{cont.}{\longrightarrow} \Re$ and $z \in [\varrho_1, \varrho_2]$, then following identities hold:*

(i) ${}_{\varrho_1}D_q \int_{\varrho_1}^{z} \Psi(\varkappa)_{\varrho_1} d_q\varkappa = \Psi(z)$;

(ii) $\int_{c}^{z} {}_{\varrho_1}D_q\Psi(\varkappa)_{\varrho_1} d_q\varkappa = \Psi(z) - \Psi(c)$ *for* $c \in (\varrho_1, z)$.

Alp et al. used convex functions to prove quantum Hermite-Hadamard inequality for q_{ϱ_1}-integrals, as shown below:

Theorem 2.5. [19] *If a convex mapping $\Psi : [\varrho_1, \varrho_2] \to \Re$ that is differentiable on $[\varrho_1, \varrho_2]$, then following inequality true:*

$$\Psi\left(\frac{q\varrho_1 + \varrho_2}{[2]_q}\right) \le \frac{1}{\varrho_2 - \varrho_1}\int_{\varrho_1}^{\varrho_2} \Psi(\varkappa)\ {}_{\varrho_1}d_q\varkappa \le \frac{q\Psi(\varrho_1) + \Psi(\varrho_2)}{[2]_q}, \tag{5}$$

where $q \in (0, 1)$.

Definition 2.6. [18] The q^{ϱ_2}-derivative of mapping $\Psi : [\varrho_1, \varrho_2] \to \Re$ is defined as:

$${}^{\varrho_2}D_q\Psi(\varkappa) = \frac{\Psi(q\varkappa + (1-q)\varrho_2) - \Psi(\varkappa)}{(1-q)(\varrho_2 - \varkappa)}, \text{ if } \varkappa \ne \varrho_2. \tag{6}$$

$$\text{If } \varkappa = \varrho_2,\ {}^{\varrho_2}D_q\Psi(\varrho_2) = \lim_{\varkappa \to \varrho_2} {}^{\varrho_2}D_q\Psi(\varkappa),$$

which is q-Jackson derivative.

Definition 2.7. [18] The q^{ϱ_2}-integral of mapping $\Psi : [\varrho_1, \varrho_2] \to \Re$ is defined as:

$$\int_{\varrho_1}^{\varrho_2} \Psi(\varkappa)\ {}^{\varrho_2}d_q\varkappa = (1-q)(\varrho_2 - \varrho_1)\sum_{n=0}^{\infty} q^n \Psi(q^n\varrho_1 + (1-q^n)\varrho_2), \tag{7}$$

which is q-Jackson integral.

Theorem 2.8. [18] *If a convex mapping $\Psi : [\varrho_1, \varrho_2] \to \Re$ that is differentiable on $[\varrho_1, \varrho_2]$, then following inequality true:*

$$\Psi\left(\frac{\varrho_1 + q\varrho_2}{[2]_q}\right) \le \frac{1}{\varrho_2 - \varrho_1}\int_{\varrho_1}^{\varrho_2} \Psi(\varkappa)\ {}^{\varrho_2}d_q\varkappa \le \frac{\Psi(\varrho_1) + q\ \Psi(\varrho_2)}{[2]_q}, \tag{8}$$

where $q \in (0, 1)$.

We will now present a Lemma that will enable us in demonstrating the identities:

LEMMA 2.9. [27] *If* $\Psi, \Upsilon : [\varrho_1, \varrho_2] \underset{cont.}{\longrightarrow} \Re$, *then following inequality true:*

$$\begin{aligned}
&\int_0^c \Upsilon(\zeta)\, {}_{\varrho_1}D_q \Psi(\zeta\varrho_2 + (1-\zeta)\varrho_1)\, d_q\zeta \\
&= \left.\frac{\Upsilon(\zeta)\Psi(\zeta\varrho_2 + (1-\zeta)\varrho_1)}{\varrho_2 - \varrho_1}\right|_0^c - \frac{1}{\varrho_2 - \varrho_1}\int_0^c D_q\Upsilon(\zeta)\Psi(q\zeta\varrho_2 + (1-q\zeta)\varrho_1)\, d_q\zeta,
\end{aligned}$$

for $c \in [0, 1]$

3. Identities

New Simpson-Newton type identities for q_{ϱ_1}-integral is presented in this section. Throughout the paper $q \in (0, 1)$.

LEMMA 3.1. *Let* $\Psi : [\varrho_1, \varrho_2] \to \Re$ *be* q_{ϱ_1}*-differentiable function on* (ϱ_1, ϱ_2). *If* ${}_{\varrho_1}D_q\Psi$ *is continuous and integrable on* $[\varrho_1, \varrho_2]$, *we have following identity*

$$\begin{aligned}
&\frac{1}{\varrho_2 - m\varrho_1}\int_{m\varrho_1}^{\varrho_2} \Psi(\zeta)\ {}_{m\varrho_1}D_q\zeta - \frac{1}{[6]_q}\left[q\Psi(m\varrho_1) + q^2[4]_q\Psi\left(\frac{qm\varrho_1 + \varrho_2}{[2]_q}\right) + \Psi(\varrho_2)\right] \\
&= q(\varrho_2 - m\varrho_1)\int_0^1 \wp(\zeta)\ {}_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)\ d_q\zeta,
\end{aligned}$$

where

$$\wp(\zeta) = \begin{cases} \frac{1}{[6]_q}, & \zeta \in \left[0, \frac{1}{[2]_q}\right) \\ \frac{[5]_q}{[6]_q}, & \zeta \in \left[\frac{1}{[2]_q}, 1\right]. \end{cases}$$

Proof. By definition of $\wp$, we have

$$\begin{aligned}
&\int_0^1 \wp(\zeta)\ {}_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)\ d_q\zeta \\
&= \int_0^{\frac{1}{[2]_q}} \left(\zeta - \frac{1}{[6]_q}\right) {}_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)\ d_q\zeta \\
&+ \int_{\frac{1}{[2]_q}}^1 \left(\zeta - \frac{[5]_q}{[6]_q}\right) {}_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)\ d_q\zeta \\
&= \int_0^{\frac{1}{[2]_q}} \left(\zeta - \frac{1}{[6]_q}\right) {}_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1) d_q\zeta \\
&+ \int_0^1 \left(\zeta - \frac{[5]_q}{[6]_q}\right) {}_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)\ d_q\zeta \\
&- \int_0^{\frac{1}{[2]_q}} \left(\zeta - \frac{[5]_q}{[6]_q}\right) {}_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)\ d_q\zeta \\
&= \int_0^{\frac{1}{[2]_q}} \zeta\ {}_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1) d_q\zeta - \frac{1}{[6]_q}\int_0^{\frac{1}{[2]_q}} {}_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)\ d_q\zeta \\
&+ \int_0^1 \left(\zeta - \frac{[5]_q}{[6]_q}\right) {}_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1) d_q\zeta - \int_0^{\frac{1}{[2]_q}} \zeta\ {}_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)\ d_q\zeta
\end{aligned}$$

$$
\begin{aligned}
&+\frac{[5]_q}{[6]_q}\int_0^{\frac{1}{[2]_q}} {}_{\varrho_1}D_q\Psi(\zeta\varrho_2+m(1-\zeta)\varrho_1)\ d_q\zeta\\
&=\frac{[5]_q-1}{[6]_q}\int_0^{\frac{1}{[2]_q}} {}_{\varrho_1}D_q\Psi(\zeta\varrho_2+m(1-\zeta)\varrho_1)\ d_q\zeta\\
&+\int_0^1\left(\zeta-\frac{[5]_q}{[6]_q}\right){}_{\varrho_1}D_q\Psi(\zeta\varrho_2+m(1-\zeta)\varrho_1)\ d_q\zeta\\
&=I_1+I_2.
\end{aligned}
$$

By using Lemma 2.9, we get

$$
\begin{aligned}
I_1&=\left.\frac{\Psi(\zeta b+m(1-\zeta)\varrho_1)}{\varrho_2-m\varrho_1}\right|_0^{\frac{1}{[2]_q}}-\frac{1}{\varrho_2-m\varrho_1}\int_0^{\frac{1}{[2]_q}} D_q(1)\Psi(q\zeta\varrho_2+m(1-q\zeta)\varrho_1)\ d_q\zeta\\
&=\frac{\Psi[(\frac{1}{[2]_q})\varrho_2+m(1-\frac{1}{[2]_q})\varrho_1]}{(\varrho_2-m\varrho_1)}-\frac{\Psi(\varrho_1 m)}{(\varrho_2-m\varrho_1)}\\
&=\left[\Psi\left(\frac{\varrho_2+mq\varrho_1}{[2]_q}\right)-\Psi(m\varrho_1)\right]\frac{[5]_q-1}{[6]_q(\varrho_2-m\varrho_1)}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
I_2&=\int_0^1\left(\zeta-\frac{[5]_q}{[6]_q}\right){}_{\varrho_1}D_q\Psi(\zeta\varrho_2+m(1-\zeta)\varrho_1)\ d_q\zeta\\
&=\left(\zeta-\frac{[5]_q}{[6]_q}\right)\left.\frac{\Psi(\zeta\varrho_2+m(1-\zeta)\varrho_1)}{\varrho_2-m\varrho_1}\right|_0^1\\
&-\frac{1}{\varrho_2-m\varrho_1}\int_0^1 D_q(\zeta-\frac{[5]_q}{[6]_q})\Psi(q\zeta\varrho_2+m(1-q\zeta)\varrho_1)d_q\zeta\\
&=(1-\frac{[5]_q}{[6]_q})\frac{\Psi(\varrho_1)}{\varrho_2-m\varrho_1}+\frac{[5]_q}{[6]_q}.\frac{\Psi(m\varrho_1)}{\varrho_2-m\varrho_1}-\frac{1}{\varrho_2-m\varrho_1}\int_0^1\Psi(q\zeta\varrho_2+m(1-q\zeta)\varrho_1)\ d_q\zeta\\
&=\frac{[6]_q-[5]_q}{[6]_q}\frac{\Psi(\varrho_1)}{\varrho_2-m\varrho_1}+\frac{[5]_q}{[6]_q}.\frac{\Psi(m\varrho_1)}{\varrho_2-m\varrho_1}-\frac{(1-q)\Sigma_{n=0}^{\infty}q^n\Psi(q^{n+1}\varrho_2+m(1-q^{n+1})\varrho_1)}{(\varrho_2-m\varrho_1)}\\
&=\frac{[6]_q-[5]_q}{[6]_q}\frac{\Psi(\varrho_1)}{\varrho_2-m\varrho_1}+\frac{[5]_q}{[6]_q}.\frac{\Psi(m\varrho_1)}{\varrho_2-m\varrho_1}-\frac{(1-q)\Sigma_{n=0}^{\infty}q^{n+1}\Psi(q^{n+1}\varrho_2+m(1-q^{n+1})\varrho_1)}{q(\varrho_2-m\varrho_1)}\\
&=\frac{[6]_q-[5]_q}{[6]_q}\frac{\Psi(\varrho_1)}{\varrho_1-m\varrho_2}+\frac{[5]_q}{[6]_q}.\frac{\Psi(m\varrho_1)}{\varrho_2-m\varrho_1}-\frac{(1-q)\sum_{k=1}^{\infty}q^k\Psi(q^k\varrho_1+m(1-q^k)\varrho_2)}{q(\varrho_2-m\varrho_1)}\\
&-\frac{(1-q)\Psi(\varrho_1)}{q(\varrho_2-m\varrho_1)}+\frac{(1-q)\Psi(\varrho_1)}{q(\varrho_2-m\varrho_1)}\\
&=\frac{[6]_q-[5]_q}{[6]_q}\frac{\Psi(\varrho_1)}{\varrho_2-m\varrho_1}+\frac{[5]_q}{[6]_q}.\frac{\Psi(m\varrho_1)}{\varrho_1-m\varrho_2}-\frac{(1-q)\Sigma_{n=0}^{\infty}q^n\Psi(q^n\varrho_1+m(1-q^n)\varrho_2)}{q(\varrho_2-m\varrho_1)}\\
&+\frac{(1-q)\Psi(\varrho_1)}{q(\varrho_2-m\varrho_1)}\\
&=\frac{[6]_q-[5]_q}{[6]_q}\frac{\Psi(\varrho_1)}{\varrho_2-m\varrho_1}+\frac{[5]_q}{[6]_q}.\frac{\Psi(m\varrho_1)}{\varrho_1-m\varrho_2}+\frac{1}{q(\varrho_2-m\varrho_1)^2}\int_{\varrho_1}^{m\varrho_2}\Psi(\varkappa)\ d_q\zeta
\end{aligned}
$$

$$-\frac{(1-q)\Psi(\varrho_1)}{q(\varrho_2 - m\varrho_1)}.$$

Then it follows that

$$\begin{aligned}
I_1 + I_2 &= \frac{[5]_q - 1}{[6]_q(\varrho_2 - m\varrho_1)}\Psi\left(\frac{\varrho_2 + mq\varrho_1}{[2]_q}\right) - \frac{[5]_q - 1}{[6]_q}\frac{\Psi(m\varrho_1)}{(\varrho_2 - m\varrho_1)} \\
&+ \frac{[6]_q - [5]_q}{[6]_q}\frac{\Psi(\varrho_2)}{\varrho_2 - m\varrho_1} + \frac{[5]_q}{[6]_q}.\frac{\Psi(m\varrho_1)}{\varrho_2 - m\varrho_1} - \frac{1}{q(\varrho_2 - m\varrho_1)^2}\int_{m\varrho_1}^{\varrho_2}\Psi(\varkappa)\, d_q\zeta \\
&+ \frac{(1-q)\Psi(\varrho_2)}{q(\varrho_2 - m\varrho_1)} \\
&= \frac{(1+q+q^2+q^3+q^4) - 1}{[6]_q(\varrho_2 - m\varrho_1)}\ \Psi\left(\frac{\varrho_2 + mq\varrho_1}{[2]_q}\right) \\
&+ \frac{1 - [5]_q + [5]_q}{[6]_q(\varrho_2 - m\varrho_1)}\ \Psi(\varrho_1 m) + \frac{q[6]_q - q[5]_q + (1-q)[6]_q}{q[6]_q(\varrho_2 - m\varrho_1)}\ \Psi(\varrho_2) \\
&- \frac{1}{q(\varrho_2 - m\varrho_1)^2}\int_{m\varrho_1}^{\varrho_2}\Psi(\varkappa)\, d_q\zeta \\
&= \frac{q[4]_q}{[6]_q(\varrho_2 - m\varrho_1)}\ \Psi\left(\frac{\varrho_2 + mq\varrho_1}{[2]_q}\right) \\
&+ \frac{1}{[6]_q(\varrho_2 - m\varrho_1)}\ \Psi(m\varrho_1) + \frac{1}{[6]_q q(\varrho_2 - m\varrho_1)}\ \Psi(\varrho_2) \\
&- \frac{1}{q(\varrho_2 - m\varrho_1)^2}\int_{m\varrho_1}^{\varrho_2}\Psi(\varkappa)\, d_q\zeta.
\end{aligned}$$

By multiplying $-q(\varrho_2 - m\varrho_1)$, we get the required results. □

REMARK 3.2. Considering Lemma 3.1, we get:

(i) We recaptures [[28], Lemma 2.1] when we set $q \to 1^-$.

(ii) We recaptures [[29], Lemma 1] when we have $m = 1$ and limit as $q \to 1^-$.

LEMMA 3.3. *Under the assumptions of Lemma 3.1 we have the following identity*

(9)

$$\begin{aligned}
&\frac{1}{\varrho_2 - m\varrho_1}\int_{m\varrho_1}^{\varrho_2}\Psi(\zeta)\ {}_{m\varrho_1}d_q\zeta \\
&- \frac{1}{[8]_q}\left[q\Psi(m\varrho_1) + \frac{q^3[6]_q}{[2]_q}\Psi\left(\frac{\varrho_2 + mq[2]_q\ \varrho_1}{[2]_q}\right) + \frac{q^2[6]_q}{[2]_q}\Psi\left(\frac{[2]_q\varrho_2 + mq^2\varrho_1}{[3]_q}\right) + \Psi(\varrho_2)\right] \\
&= q(\varrho_2 - m\varrho_1)\int_0^1 \wp(\zeta)\ {}_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)\ d_q\zeta
\end{aligned}$$

$$\wp(\zeta) = \begin{cases} \zeta - \frac{1}{[8]_q}, & \zeta \in \left[0, \frac{1}{[3]_q}\right) \\ \zeta - \frac{1}{[2]_q}, & \zeta \in \left[\frac{1}{[3]_q}, \frac{[2]_q}{[3]_q}\right) \\ \zeta - \frac{[7]_q}{[8]_q}, & \zeta \in \left[\frac{[2]_q}{[3]_q}, 1\right]. \end{cases}$$

Proof. If we use the same steps of Lemma 3.1, required result can be obtained. □

Before the main results, we should give the following results:

$$C_{1q} = \frac{2[2]_q^{2+\alpha}([2+\alpha]_q - [1+\alpha]_q) + [6]_q^{1+\alpha}([6]_q[1+\alpha]_q - [2+\alpha]_q[2]_q)}{[2]_q^{2+\alpha}[6]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q}, \tag{10}$$

$$\begin{aligned} D_{1q} = {} & 2\frac{q[1+\alpha]_q[2+\alpha]_q[6]_q^{\alpha} - [2]_q([2+\alpha]_q - [1+\alpha]_q)}{[2]_q[6]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q} \\ & + \frac{1}{[2]_q^3}\left(\frac{([6]_q - [2]_q^2)[2]_q^{3+\alpha}[1+\alpha]_q[2+\alpha]_q - [2]_q^4[6]_q[1+\alpha]_q + [2]_q^5[2+\alpha]_q}{[6]_q[2]_q^{3+\alpha}[1+\alpha]_q[2+\alpha]_q}\right), \end{aligned} \tag{11}$$

$$\begin{aligned} C_{2q} = {} & \frac{2[5]_q^{2+\alpha}([2+\alpha]_q - [1+\alpha]_q)}{[6]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q} \\ & + \frac{[6]_q[1+\alpha]_q(1+[2]_q^{2+\alpha}) - [5]_q[2]_q[2+\alpha]_q(1+[2]_q^{1+\alpha})}{[6]_q[2]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q}, \end{aligned} \tag{12}$$

$$\begin{aligned} D_{2q} = {} & 2\frac{q[5]_q^2[6]_q^{\alpha}[1+\alpha]_q[2+\alpha]_q - [5]_q^{2+\alpha}([2+\alpha]_q - [1+\alpha]_q)}{[2]_q[6]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q} \\ & + \frac{[1+\alpha]_q[2+\alpha]_q([6]_q - [5]_q[2]_q) - [2]_q([6]_q[1+\alpha]_q - [5]_q[2+\alpha]_q)}{[6]_q[2]_q[1+\alpha]_q[2+\alpha]_q} \\ & - \frac{1}{[2]_q^3}\left[\frac{[2]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q([5]_q[2]_q^2 - [6]_q)}{[6]_q[2]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q}\right] \\ & + \frac{1}{[2]_q^3}\left[\frac{[2]_q^3([5]_q[2]_q[2+\alpha]_q - [6]_q[1+\alpha]_q)}{[6]_q[2]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q}\right], \end{aligned} \tag{13}$$

$$C_{3q} = \frac{2[3]_q^{2+\alpha}([2+\alpha]_q - [1+\alpha]_q) + [8]_q^{1+\alpha}([8]_q[1+\alpha]_q - [3]_q[2+\alpha]_q)}{[8]_q^{2+\alpha}[3]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q}, \tag{14}$$

$$\begin{aligned} D_{3q} = {} & 2\frac{q[1+\alpha]_q[2+\alpha]_q[8]_q^{\alpha} - [2]_q([2+\alpha]_q - [1+\alpha]_q)}{[2]_q[8]_q^{2+\alpha}[2+\alpha]_q[1+\alpha]_q} \\ & + \frac{[3]_q^{\alpha}[1+\alpha]_q[2+\alpha]_q([8]_q - [2]_q[3]_q) - [8]_q[2]_q[1+\alpha]_q + [2]_q[3]_q[2+\alpha]_q}{[8]_q[2]_q[3]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q}, \end{aligned} \tag{15}$$

$$C_{4q} = \frac{2([2+\alpha]_q - [1+\alpha]_q)}{[2]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q} + \frac{[2]_q[1+\alpha]_q(1+[2]_q^{2+\alpha}) - [3]_q[2+\alpha]_q(1+[2]_q^{1+\alpha})}{[3]_q^{2+\alpha}[2]_q[1+\alpha]_q[2+\alpha]_q}, \tag{16}$$

$$D_{4q} = \frac{2q}{[2]_q^3} - \frac{q}{[3]_q^2} - \frac{q^2}{[3]_q^2} - C_{4q}, \tag{17}$$

$$\begin{aligned} C_{5q} = {} & 2\frac{[7]_q^{2+\alpha}([2+\alpha]_q - [1+\alpha]_q)}{[8]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q} \\ & + \frac{[8]_q[1+\alpha]_q([3]_q^{2+\alpha} + [2]_q^{2+\alpha}) - [7]_q[3]_q[2+\alpha]_q([2]_q^{1+\alpha} + [3]_q^{1+\alpha})}{[8]_q[3]_q^{2+\alpha}[2+\alpha]_q[1+\alpha]_q}, \end{aligned} \tag{18}$$

and

$$D_{5q} = \frac{2q[7]_q^2[8]_q^{\alpha}[1+\alpha]_q[2+\alpha]_q - [7]_q^{2+\alpha}[2]_q([2+\alpha]_q - [1+\alpha]_q)}{[8]_q^{2+\alpha}[2]_q[1+\alpha]_q[2+\alpha]_q} - \frac{[7]_q[2]_q[3]_q[2+\alpha]_q([2]_q^{\alpha} - [3]_q^{\alpha}[1+\alpha]_q)}{[8]_q[3]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q} + \frac{[8]_q[2]_q[1+\alpha]_q([3]_q^{\alpha}[2+\alpha]_q - [2]_q^{1+\alpha})}{[8]_q[3]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q} + \frac{[8]_q[1+\alpha]_q([2+\alpha]_q - [2]_q)}{[2]_q[8]_q[1+\alpha]_q[2+\alpha]_q} - \frac{[7]_q[2]_q[2+\alpha]_q([1+\alpha]_q - 1)}{[2]_q[8]_q[1+\alpha]_q[2+\alpha]_q}. \tag{19}$$

4. New Quantum Simpson's 1/3 Type Inequalities

THEOREM 4.1. *Under the assumptions of Lemma 3.1 if* $|\,{}_{\theta_1}D_q\Psi\,|$ *is* (α,m)*-convex mapping over* $[\theta_1,\varrho_2]$ *then we have the following identity*

$$\left|\frac{1}{(\varrho_2 - m\varrho_1)}\int_{m\varrho_1}^{\varrho_2}\Psi(\zeta)\ {}_{m\varrho_1}d_q\zeta - \frac{1}{[6]_q}\left[q\ \Psi(m\varrho_1) + q^2[4]_q\Phi\left(\frac{\varrho_2 + mq\varrho_1}{[2]_q}\right) + \Psi(\varrho_2)\right]\right| \tag{20}$$
$$\leq q(\varrho_2 - m\varrho_1)\{|{}_{\varrho_1}D_q\Psi(\varrho_2)|[C_{1q} + C_{2q}] + m|{}_{\varrho_1}D_q\Psi(\varrho_1)|[D_{1q} + D_{2q}]\},$$

where $C_{1q}, C_{2q}, D_{1q}, D_{2q}$ *are given in (10)-(14), respectively.*

Proof. Taking modulus on Lemma 3.1 along (α,m)-convexity of $|{}_{\varrho_1}D_q\Psi|$, we attain

$$\left|\frac{1}{(\varrho_2 - m\varrho_1)}\int_{m\varrho_1}^{\varrho_2}\Psi(\zeta)\ {}_{m\varrho_1}d_q\zeta - \frac{1}{[6]_q}\left[q\Psi(m\varrho_1) + q^2[4]_q\Psi\left(\frac{\varrho_2 + mq\varrho_1}{[2]_q}\right) + \Psi(\varrho_2)\right]\right| \tag{21}$$
$$\leq q(\varrho_2 - m\varrho_1)\int_0^{\frac{1}{[2]_q}}\left|\zeta - \frac{1}{[6]_q}\right||{}_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)|\ d_q\zeta$$
$$+ q(\varrho_2 - m\varrho_1)\int_{\frac{1}{[2]_q}}^1\left|\zeta - \frac{[5]_q}{[6]_q}\right||{}_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)|\ d_q\zeta$$
$$\leq |{}_{\varrho_1}D_q\Psi(\varrho_2)|\int_0^{\frac{1}{[2]_q}}\left|\zeta - \frac{1}{[6]_q}\right|\zeta^{\alpha}\ d_q\zeta + m|{}_{\varrho_1}D_q\Psi(\varrho_1)|\int_0^{\frac{1}{[2]_q}}(1-\zeta^{\alpha})\left|\zeta - \frac{1}{[6]_q}\right|\ d_q\zeta$$
$$+ |{}_{\varrho_1}D_q\Psi(\varrho_2)|\int_{\frac{1}{[2]_q}}^1\left|\zeta - \frac{[5]_q}{[6]_q}\right|\zeta^{\alpha}\ d_q\zeta + m|{}_{\varrho_1}D_q\Psi(\varrho_1)|\int_{\frac{1}{[2]_q}}^1(1-\zeta^{\alpha})\left|\zeta - \frac{1}{[6]_q}\right|\ d_q\zeta.$$

Here, we have

$$\int_0^{\frac{1}{[2]_q}}\zeta^{\alpha}\left|\zeta - \frac{1}{[6]_q}\right|\ d_q\zeta = \int_0^{\frac{1}{[6]_q}}\zeta^{\alpha}\left(\frac{1}{[6]_q} - \zeta\right)\ d_q\zeta + \int_{\frac{1}{[6]_q}}^{\frac{1}{[2]_q}}\zeta^{\alpha}\left(\zeta - \frac{1}{[6]_q}\right)\ d_q\zeta$$
$$=2\int_0^{\frac{1}{[6]_q}}\zeta^{\alpha}\left(\frac{1}{[6]_q} - \zeta\right)\ d_q\zeta + \int_0^{\frac{1}{[2]_q}}\zeta^{\alpha}\left(\zeta - \frac{1}{[6]_q}\right)\ d_q\zeta$$

$$
\begin{aligned}
&=2\int_0^{\frac{1}{[6]_q}}\left(\frac{\zeta^\alpha}{[6]_q}-\zeta^{1+\alpha}\right)d_q\zeta+\int_0^{\frac{1}{[2]_q}}\left(\zeta^{1+\alpha}-\frac{\zeta^\alpha}{[6]_q}\right)d_q\zeta\\
&=2\left[\frac{1}{[6]_q^{2+\alpha}[1+\alpha]_q}-\frac{1}{[6]_q^{2+\alpha}[2+\alpha]_q}\right]+\left[\frac{1}{[2]_q^{2+\alpha}[2+\alpha]_q}-\frac{1}{[2]_q^{1+\alpha}[1+\alpha]_q}\right]\\
&=\frac{2[2]_q^{2+\alpha}([2+\alpha]_q-[1+\alpha]_q)+[6]_q^{1+\alpha}([6]_q[1+\alpha]_q-[2+\alpha]_q[2]_q)}{[2]_q^{2+\alpha}[6]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q},
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^{\frac{1}{[2]_q}}(1-\zeta^\alpha)\left|\zeta-\frac{1}{[6]_q}\right|d_q\zeta\\
&=\int_0^{\frac{1}{[6]_q}}(1-\zeta^\alpha)\left(\frac{1}{[6]_q}-\zeta\right)d_q\zeta+\int_{\frac{1}{[6]_q}}^{\frac{1}{[2]_q}}(1-\zeta^\alpha)\left(\zeta-\frac{1}{[6]_q}\right)d_q\zeta\\
&=2\int_0^{\frac{1}{[6]_q}}(1-\zeta^\alpha)\left(\frac{1}{[6]_q}-\zeta\right)d_q\zeta+\int_0^{\frac{1}{[2]_q}}(1-\zeta^\alpha)\left(\zeta-\frac{1}{[6]_q}\right)d_q\zeta\\
&=2\left[\frac{q}{[2]_q{[6]_q}^2}-\frac{1}{[6]_q^{2+\alpha}[1+\alpha]_q}+\frac{1}{[6]_q^{2+\alpha}[2+\alpha]_q}\right]\\
&+\left[\frac{[6]_q-[2]_q^2}{{[2]_q}^3[6]_q}-\frac{1}{[2]_q^{2+\alpha}[2+\alpha]_q}+\frac{1}{[2]_q^{1+\alpha}[6]_q[1+\alpha]_q}\right]\\
&=\frac{2q[1+\alpha]_q[2+\alpha]_q[6]_q^\alpha-[2]_q([2+\alpha]_q-[1+\alpha]_q)}{[2]_q[6]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q}\\
&+\frac{1}{[2]_q^3}\left(\frac{([6]_q-[2]_q^2)[2]_q^{3+\alpha}[1+\alpha]_q[2+\alpha]_q-[2]_q^4[6]_q[1+\alpha]_q+_q^5[2+\alpha]_q}{[6]_q[2]_q^{3+\alpha}[1+\alpha]_q[2+\alpha]_q}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\frac{1}{[2]_q}}^1\zeta^\alpha\left|\zeta-\frac{[5]_q}{[6]_q}\right|d_q\zeta\\
&=\int_{\frac{1}{[2]_q}}^{\frac{[5]_q}{[6]_q}}\zeta^\alpha\left(\frac{[5]_q}{[6]_q}-\zeta\right)d_q\zeta+\int_{\frac{[5]_q}{[6]_q}}^1\zeta^\alpha\left(\zeta-\frac{[5]_q}{[6]_q}\right)d_q\zeta\\
&=2\int_0^{\frac{[5]_q}{[6]_q}}\zeta^\alpha\left(\frac{[5]_q}{[6]_q}-\zeta\right)d_q\zeta-\int_0^{\frac{1}{[2]_q}}\zeta^\alpha\left(\frac{[5]_q}{[6]_q}-\zeta\right)d_q\zeta+\int_0^1\zeta^\alpha\left(\zeta-\frac{[5]_q}{[6]_q}\right)d_q\zeta\\
&=2\left[\frac{[5]_q^{2+\alpha}}{[6]_q^{2+\alpha}[1+\alpha]_q}-\frac{[5]_q^{2+\alpha}}{[6]_q^{2+\alpha}[2+\alpha]_q}\right]\\
&-\left[\frac{[5]_q}{[6]_q[2]_q^{1+\alpha}[1+\alpha]_q}-\frac{1}{[2]_q^{2+\alpha}[2+\alpha]_q}\right]\\
&+\left[\frac{1}{[2+\alpha]_q}-\frac{[5]_q}{[6]_q[1+\alpha]_q}\right]\\
&=\frac{2[5]_q^{2+\alpha}([2+\alpha]_q-[1+\alpha]_q)}{[6]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q}
\end{aligned}
$$

$$+\frac{[6]_q[1+\alpha]_q(1+[2]_q^{2+\alpha})-[5]_q[2]_q[2+\alpha]_q(1+[2]_q^{1+\alpha})}{[6]_q[2]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q},$$

and

$$\int_{\frac{1}{[2]_q}}^{1}(1-\zeta^\alpha)\left|\zeta-\frac{[5]_q}{[6]_q}\right| d_q\zeta$$
$$=\int_{\frac{1}{[2]_q}}^{\frac{[5]_q}{[6]_q}}(1-\zeta^\alpha)\left(\frac{[5]_q}{[6]_q}-\zeta\right) d_q\zeta+\int_{\frac{[5]_q}{[6]_q}}^{1}(1-\zeta^\alpha)\left(\zeta-\frac{[5]_q}{[6]_q}\right) d_q\zeta$$
$$=2\int_{0}^{\frac{[5]_q}{[6]_q}}(1-\zeta^\alpha)\left(\frac{[5]_q}{[6]_q}-\zeta\right) d_q\zeta$$
$$-\int_{0}^{\frac{1}{[2]_q}}(1-\zeta^\alpha)\left(\frac{[5]_q}{[6]_q}-\zeta\right) d_q\zeta+\int_{0}^{1}(1-\zeta^\alpha)\left(\zeta-\frac{[5]_q}{[6]_q}\right) d_q\zeta$$
$$=2\left[\frac{[5]_q^2q}{[6]_q^2[2]_q}-\frac{[5]_q^{2+\alpha}}{[6]_q^{2+\alpha}[1+\alpha]_q}+\frac{[5]_q^{2+\alpha}}{[6]_q^{2+\alpha}[2+\alpha]_q}\right]$$
$$-\left[\frac{[5]_q[2]_q^2-[6]_q}{[2]_q^3[6]_q}-\frac{[5]_q}{[6]_q[2]_q^{1+\alpha}[1+\alpha]_q}+\frac{1}{[2]_q^{2+\alpha}[2+\alpha]_q}\right]$$
$$+\left[\frac{[6]_q-[5]_q[2]_q}{[6]_q[2]_q}-\frac{1}{[2+\alpha]_q}+\frac{[5]_q^{2+\alpha}}{[6]_q^{2+\alpha}[2+\alpha]_q}\right]$$
$$=\frac{2q[5]_q^2[6]_q^\alpha[1+\alpha]_q[2+\alpha]_q-[5]_q^{2+\alpha}([2+\alpha]_q-[1+\alpha]_q)}{[2]_q[6]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q}$$
$$+\frac{[1+\alpha]_q[2+\alpha]_q([6]_q-[5]_q[2]_q)-[2]_q([6]_q[1+\alpha]_q-[5]_q[2+\alpha]_q)}{[6]_q[2]_q[1+\alpha]_q[2+\alpha]_q}$$
$$-\frac{1}{[2]_q^3}\left[\frac{[2]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q([5]_q[2]_q^2-[6]_q)-[2]_q^3([5]_q[2]_q[2+\alpha]_q-[6]_q[1+\alpha]_q)}{[6]_q[2]_q^{2+\alpha}[1+\alpha]_q[2+\alpha]_q}\right]$$

By putting the values of integrals, we attain the desired results. □

REMARK 4.2. In Theorem 4.1, we get

(i) By taking $q\to 1^-$, then we regain [[28], Theorem 2.2].

(ii) By taking $m=1$, $\alpha=1$ and applying limit as $q\to 1$, we regain [[29], Corollary 1].

EXAMPLE 4.3. Let $\Psi:[0,1]\to\mathbb{R}$ defined by $\Psi(\varkappa)=\varkappa^2$ and let $m=\frac{1}{2}$ and $\alpha=1$. Under these assumptions, we have

$$\int_{m\varrho_1}^{\varrho_2}\Psi(\zeta)_{m\varrho_1}d_q\zeta=\int_0^1\zeta^2\ {}_0d_q\zeta=\frac{1}{[3]_q}$$

and

$$|_{\varrho_1}D_q\Psi(\varkappa)|=[2]_q\,\varkappa.$$

Then the left hand side of the inequality (20) reduces to

$$\left|\frac{1}{(\varrho_2-m\varrho_1)}\int_{m\varrho_1}^{\varrho_2}\Psi(\zeta)_{m\varrho_1}d_q\zeta-\frac{1}{[6]_q}\left[q\ \Psi(m\varrho_1)+q^2[4]_q\Psi\left(\frac{\varrho_2+mq\varrho_1}{[2]_q}\right)+\Psi(\varrho_2)\right]\right|$$

$$\left| \frac{1}{[3]_q} - \frac{1}{[6]_q} \left[\frac{q^2[4]_q}{[2]_q^2} + 1 \right] \right|.$$

Therefore, the right hand side of the inequality (20) reduces to

$$\begin{aligned} & q(\varrho_2 - m\varrho_1)\{|_{\varrho_1}D_q\Psi(\varrho_2)|[C_{1q} + C_{2q}] + m|_{\varrho_1}D_q\Psi(\varrho_1)|[D_{1q} + D_{2q}]\}, \\ = \; & q\,[2]_q\,[C_{1q} + C_{2q}] \end{aligned}$$

By the inequality (20), we have the inequality

$$\left| \frac{1}{[3]_q} - \frac{1}{[6]_q} \left[\frac{q^2[4]_q}{[2]_q^2} + 1 \right] \right| \leq q\,[2]_q\,[C_{1q} + C_{2q}]. \tag{22}$$

One can see the validity of the inequality (22) in Figure 1.

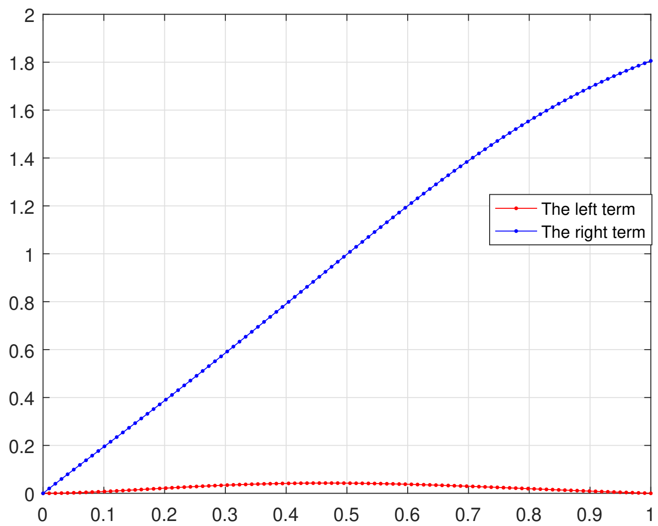

FIGURE 1. An example to Theorem 4.1

THEOREM 4.4. *Under the assumptions of Lemma 3.1 if* $|_{\theta_1}D_q\Psi|^{\wp_1}$, $\wp_1 > 1$, *is* (α, m)*-convex mapping on* $[\varrho_1, \varrho_2]$*, the following inequality*

$$\left| \frac{1}{\varrho_2 - m\varrho_1} \int_{m\varrho_1}^{\varrho_2} \Psi(\zeta)\ {}_{\varrho_1 m}d_q\zeta - \frac{1}{[6]_q} \left[q\Psi(m\varrho_1) + q^2[4]_q\Psi\left(\frac{\varrho_2 + mq\varrho_1}{[2]_q}\right) + \Psi(\varrho_2) \right] \right|$$

$$\leq q(\varrho_2 - m\varrho_1)\left(\frac{q^{\wp_2}[4]_q^{\wp_2}}{[2]_q^{\wp_2+1}[6]_q^{\wp_2}}\right)^{\frac{1}{\wp_2}} \left(\frac{1}{[2]_q^{1+\alpha}[2]_q}|_{\theta_1}D_q\Psi(\varrho_2)|^{\wp_1} + \frac{[2]_q^{\alpha}[1+\alpha]_q - 1}{[2]_q^{1+\alpha}[1+\alpha]_q} m|_{\theta_1}D_q\Psi(\varrho_1)|^{\wp_1}\right)^{\frac{1}{\wp_1}}$$

$$+q(\varrho_2 - m\varrho_1)\left(\frac{[2]_q^{\wp_2+1}q^{5\wp_2} - q^{\wp_2}[4]_q^{\wp_2}}{[2]_q^{\wp_2+1}[6]_q^{\wp_2}}\right)^{\frac{1}{\wp_2}}$$

$$\left(\frac{[2]_q^{1+\alpha} - 1}{[2]_q^{1+\alpha}[1+\alpha]_q}|_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1} + \frac{q[2]_q^{\alpha}[1+\alpha]_q - [2]_q^{1+\alpha} + 1}{[2]_q^{\alpha+1}[1+\alpha]_q} m|_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1}\right)^{\frac{1}{\wp_1}},$$

where $\frac{1}{\wp_1} + \frac{1}{\wp_2} = 1$.

Proof. Utilizing Hölder's inequality on R.H.S. of Lemma 3.1, then we have

$$\left|\frac{1}{(\varrho_2 - m\varrho_1)}\int_{m\varrho_1}^{\varrho_2}\Psi(\zeta)\ {}_{\varrho_1 m}d_q\zeta - \frac{1}{[6]_q}\left[q\Psi(m\varrho_1) + q^2[4]_q\Psi\left(\frac{\varrho_2 + q\varrho_1 m}{[2]_q}\right) + \Psi(\varrho_2)\right]\right|$$

$$\leq q(\varrho_2 - m\varrho_1)\left(\int_0^{\frac{1}{[2]_q}}\left|\zeta - \frac{1}{[6]_q}\right|^{\wp_2} d_q\zeta\right)^{\frac{1}{\wp_2}}\left(\int_0^{\frac{1}{[2]_q}}\left|{}_{\theta_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)\right|^{\wp_1} d_q\zeta\right)^{\frac{1}{\wp_1}}$$

$$+q(\varrho_2 - m\varrho_1)\left(\int_{\frac{1}{[2]_q}}^1\left|\zeta - \frac{[5]_q}{[6]_q}\right|^{\wp_2} d_q\zeta\right)^{\frac{1}{\wp_2}}\left(\int_{\frac{1}{[2]_q}}^1\left|{}_{\theta_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)\right|^{\wp_1} d_q\zeta\right)^{\frac{1}{\wp_1}}.$$

By using (α, m)-convexity $|{}_{\varrho_1}D_q\Psi|$, we get

$$\left|\frac{1}{(\varrho_2 - m\varrho_1)}\int_{m\varrho_1}^{\varrho_2}\Psi(\zeta)\ {}_{m\varrho_1}d_q\zeta - \frac{1}{[6]_q}\left[q\Psi(m\varrho_1) + q^2[4]_q\Psi\left(\frac{\varrho_2 + q\varrho_1 m}{[2]_q}\right) + \Psi(\varrho_2)\right]\right|$$

$$\leq q(\varrho_2 - m\varrho_1)\left(\int_0^{\frac{1}{[2]_q}}\left|\zeta - \frac{1}{[6]_q}\right|^{\wp_2} d_q\zeta\right)^{\frac{1}{\wp_2}}$$

$$\times\left(|{}_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1}\int_0^{\frac{1}{[2]_q}}\zeta^\alpha d_q\zeta + m|{}_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1}\int_0^{\frac{1}{[2]_q}}(1-\zeta^\alpha)\ d_q\zeta\right)^{\frac{1}{\wp_1}}$$

$$+q(\varrho_2 - m\varrho_1)\left(\int_{\frac{1}{[2]_q}}^1\left|\zeta - \frac{[5]_q}{[6]_q}\right|^{\wp_2} d_q\zeta\right)^{\frac{1}{\wp_2}}$$

$$\times\left(|{}_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1}\int_{\frac{1}{[2]_q}}^1\zeta^\alpha d_q\zeta + m|{}_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1}\int_{\frac{1}{[2]_q}}^1(1-\zeta^\alpha)d_q\zeta\right)^{\frac{1}{\wp_1}}$$

By calculating the integrals, we have

$$\int_0^{\frac{1}{[2]_q}}\left|\zeta - \frac{1}{[6]_q}\right|^{\wp_2} d_q\zeta = \frac{(1-q)}{[2]_q}\sum_{n=0}^{\infty}q^n\left|\frac{q^n}{[2]_q} - \frac{1}{[6]_q}\right|^{\wp_2}$$

$$\leq \frac{(1-q)}{[2]_q}\sum_{n=0}^{\infty}q^n\left|\frac{1}{[2]_q} - \frac{1}{[6]_q}\right|^{\wp_2}$$

$$= \left[\frac{1}{[2]_q} - \frac{1}{[6]_q}\right]^{\wp_2}\frac{1}{[2]_q}$$

$$= \frac{q^{2\wp_2}[4]_q^{\wp_2}}{[2]_q^{\wp_2+1}[6]_q^{\wp_2}}.$$

Similarly,

$$\int_{\frac{1}{[2]_q}}^1\left|\zeta - \frac{[5]_q}{[6]_q}\right|^{\wp_2} d_q\zeta = \frac{[2]_q^{\wp_2+1}q^{5\wp_2} - q^{\wp_2}[4]_q^{\wp_2}}{[2]_q^{\wp_2+1}[6]_q^{\wp_2}}.$$

On the other hand, we have

$$\int_0^{\frac{1}{[2]_q}}\zeta^\alpha d_q\zeta = \frac{1}{[2]_q^{1+\alpha}[2]_q},$$

$$\int_0^{\frac{1}{[2]_q}}(1-\zeta^\alpha)d_q\zeta=\frac{[2]_q^\alpha[1+\alpha]_q-1}{[2]_q^{1+\alpha}[1+\alpha]_q},$$

$$\int_{\frac{1}{[2]_q}}^1\zeta^\alpha d_q\zeta=\frac{[2]_q^{1+\alpha}-1}{[2]_q^{1+\alpha}[1+\alpha]_q},$$

and

$$\int_{\frac{1}{[2]_q}}^1(1-\zeta^\alpha)d_q\zeta=\frac{q[2]_q^\alpha[1+\alpha]_q-[2]_q^{1+\alpha}+1}{[2]_q^{\alpha+1}[1+\alpha]_q}.$$

By substituting these integrals then, we get

$$\left|\frac{1}{\varrho_2-m\varrho_1}\int_{m\varrho_1}^{\varrho_2}\Psi(\zeta)\ {}_{\varrho_1 m}d_q\zeta-\frac{1}{[6]_q}\left[q\Psi(m\varrho_1)+q^2[4]_q\Psi\left(\frac{\varrho_2+mq\varrho_1}{[2]_q}\right)+\Psi(\varrho_2)\right]\right|$$

$$\le q(\varrho_2-m\varrho_1)\left(\frac{q^{2\wp_2}[4]_q^{\wp_2}}{[2]_q^{\wp_2+1}[6]_q^{\wp_2}}\right)^{\frac{1}{\wp_2}}$$

$$\times\left(\frac{1}{[2]_q^{1+\alpha}[2]_q}|_{\theta_1}D_q\Psi(\varrho_2)|^{\wp_1}+\frac{[2]_q^\alpha[1+\alpha]_q-1}{[2]_q^{1+\alpha}[1+\alpha]_q}m|_{\theta_1}D_q\Psi(\varrho_1)|^{\wp_1}\right)^{\frac{1}{\wp_1}}$$

$$+q(\varrho_2-m\varrho_1)\left(\frac{[2]_q^{\wp_2+1}[5]_q^{\wp_2}-q^{\wp_2}[4]_q^{\wp_2}}{[2]_q^{\wp_2+1}[6]_q^{\wp_2}}\right)^{\frac{1}{\wp_2}}$$

$$\left(\frac{[2]_q^{1+\alpha}-1}{[2]_q^{1+\alpha}[1+\alpha]_q}|_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1}+\frac{q[2]_q^\alpha[1+\alpha]_q-[2]_q^{1+\alpha}+1}{[2]_q^{\alpha+1}[1+\alpha]_q}m|_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1}\right)^{\frac{1}{\wp_1}}.$$

This completes the proof. □

THEOREM 4.5. *Under the assumptions of Lemma 3.1 if* $|_{\varrho_1}D_q\Psi|^{\wp_1}$, $\wp_1\ge 1$ *is* (α,m)*-convex on* $[\varrho_1,\varrho_2]$, *then have following inequality*

$$\left|\frac{1}{(\varrho_2-m\varrho_1)}\int_{m\varrho_1}^{\varrho_2}\Psi(\zeta)\ {}_{m\varrho_1}d_q\zeta-\frac{1}{[6]_q}\left[q\ \Psi(m\varrho_1)+q^2[4]_q\Psi\left(\frac{\varrho_2+mq\varrho_1}{[2]_q}\right)+\Psi(\varrho_2)\right]\right|$$

$$\le q(\varrho_2-m\varrho_1)\left(\frac{2q}{[2]_q[6]_q^2}+\frac{q^3[3]_q-q}{[6]_q[2]_q^3}\right)^{1-\frac{1}{\wp_1}}(C_{1q}|_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1}+mD_{1q}|_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1})^{\frac{1}{\wp_1}}$$

$$+\left(2q\frac{[5]_q^2}{[2]_q[6]_q^2}+\frac{1}{[2]_q}-\frac{[5]_q}{[6]_q}-\frac{[5]_q[2]_q^2-[6]_q}{[6]_q[2]_q^3}\right)^{1-\frac{1}{\wp_1}}(C_{2q}|_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1}+mD_{2q}|_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1})^{\frac{1}{\wp_1}}.$$

where C_{1q},C_{2q},D_{1q} *and* D_{2q} *are given in (10)-(14), respectively.*

Proof. Utilizing power mean inequality on right hand side of Lemma 3.1, we attain

$$\left|\frac{1}{(\varrho_2-m\varrho_1)}\int_{m\varrho_1}^{\varrho_2}\Psi(\zeta)\ {}_{m\varrho_1}d_q\zeta-\frac{1}{[6]_q}\left[q\Psi(m\varrho_1)+q^2[4]_q\Psi\left(\frac{\varrho_2+q\varrho_1 m}{[2]_q}\right)+\Psi(\varrho_2)\right]\right|$$

$$\le q(\varrho_2-m\varrho_1)\left(\int_0^{\frac{1}{[2]_q}}\left|\zeta-\frac{1}{[6]_q}\right|d_q\zeta\right)^{1-\frac{1}{\wp_1}}$$

$$\times\left(\int_0^{\frac{1}{[2]_q}}\left|\zeta-\frac{1}{[6]_q}\right||_{\varrho_1}D_q\Psi(\zeta\varrho_2+m(1-\zeta)\varrho_1)^{\wp_1}|d_q\zeta\right)^{\frac{1}{\wp_1}}$$

$$+q(\varrho_2 - m\varrho_1)\left(\int_{\frac{1}{[2]_q}}^{1}\left|\zeta - \frac{[5]_q}{[6]_q}\right| d_q\zeta\right)^{1-\frac{1}{\wp_1}}\left(\int_{\frac{1}{[2]_q}}^{1}\left|\zeta - \frac{[5]_q}{[6]_q}\right| |_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)^{\wp_1}| d_q\zeta\right)^{\frac{1}{\wp_1}}.$$

Applying (α, m)-convexity of $|_{\varrho_1}D_q\Psi|^{\wp_1}$, we get

$$\left|\frac{1}{(\varrho_2 - m\varrho_1)}\int_{m\varrho_1}^{\varrho_2}\Psi(\zeta)\ {}_{m\varrho_1}d_q\zeta - \frac{1}{[6]_q}\left[q\Psi(m\varrho_1) + q^2[4]_q\Psi\left(\frac{\varrho_2 + q\varrho_1 m}{[2]_q}\right) + \Psi(\varrho_2)\right]\right|$$

$$\leq q(\varrho_2 - m\varrho_1)\left(\int_0^{\frac{1}{[2]_q}}\left|\zeta - \frac{1}{[6]_q}\right| d_q\zeta\right)^{1-\frac{1}{\wp_1}}$$

$$\times\left(|_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1}\int_0^{\frac{1}{[2]_q}}\left|\zeta - \frac{1}{[6]_q}\right|\zeta^{\alpha}d_q\zeta + m|_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1}\int_0^{\frac{1}{[2]_q}}\left|\zeta - \frac{1}{[6]_q}\right|(1-\zeta^{\alpha})d_q\zeta\right)^{\frac{1}{\wp_1}}$$

$$+q(\varrho_2 - m\varrho_1)\left(\int_{\frac{1}{[2]_q}}^{1}\left|\zeta - \frac{[5]_q}{[6]_q}\right| d_q\zeta\right)^{1-\frac{1}{\wp_1}}$$

$$\times\left(|_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1}\int_{\frac{1}{[2]_q}}^{1}\left|\zeta - \frac{[5]_q}{[6]_q}\right|\zeta^{\alpha}d_q\zeta + m|_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1}\int_{\frac{1}{[2]_q}}^{1}\left|\zeta - \frac{1}{[6]_q}\right|(1-\zeta^{\alpha})d_q\zeta\right)^{\frac{1}{\wp_1}}$$

$$= q(\varrho_2 - m\varrho_1)\left(\int_0^{\frac{1}{[2]_q}}\left|\zeta - \frac{1}{[6]_q}\right| d_q\zeta\right)^{1-\frac{1}{\wp_1}}\left(C_{1q}|_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1}d_q\zeta + mD_{1q}|_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1}d_q\zeta\right)^{\frac{1}{\wp_1}}$$

$$+q(\varrho_2 - m\varrho_1)\left(\int_{\frac{1}{[2]_q}}^{1}\left|\zeta - \frac{[5]_q}{[6]_q}\right| d_q\zeta\right)^{1-\frac{1}{\wp_1}}\left(C_{2q}|_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1}d_q\zeta + mD_{2q}|_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1}d_q\zeta\right)^{\frac{1}{\wp_1}}.$$

Here we have

$$\int_0^{\frac{1}{[2]_q}}\left|\zeta - \frac{1}{[6]_q}\right| d_q\zeta = 2\int_0^{\frac{1}{[6]_q}}\left(\frac{1}{[6]_q} - \zeta\right)d_q\zeta + \int_0^{\frac{1}{[2]_q}}\left(\zeta - \frac{1}{[6]_q}\right)d_q\zeta$$
$$= \frac{2q}{[6]_q^2[2]_q} + \frac{q^3[3]_q - q}{[6]_q[2]_q^3}$$

and similarly

$$\int_{\frac{1}{[2]_q}}^{1}\left|\zeta - \frac{[5]_q}{[6]_q}\right| d_q\zeta$$
$$= \frac{2q[5]_q^2}{[6]_q^2[2]_q} + \frac{1}{[2]_q} - \frac{[5]_q}{[6]_q} - \frac{[5]_q[2]_q^2 - [6]_q}{[2]_q^3[6]_q}$$

By putting the values of integrals we get the required results. □

Remark 4.6. In Theorem 4.5, we get

(i) By taking $q \to 1^-$, then we regain [[28], Theorem 2.10].

(ii) By taking $m = 1$, $\alpha = 1$ and taking limit as $q \to 1^-$, then we regain [[29], Theorem 7 for $s = 1$].

5. New Quantum Simpson's $3/8$ Type Inequalities

THEOREM 5.1. *Under the assumptions of Lemma 3.1 if* $|_{\varrho_1}D_q\Psi|$ *is* (α, m)*-convex on* $[\varrho_1, \varrho_2]$*, then we have following inequality*

(23)
$$\left|\frac{1}{\varrho_2 - m\varrho_1}\int_{\varrho_1 m}^{\varrho_2}\Psi(\zeta)\ {}_{m\varrho_1}d_q\zeta - \frac{1}{[8]_q}\left[q\Psi(m\varrho_1) + \frac{q^3[6]_q}{[2]_q}\Psi\left(\frac{\varrho_2 + m\varrho_1 q[2]_q}{[3]_q}\right) + \frac{q^2[6]_q}{[2]_q}\Psi\left(\frac{[2]_q\ \varrho_2 + m\varrho_1 q^2}{[3]_q}\right) + \Psi(\varrho_2)\right]\right|$$
$$\leq q(m\varrho_2 - \varrho_1)\left[|_{\varrho_1}D_q\Psi(\varrho_2)|[C_{3q} + C_{4q} + C_{5q}] + |_{\theta_1}D_q\Psi(\varrho_1)|[D_{3q} + D_{4q} + D_{5q}]\right],$$

where:
$C_{3q}, C_{4q}, C_{5q}, D_{3q}, D_{4q}$ *and* D_{5q} *are given as in (14)-(19) respectively.*

Proof. Utilizing Lemma 3.3, we have

$$\left|\frac{1}{\varrho_2 - m\varrho_1}\int_{\varrho_1 m}^{\varrho_2}\Psi(\zeta)\ {}_{m\varrho_1}d_q\zeta - \frac{1}{[8]_q}\left[q\Psi(m\varrho_1) + \frac{q^3[6]_q}{[2]_q}\Psi\left(\frac{\varrho_2 + mq[2]_q\ \varrho_1}{[3]_q}\right) + \frac{q^2[6]_q}{[2]_q}\Psi\left(\frac{[2]_q\varrho_2 + mq^2\varrho_1}{[3]_q}\right) + \Psi(\varrho_2)\right]\right|$$
$$\leq q(\varrho_2 - m\varrho_1)\int_0^{\frac{1}{[3]_q}}\left|\zeta - \frac{1}{[8]_q}\right||_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)|\ d_q\zeta$$
$$+ q(\varrho_2 - m\varrho_1)\int_{\frac{1}{[3]_q}}^{\frac{[2]_q}{[3]_q}}\left|\zeta - \frac{1}{[8]_q}\right||_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)|\ d_q\zeta$$
$$+ q(\varrho_2 - m\varrho_1)\int_{\frac{[2]_q}{[3]_q}}^{1}\left|\zeta - \frac{[7]_q}{[8]_q}\right||_{\varrho_1}D_q\Psi(\zeta\varrho_2 + m(1-\zeta)\varrho_1)|\ d_q\zeta.$$

It is sufficient to use the same methods as used in Theorem 4.1, to get the required result. □

EXAMPLE 5.2. We assume that the assumptions of Example 4.3 hold. We can calculate L. H. S. and R. H. S. of the inequality (23) as follows

$$\left|\frac{1}{\varrho_2 - m\varrho_1}\int_{\varrho_1 m}^{\varrho_2}\Psi(\zeta)\ {}_{m\varrho_1}d_q\zeta - \frac{1}{[8]_q}\left[q\Psi(m\varrho_1) + \frac{q^3[6]_q}{[2]_q}\Psi\left(\frac{\varrho_2 + m\varrho_1 q[2]_q}{[3]_q}\right) + \frac{q^2[6]_q}{[2]_q}\Psi\left(\frac{[2]_q\ \varrho_2 + m\varrho_1 q^2}{[3]_q}\right) + \Psi(\varrho_2)\right]\right|$$
$$\left|\frac{1}{[3]_q} - \frac{1}{[8]_q}\left[\frac{q^3[6]_q}{[2]_q[3]_q^2} + \frac{q^2[6]_q[2]_q}{[3]_q^2} + 1\right]\right|$$

and

$$\begin{aligned} & q(m\varrho_2 - \varrho_1)\left[|_{\varrho_1}D_q\Psi(\varrho_2)|[C_{3q} + C_{4q} + C_{5q}] + |_{\varrho_1}D_q\mathfrak{g}(\varrho_1)|[D_{3q} + D_{4q} + D_{5q}]\right] \\ = & \frac{q[2]_q}{2}[C_{3q} + C_{4q} + C_{5q}], \end{aligned}$$

respectively. By the inequality (23), we have the inequality

$$\left|\frac{1}{[3]_q}-\frac{1}{[8]_q}\left[\frac{q^3[6]_q}{[2]_q[3]_q^2}+\frac{q^2[6]_q[2]_q}{[3]_q^2}+1\right]\right| \leq \frac{q[2]_q}{2}[C_{3q}+C_{4q}+C_{5q}]. \tag{24}$$

One can see the validity of the inequality (24) in Figure 2.

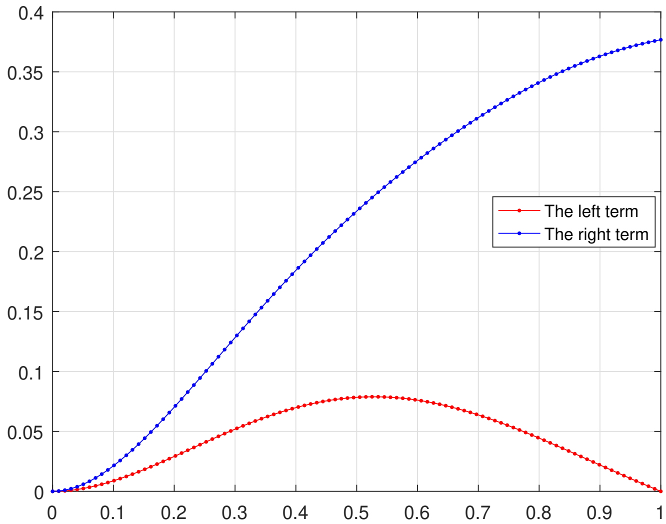

FIGURE 2. An example to Theorem 5.1

THEOREM 5.3. *Under the assumptions of Lemma 3.1 if* $|_{\varrho_1}D_q\Psi|$, $\wp_1 > 1$ *is* (α, m)*-convex on* $[\varrho_1, \varrho_2]$, *then we have following inequality*

$$\begin{aligned}
&\left|\frac{1}{\varrho_2-m\varrho_1}\int_{m\varrho_1}^{\varrho_2}\Psi(\zeta)\ {}_{m\varrho_1}d_q\zeta\right.\\
&\left.-\frac{1}{[8]_q}\left[q\Psi(m\varrho_1)+\frac{q^3[6]_q}{[2]_q}\Psi\left(\frac{\varrho_2+mq[2]_q\varrho_1}{[3]_q}\right)+\frac{q^2[6]_q}{[2]_q}\Psi\left(\frac{[2]_q\varrho_2+mq^2\ \varrho_1}{[3]_q}\right)+\Psi(\varrho_2)\right]\right|\\
&\leq q(\varrho_2-m\varrho_1)\left(\frac{q^{3\wp_2}[5]_q^{\wp_2}}{[3]_q^{\wp_2+1}[8]_q^{\wp_2}}\right)^{\frac{1}{\wp_2}}\\
&\times\left(\frac{1}{[3]_q^{1+\alpha}[2]_q}|_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1}+\frac{[3]_q^{\alpha}[2]_q-1}{[3]_q^{1+\alpha}[2]_q}m|_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1}\right)^{\frac{1}{\wp_1}}\\
&+q(\varrho_2-m\varrho_1)\left(\frac{q^{\wp_2}[2]_q-q^{2\wp_2}}{[3]_q^{\wp_2+1}[2]_q^{\wp_2}}\right)^{\frac{1}{\wp_2}}\\
&\times\left(\frac{[2]_q^{1+\alpha}-1}{[3]_q^{1+\alpha}[1+\alpha]_q}|_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1}+\frac{[3]_q^{\alpha}[1+\alpha]_qq^2-[2]_q^{1+\alpha}+1}{[3]_q^{1+\alpha}[1+\alpha]_q}m|_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1}\right)^{\frac{1}{\wp_1}}\\
&+q(\varrho_2-m\varrho_1)\left(\frac{q^{7\wp_2}}{[8]_q^{\wp_2}}-\frac{[2]_q([7]_q[3]_q-[8]_q[2]_q)^{\wp_2}}{[8]_q^{\wp_2}[3]_q^{\wp_2+1}}\right)^{\frac{1}{\wp_2}}
\end{aligned}$$

$$\times\left(\frac{[3]_q^{1+\alpha}-[2]_q^{1+\alpha}}{[1+\alpha]_q[3]_q^{1+\alpha}}|_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1}+\frac{[1+\alpha]_q[3]_q^{\alpha}q^2+[2]_q^{1+\alpha}-[3]_q^{1+\alpha}}{[3]_q^{1+\alpha}[1+\alpha]_q}m|_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1}\right)^{\frac{1}{\wp_1}},$$

where $\frac{1}{\wp_1}+\frac{1}{\wp_2}=1$.

Proof. Utilizing Hölder's inequality in Lemma 3.3 then we get

$$\begin{aligned}&\Bigg|\frac{1}{\varrho_2-m\varrho_1}\int_{m\varrho_1}^{\varrho_2}\Psi(\zeta)\ _{m\varrho_1}d_q\zeta\\&-\frac{1}{[8]_q}\left[q\Psi(m\varrho_1)+\frac{q^3[6]_q}{[2]_q}\Psi\left(\frac{\varrho_2+mq[2]_q\ \varrho_1}{[3]_q}\right)+\frac{q^2[6]_q}{[2]_q}\Psi\left(\frac{[2]_q\ \varrho_2+m\varrho_1q^2}{[3]_q}\right)+\Psi(\varrho_2)\right]\Bigg|\\&\le q(\varrho_2-m\varrho_1)\left(\int_0^{\frac{1}{[3]_q}}\left|\zeta-\frac{1}{[8]_q}\right|^{\wp_2}d_q\zeta\right)^{\frac{1}{\wp_2}}\left(\int_0^{\frac{1}{[3]_q}}|_{\varrho_1}D_q\Psi(\zeta\varrho_2+m(1-\zeta)\varrho_1)|^{\wp_1}\ d_q\zeta\right)^{\frac{1}{\wp_1}}\\&+q(\varrho_2-m\varrho_1)\left(\int_{\frac{1}{[3]_q}}^{\frac{[2]_q}{[3]_q}}\left|\zeta-\frac{1}{[8]_q}\right|^{\wp_2}d_q\zeta\right)^{\frac{1}{\wp_2}}\left(\int_{\frac{1}{[3]_q}}^{\frac{[2]_q}{[3]_q}}|_{\varrho_1}D_q\Psi(\zeta\varrho_2+m(1-\zeta)\varrho_1)|^{\wp_1}\ d_q\zeta\right)^{\frac{1}{\wp_1}}\\&+q(\varrho_2-m\varrho_1)\left(\int_{\frac{[2]_q}{[3]_q}}^{1}\left|\zeta-\frac{[7]_q}{[8]_q}\right|^{\wp_2}d_q\zeta\right)^{\frac{1}{\wp_2}}\left(\int_{\frac{[2]_q}{[3]_q}}^{1}|_{\varrho_1}D_q\Psi(\zeta\varrho_2+m(1-\zeta)\varrho_1)|\ d_q\zeta\right)^{\frac{1}{\wp_1}}.\end{aligned}$$

To achieve the desired outcomes, it is sufficient to employ the same techniques as in the proof of Theorem 4.4. □

THEOREM 5.4. *Under the assumptions of Lemma 3.1 if* $|_{\varrho_1}D_q\Psi|$, $\wp_1\ge 1$ *is* (α,m)-*convex on* $[\varrho_1,\varrho_2]$, *then we have following inequality*

$$\begin{aligned}&\Bigg|\frac{1}{\varrho_2-m\varrho_1}\int_{m\varrho_1}^{\varrho_2}\Psi(\zeta)\ _{m\varrho_1}d_q\zeta\\&-\frac{1}{[8]_q}\left[q\Psi(m\varrho_2)+\frac{q^3[6]_q}{[2]_q}\Psi\left(\frac{\varrho_2+mq[2]_q\ \varrho_1}{[3]_q}\right)+\frac{q^2[6]_q}{[2]_q}\Psi\left(\frac{[2]_q\varrho_2+mq^2\ \varrho_1}{[3]_q}\right)+\Psi(\varrho_2)\right]\Bigg|\\&\le q(\varrho_2-m\varrho_1)\left(\frac{2q}{[8]_q^2[2]_q}+\frac{[8]_q-[3]_q[2]_q}{[3]_q^2[2]_q[8]_q}\right)^{1-\frac{1}{\wp_1}}(C_{3q}|_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1}+D_{3q}m|_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1})^{\frac{1}{\wp_1}}\\&+\left(\frac{2q}{[2]_q^3}+\frac{q}{[3]_q^2[2]_q}+\frac{1-[3]_q[2]_q}{[3]_q^2[2]_q}\right)^{1-\frac{1}{\wp_1}}(C_{4q}|_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1}+D_{4q}m|_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1})^{\frac{1}{\wp_1}}\\&+\left(\frac{2q[7]_q^2}{[8]_q^2[2]_q}+\frac{[3]_q^2+[2]_q^2}{[3]_q^2[2]_q}-\frac{[7]_q([3]_q+[2]_q)}{[3]_q[8]_q}\right)^{1-\frac{1}{\wp_1}}(C_{5q}|_{\varrho_1}D_q\Psi(\varrho_2)|^{\wp_1}+D_{5q}m|_{\varrho_1}D_q\Psi(\varrho_1)|^{\wp_1})^{\frac{1}{\wp_1}},\end{aligned}$$

where:
$C_{3q}, C_{4q}, C_{5q}, D_{3q}, D_{4q}$ *and* D_{5q} *are given as in (14)-(19) respectively.*

Proof. If the strategy was used in the proof Theorem 4.5 is by taking into account Lemma 3.3, the required inequality can be obtained. □

Conclusion

We used two quantum integral identities to develop some novel inequalities of Newton and Simpson for convex functions and also use the concept of integration and derivation in quantum, which served as the primary source of inspiration for this article. We also demonstrated that newly discovered inequalities might be converted into quantum Newton and quantum Simpson inequalities. We establish connections between our findings and several well-known scientific discoveries. It's a novel and important topic, researchers can use it to find inequalities in future. This study's conclusion summarises and interprets earlier findings.

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