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AFFINE HOMOGENEOUS DOMAINS IN THE COMPLEX PLANE

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Abstract. In this paper, we will describe affine homogeneous domains in the complex plane. For this study, we deal with the Lie algebra of infinitesimal affine transformations, a structure of the hyperbolic metric involved with affine automorphisms. As a consequence, an affine homogeneous domain is affine equivalent to the complex plane, the punctured plane or the half plane.

1. Introduction

For a domain (connected open set) Ω in the complex plane $\mathbb{C}$, the (holomorphic) *automorphism group* of Ω, denoted by $\mathsf{Aut}(\Omega)$, is the group of self-biholomorphisms of Ω under the law of the mapping composition. If a given domain is the whole $\mathbb{C}$, then the automorphism group $\mathsf{Aut}(\mathbb{C})$ is exactly the holomorphic affine transformation group $\mathsf{Aff}(\mathbb{C})$ of $\mathbb{C}$:

$$\mathsf{Aff}(\mathbb{C}) = \{z \mapsto az + b : a \in \mathbb{C}^*, b \in \mathbb{C}\} \ .$$

Here $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For a proper domain Ω, the *affine automorphism group*

$$\mathsf{Aff}(\Omega) = \mathsf{Aut}(\Omega) \cap \mathsf{Aff}(\mathbb{C})$$

should be a proper subgroup of $\mathsf{Aut}(\Omega)$ since an orbit $\mathsf{Aff}(\mathbb{C}) \cdot p = \{f(p) : f \in \mathsf{Aff}(\mathbb{C})\}$ is $\mathbb{C}$ for any $p \in \Omega$. If $\mathsf{Aff}(\Omega)$ acts transitively on Ω, i.e. $\mathsf{Aff}(\Omega) \cdot p = \Omega$ for $p \in \Omega$, then we say that Ω is *affine homogeneous*. The complex plane $\mathbb{C}$ is affine homogeneous and there are two typical affine homogeneous domains. One is the *punctured plane* $\mathbb{C}^*$ whose automorphism group is of the form,

$$\mathsf{Aut}(\mathbb{C}^*) = \{z \mapsto az : a \in \mathbb{C}^*\} \cup \{z \mapsto a/z : a \in \mathbb{C}^*\}$$

So $\mathsf{Aff}(\mathbb{C}^*) = \{z \mapsto az : a \in \mathbb{C}^*\}$ acts on Ω transitively. Note that $\mathsf{Aff}(\mathbb{C}^*)$ is generated by the *rotation*

$$\mathcal{R}_t : z \mapsto e^{it} z \quad (t \in \mathbb{R}) \ , \tag{1}$$

and the *dilation*

$$\mathcal{D}_t : z \mapsto e^{t} z \quad (t \in \mathbb{R}) \ . \tag{2}$$

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Another affine homogeneous domain is the left half-plane $\mathbf{H} = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ whose automorphism group is the projective special linear group $\mathrm{PSL}(2, \mathbb{R})$ which is a 3-dimensional Lie group. The affine automorphism group $\mathsf{Aff}(\mathbf{H})$ is the 2-dimensional subgroup generated by the dilation $\mathcal{D}_t$ in (2) and the *translation*

$$\mathcal{T}_{it} : z \mapsto z + it \quad (t \in \mathbb{R}) \tag{3}$$

whose action on $\mathbf{H}$ is also transitive.

If a domain Ω is *affine equivalent* to another domain Ω', that is, there is an affine mapping $F \in \mathsf{Aff}(\mathbb{C})$ with $F(\Omega) = \Omega'$, then the pushing-forward group

$$F_* (\mathsf{Aff}(\Omega)) = \left\{F \circ f \circ F^{-1} : f \in \mathsf{Aff}(\Omega)\right\}$$

coincides with $\mathsf{Aff}(\Omega')$. Therefore a domain which is affine equivalent to $\mathbb{C}^*$ or $\mathbf{H}$, is affine homogeneous.

In this paper, we will show that $\mathbb{C}$, $\mathbb{C}^*$ and $\mathbf{H}$ constitute the complete list of affine homogeneous domains in $\mathbb{C}$ up to the affine equivalence.

THEOREM 1.1. *An affine homogeneous, proper domain in $\mathbb{C}$ is affine equivalent to $\mathbb{C}^*$ or $\mathbf{H}$.*

Maybe, there are many approaches to get this result. The aim of the paper is indeed to give a geometric approach that can be extended even in the case of the multi-dimensional complex affine spaces.

For a simply connected, proper domain Ω which is so biholomorphic to $\mathbf{H}$, the domain Ω is affine equivalent to $\mathbf{H}$ if Ω admits a potential function of the hyperbolic metric whose gradient length is identically constant by Corollary 1.3 of [4]. The existence of such potential function allows us to construct a complete holomorphic vector field on Ω. Realizing this vector field as an infinitesimal parabolic automorphisms of the unit disc Δ, the affine equivalence to $\mathbf{H}$ follows. The relation between potential functions and complete holomorphic vector fields has been improved for the multi-dimensional complex geometry under the name of potential rescaling (see [3, 7]). But this approach is not applicable for the characterization of affine homogeneous domains in $\mathbb{C}^n$. Thus we will improve the method of [4] using the affine geometry of $\mathbb{C}$. Especially, we will mainly use the hyperbolic metric of a domain in $\mathbb{C}$ omitting at least two points and a structure of the Lie algebra of infinitesimal affine automorphisms.

In Section 2, we will introduce the Lie algebra of infinitesimal affine automorphisms, the hyperbolic structure of domains, and their relations in case of the unit disc. Then Theorem 1.1 will be proved in Section 3.

2. Lie algebra of the affine automorphism groups

In this section, we will consecutively introduce infinitesimal automorphisms, a structure of the affine transformation group $\mathsf{Aff}(\mathbb{C})$, the hyperbolicity of domains, a certain structure of Poincaré metric, and the affine geometry of the left half-plane.

2.1. Infinitesimal automorphisms. Let X be a Riemann surface. An 1-*parameter family of automorphisms* of X is a family $\{\mathcal{V}_s : s \in \mathbb{R}\} \subset \mathsf{Aut}(X)$ with parameter s satisfying

$$\mathcal{V}_s \circ \mathcal{V}_t = \mathcal{V}_{s+t}$$

for any $s, t \in \mathbb{R}$. Then the family $\{\mathcal{V}_s : s \in \mathbb{R}\}$ is a subgroup of $\mathsf{Aut}(X)$. By an *infinitesimal generator* of $\{\mathcal{V}_s\}$, we means that a $(1,0)$-vector field $\mathcal{V}$ of X such that $\mathcal{V}_s$ is the flow of the corresponding real tangent vector field $\operatorname{Re}\mathcal{V} = \mathcal{V} + \overline{\mathcal{V}}$, that is,

$$(\operatorname{Re}\mathcal{V})(x) = d\gamma_x \left(\left.\frac{d}{ds}\right|_{s=0} \right) \quad \text{for any } x \in X,$$

where $\gamma_x : \mathbb{R} \to X$ is the curve given by $\gamma_x(s) = \mathcal{V}_s(x)$. For a local coordinate function z near $x \in X$, the value of $\mathcal{V}$ at x is written by

$$\mathcal{V}(x) = \left(\left.\frac{d}{ds}\right|_{s=0} \mathcal{V}_s(x) \right) \left.\frac{\partial}{\partial z}\right|_x .$$

Here we regards $\mathcal{V}_s$ as a s-parametrized local holomorphic functions near x.

An $(1,0)$-vector field $\mathcal{V}$ of X is an *infinitesimal automorphisms* of X if it is a complete holomorphic vector field, more precisely,

1. the corresponding real vector field $\operatorname{Re}\mathcal{V} = \mathcal{V} + \overline{\mathcal{V}}$ is complete,
2. in any local holomorphic coordinate function z, $\mathcal{V}$ can be written by

$$\mathcal{V} = f \frac{\partial}{\partial z}$$

for some holomorphic function f.

One can easily see that a flow of an infinitesimal automorphism of X is an 1-parameter family of automorphisms of X. Therefore the set of complete holomorphic vector fields of X, denoted by $\mathfrak{aut}(X)$, can be identified with the set of 1-parameter automorphism families. Theories of Riemann surfaces say that the automorphism group $\mathsf{Aut}(X)$ is always a Lie group. Therefore we can regard $\mathfrak{aut}(X)$ as the Lie algebra of $\mathsf{Aut}(X)$ when a bracket structure of $\mathfrak{aut}(X)$ is given by the Lie bracket of $(1,0)$-vector fields.

If two $(1,0)$-vector fields $\mathcal{V}$ and $i\mathcal{V}$ are complete and holomorphic ($\mathcal{V}, i\mathcal{V} \in \mathfrak{aut}(X)$), then $[\operatorname{Re}\mathcal{V}, \operatorname{Re} i\mathcal{V}] = 0$ implies that $\mathcal{V}_s \circ (i\mathcal{V})_t = (i\mathcal{V})_t \circ \mathcal{V}_s$ for any $s, t \in \mathbb{R}$. So we have a $\mathbb{C}$-parameter subgroup of $\mathsf{Aut}(X)$ generated by

$$\mathcal{V}_\zeta = \mathcal{V}_s \circ (i\mathcal{V})_t$$

for $\zeta = s + it$. Simultaneously, each $x \in X$ gives a entire curve

$$u_x : \zeta \mapsto \mathcal{V}_\zeta(x) ,$$

because

1. $\mathcal{V}_\zeta \circ \mathcal{V}_\xi = \mathcal{V}_{\zeta+\xi}$ for any $\zeta, \xi \in \mathbb{C}$,
2. the Cauchy-Riemann equation holds:

$$\begin{aligned} du_x \left(\left.\frac{\partial}{\partial \zeta}\right|_0 \right) &= \frac{1}{2} \left(du_x \left(\left.\frac{\partial}{\partial s}\right|_0 \right) - i du_x \left(\left.\frac{\partial}{\partial t}\right|_0 \right) \right) \\ &= \frac{1}{2} \left((\operatorname{Re}\mathcal{V})(x) - i(\operatorname{Re} i\mathcal{V})(x) \right) = \frac{1}{2} \left((\operatorname{Re}\mathcal{V})(x) - iJ(\operatorname{Re}\mathcal{V})(x) \right) \end{aligned}$$

where J is the complex structure of X.

As we will see in Section 2.3, there is no nontrivial entire curve in a hyperbolic surface X; thus $\mathfrak{aut}(X)$ for hyperbolic X is a real Lie algebra without nontrivial complex subalgebra. This is a quite distinguished phenomenon from the non-hyperbolic surface, for instance $\mathbb{C}$.

2.2. The automorphism group of $\mathbb{C}$: the group of affine transformations. By a classical theory of Complex Analysis, the automorphism group of the complex plane $\mathbb{C}$ coincides with the group of affine transformations of $\mathbb{C}$. Since the aim of this research is the affine homogenuity, we will especially denote the automorphism group of $\mathbb{C}$ by $\mathsf{Aff}(\mathbb{C})$. Typical affine transformations are the complex multiplication

$$\mathcal{M}_a : z \mapsto az \quad (a \in \mathbb{C}^*)$$

and the translation

$$\mathcal{T}_b : z \mapsto z + b \quad (b \in \mathbb{C}) .$$

The group of complex multiplications is the $\mathbb{C}^*$ parameter family, so isomorphic to the multiplicative group $\mathbb{C}^*$. And the group of translations is isomorphic to the additive group $\mathbb{C}$. One can easily see that the translation group is normal so $\mathsf{Aff}(\mathbb{C}) \simeq \mathbb{C}^* \ltimes \mathbb{C}$. The linear group expression of $\mathsf{Aff}(\mathbb{C})$ is the matrix multiplication group

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^*, b \in \mathbb{C} \right\} ,$$

where $\mathcal{M}_a$ and $\mathcal{T}_b$ correspond to $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, with respectively. Therefore $\mathsf{Aff}(\mathbb{C})$ is a complex Lie group biholomorphic to $\mathbb{C}^* \times \mathbb{C}$. As a real Lie group, $\mathsf{Aut}(\mathbb{C})$ is generated by

$$\mathcal{R}_t , \quad \mathcal{D}_t , \quad \mathcal{T}_t , \quad \mathcal{T}_{it} \quad (t \in \mathbb{R})$$

where $\mathcal{R}_t$ and $\mathcal{D}_t$ are the rotation and the dilation as in (1) and (2), respectively. Their infinitesimal generators are

$$\mathcal{R} = iz\frac{\partial}{\partial z} , \quad \mathcal{D} = z\frac{\partial}{\partial z} , \quad \mathcal{T} = \frac{\partial}{\partial z} , \quad i\mathcal{T} = i\frac{\partial}{\partial z} \tag{4}$$

which form a real basis of the real 4-dimensional Lie algebra $\mathfrak{aff}(\mathbb{C}) = \mathfrak{aut}(\mathbb{C})$. Note that $\mathcal{D}$ and $\mathcal{T}$ form a complex basis of $\mathfrak{aff}(\mathbb{C})$.

In case of the Riemann sphere $\mathbb{CP}^1$, the automorphism group is the Möbius transformation group, usually denoted also by $\mathrm{PSL}(2,\mathbb{C})$, which is a complex 3-dimensional Lie group.

2.3. Hyperbolic Riemann surfaces and their automorphism groups. Let X be a Riemann surface with a hermitian metric ds^2. For an expression

$$ds^2 = \lambda\, |dz|^2$$

by a local coordinate function z, the gaussian curvature κ of ds^2 is written by

$$\kappa = -\frac{1}{\lambda}\frac{\partial^2}{\partial z \partial \bar{z}} \log \lambda . \tag{5}$$

If there is a complete hermitian metric with $\kappa \equiv -2$, then the metric is called a *hyperbolic metric* and the surface is *hyperbolic*. By Ahlfors' Schwarz lemma in [1], a hyperbolic metric is uniquely determined so there is no confusion of notation when we denote by ds^2_X the hyperbolic metric of a given hyperbolic surface X.

By the uniformization theorem for Riemann surfaces (see [6,8]), a simply connected, hyperbolic Riemann surface is biholomorphic and isometric to the *Poincaré disc model*

(Δ, ds^2_Δ) which is the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ endowed with the *Poincaré metric*

$$ds^2_\Delta = \frac{1}{\left(1 - |z|^2\right)^2} |dz|^2 \ . \tag{6}$$

Moreover, a Riemann surface X is hyperbolic if and only if X is a quotient of Δ, i.e. there is a discrete subgroup $\Gamma \subset \mathsf{Aut}(\Delta)$ such that X is biholomorphic to the quotient space Δ/Γ. Since every automorphism of Δ is an isometry of the Poincaré metric, the quotient Δ/Γ is a Riemannian quotient space, so the hyperbolic metric of $X = \Delta/\Gamma$ is the pushing-forward metric of the Poincaré metric,

$$ds^2_X = \pi_* ds^2_\Delta$$

where $\pi : \Delta \to \Delta/\Gamma = X$ is the quotient mapping.

The Riemann sphere $\mathbb{CP}^1$ and the complex plane $\mathbb{C}$, the other models in the uniformization theorem, can not admit a complete hermitian metric with strictly negative curvature. And their automorphism groups are complex Lie groups. Since there is no nontrivial entire curve in a hyperbolic Riemann surface X by Ahlfors' Schwarz lemma, the automorphism group of X is purely real, that is, there is no complex subalgebra in $\mathfrak{aut}(X)$.

Let us see $\mathsf{Aut}(\Delta)$, the automorphism group of Δ. This is a real 3-dimensional, connected subgroup of the isometry group $\mathsf{Isom}(\Delta, ds^2_\Delta)$. The Lie algebra $\mathfrak{aut}(\Delta)$ has three nontrivial elements $\mathcal{R}, \mathcal{H}, \mathcal{P}$ where

1. $\mathcal{R}$ is the infinitesimal generator of the rotation as in (4),
2. $\mathcal{H}$ and $\mathcal{P}$ are defined by

$$\mathcal{H} = (z^2 - 1)\frac{\partial}{\partial z} \quad \text{and} \quad \mathcal{P} = i(z+1)^2\frac{\partial}{\partial z} \ . \tag{7}$$

The vector field $\mathcal{H}$ generates hyperbolic automorphisms leaving two boundary points $1, -1$ fixed and $\mathcal{P}$ generates parabolic automorphisms leaving -1 fixed (see [4]). One can easily see that $\mathcal{R}, \mathcal{H}, \mathcal{P}$ are linearly independent over $\mathbb{R}$ so form a basis of $\mathfrak{aut}(\Delta)$.

2.4. A certain potential function of the Poincaré metric and the automorphism group of the unit disc. Let X be a hyperbolic Riemann surface with the hyperbolic metric ds^2_X. When we write $ds^2_X = \lambda |dz|^2$ in the local holomorphic coordinate function z, Equation (5) and the curvature condition $\kappa \equiv -2$ implies that

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \lambda = 2\lambda \ ,$$

so the function $(1/2)\log\lambda$ is a local Kähler potential of ds^2_X. Any other potential of ds^2_X is always locally given by $(1/2)\log\lambda + |f|^2$ where f is a local holomorphic function on the domain of z. We call $(1/2)\log\lambda$ the *local canonical potential* with respect to the coordinate function z. Especially, the *canonical potential* of a domain $\Omega \subset \mathbb{C}$ is the canonical potential of ds^2_Ω which is globally given by the standard coordinate function of $\mathbb{C}$.

For a (local) real-valued differentiable function ψ on X, the *gradient length* $\|\partial\psi\|_X$ of ψ is the length of gradient vector field measured by the hyperbolic metric which can be locally written by

$$\|\partial\psi\|^2_X = \left\| \frac{\partial\psi}{\partial z} dz \right\|^2_X = \frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial \bar{z}}\frac{1}{\lambda} \ .$$

We will describe the structure of $\mathfrak{aut}(\Delta)$ in terms of a certain potential function φ of the Poincaré metric whose gradient length $\|\partial\varphi\|_\Delta$ is constant.

On the unit disc, the canonical potential function of the Poincaré metric ds^2_Δ in (6) is

$$\varphi_\Delta = \frac{1}{2}\log\frac{1}{\left(1-|z|^2\right)^2} = -\log\left(1-|z|^2\right) .$$

Let us define

$$\varphi_0 = \frac{1}{2}\log\frac{|1+z|^4}{\left(1-|z|^2\right)^2} = \varphi_\Delta + \log|1+z|^2 \tag{8}$$

which is also a global potential of the Poincaré metric ds^2_Δ. The *gradient length* of the canonical potential φ_Δ is not constant but that of φ_0 is constant:

$$\|\partial\varphi_\Delta\|^2_\Delta = \frac{\partial\varphi_\Delta}{\partial z}\frac{\partial\varphi_\Delta}{\partial\bar z}\frac{1}{\lambda_\Delta} = |z|^2 \ , \quad \|\partial\varphi_0\|^2_\Delta \equiv 1 \ .$$

By a result of Kai-Ohsawa [5], the constant 1 is invariant, that is, if there is a potential function ψ of ds^2_Δ with constant $\|\partial\psi\|_\Delta \equiv c$, then $c = 1$. The theorem of Kai-Ohsawa holds for bounded homogeneous domains in $\mathbb{C}^n$ and their Bergman potentials. In the 1-dimensional case, a bounded homogeneous domain is only the unit disc Δ up to the biholomorphic equivalence and its Bergman metric is just the Poincaré metric. When we focus on the unit disc only, we have proved that φ_0 is the unique potential of constant gradient length, up to constant and up to rotational symmetry.

THEOREM 2.1 (Theorem 2.2 in [4]). *If a function ψ on Δ is a potential of ds^2_Δ with $\|\partial\psi\|_\Delta \equiv 1$, then there is a constant c and $\theta \in \mathbb{R}$ such that*

$$\psi = \varphi_0 \circ \mathcal{R}_\theta + c = \frac{1}{2}\log\frac{\left|1+e^{i\theta}z\right|^4}{\left(1-|z|^2\right)^2} + c \ .$$

For the basis $\{\mathcal{H},\mathcal{P},\mathcal{R}\}$ of $\mathfrak{aut}(\Delta)$ as we introduced in Section 2.3, $\mathcal{R} = iz\partial/\partial z$ gives a zero vector at $z = 0$ but $\mathcal{H} = (z^2-1)\partial/\partial z$ and $\mathcal{P} = i(z+1)^2\partial/\partial z$ are nowhere vanishing on Δ. The potential function φ_0 can characterize these two elements $\mathcal{H},\mathcal{P}$. Since

$$\mathcal{H}\varphi_0 = (z^2-1)\frac{(1+\bar z)}{(1+z)(1-|z|^2)} = \frac{|z|^2+z-\bar z-1}{1-|z|^2} \ ,$$

$$\mathcal{P}\varphi_0 = i(z+1)^2\frac{(1+\bar z)}{(1+z)(1-|z|^2)} = i\frac{|1+z|^2}{1-|z|^2}$$

so we have

$$(\operatorname{Re}\mathcal{H})\varphi_0 \equiv -2 \quad \text{and} \quad (\operatorname{Re}\mathcal{P})\varphi_0 \equiv 0 \ .$$

But

$$(\operatorname{Re}\mathcal{R})\varphi_0 = i\frac{z(1+\bar z)^2}{|1+z|^2\left(1-|z|^2\right)} - i\frac{\bar z(1+z)^2}{|1+z|^2\left(1-|z|^2\right)} = \frac{z-\bar z}{|1+z|^2}$$

is not identically constant. Moreover

THEOREM 2.2 (Section 4 in [4]). *A complete holomorphic vector field* $\mathcal{V} \in \mathfrak{aut}(\Delta)$ *satisfies* $(\operatorname{Re} \mathcal{V})\varphi_0 \equiv c$ *for some constant* c *if and only if*

$$\mathcal{V} = -\frac{c}{2}\mathcal{H} + t\mathcal{P}$$

for some $t \in \mathbb{R}$.

2.5. Affine automorphism group of the left half-plane H. The left half-plane $\mathbf{H} = \{w \in \mathbb{C} : \operatorname{Re} w < 0\}$ which is affine homogeneous, is biholomorphic to the unit disc Δ by the *Cayley transform* $F : \mathbf{H} \to \Delta$ defined by

$$\begin{aligned} F : \mathbf{H} &\longrightarrow \Delta \\ w &\longmapsto z = \frac{1+w}{1-w} \,. \end{aligned} \tag{9}$$

Then the hyperbolic metric of $\mathbf{H}$ is

$$ds^2_{\mathbf{H}} = F^* ds^2_{\Delta} = \frac{1}{(w+\bar{w})^2}\,|dw|^2$$

and the canonical potential $\varphi_{\mathbf{H}}$ is

$$\varphi_{\mathbf{H}} = \frac{1}{2}\log\frac{1}{(w+\bar{w})^2} = -\log(-(w+\bar{w}))$$

which satisfies $\|\partial\varphi_{\mathbf{H}}\|_{\mathbf{H}} \equiv 1$. In fact, the canonical potential $\varphi_{\mathbf{H}}$ coincides with the pulling back of φ_0 (Equation (8)) on Δ by F, i.e.

$$F^*\varphi_0 = \varphi_{\mathbf{H}} \,.$$

In [4], we showed that

$$(F^{-1})_*\mathcal{H} = \mathcal{D} = z\frac{\partial}{\partial z}\,, \quad (F^{-1})_*\mathcal{P} = i\mathcal{T} = i\frac{\partial}{\partial z}\,. \tag{10}$$

These two vector fields $\mathcal{D}$ and $\mathcal{P}$ generate affine transformations $\mathcal{D}_t$ and $\mathcal{T}_{it}$ on $\mathbb{C}$, respectively, thus

$$\mathcal{D}\,, i\mathcal{T} \ \in \mathfrak{aff}(\mathbf{H}) = \mathfrak{aut}(\mathbf{H}) \cap \mathfrak{aff}(\mathbb{C}) \,.$$

In the proof of Theorem 1.1, we can easily see that the $\mathfrak{aff}(\mathbf{H})$ is of 2-dimension, so $\{\mathcal{D}, i\mathcal{T}\}$ is a basis of $\mathfrak{aff}(\mathbf{H})$.

3. Proof of Theorem 1.1

Let Ω be an affine homogeneous, proper domain in $\mathbb{C}$. If $\mathbb{C} \setminus \Omega$ is just an one point set, Ω is affine equivalent to the punctured plane $\mathbb{C}^*$; thus we may assume that Ω omits at least two points of $\mathbb{C}$. By the little Picard theorem (Theorem 5 of Chapter 8 in [2]), there is no nonconstant entire curve in Ω. This means that Ω is not covered by $\mathbb{C}$, so the universal covering of Ω is the unit disc. Therefore Ω is hyperbolic and admits the hyperbolic metric ds^2_{Ω}. We will show consecutively

1. there is an affine translation in $\mathsf{Aff}(\Omega)$, i.e. $a\mathcal{T} \in \mathfrak{aff}(\Omega)$ for some $a \in \mathbb{C}$, and Ω is simply connected so biholomorphic to the unit disc,
2. $a\mathcal{T}$ annihilates the canonical potential function φ_{Ω} of the hyperbolic metric and $\|\partial\varphi_{\Omega}\|_{\Omega}$ is constant,
3. there is a biholomorphism $G : \Omega \to \Delta$ which sends $a\mathcal{T}$ to a parabolic vector field, i.e. $G_*(a\mathcal{T}) = t\mathcal{P}$ for some $t \in \mathbb{R}$,

4. the biholomorphism $G^{-1} \circ F : \mathbf{H} \to \Omega$ is affine where F is the Cayley transform of (9).

Step 1. Let us consider the Lie algebra $\mathfrak{aff}(\Omega) = \mathfrak{aut}(\Omega) \cap \mathfrak{aff}(\mathbb{C})$ of $\mathsf{Aff}(\Omega)$ whose real dimension is less than or equal to $4 = \dim_{\mathbb{R}} \mathfrak{aff}(\mathbb{C})$. The transitivity of the action by $\mathsf{Aff}(\Omega)$ on Ω implies that the real dimension of $\mathfrak{aff}(\Omega)$ is at least real 2. Since $\mathfrak{aff}(\mathbb{C})$ is an algebra over $\mathbb{C}$, $i\mathfrak{aff}(\Omega) = \{i\mathcal{V} : \mathcal{V} \in \mathfrak{aff}(\Omega)\}$ is a real subalgebra whose real dimension is the same as $\dim_{\mathbb{R}} \mathfrak{aff}(\Omega)$. When we assume that $\dim_{\mathbb{R}} \mathfrak{aff}(\Omega) \geq 3$, the complex subalgebra $\mathfrak{aff}(\Omega) \cap i\mathfrak{aff}(\Omega)$ of $\mathfrak{aff}(\Omega)$ has positive dimension. This is a contradiction to the hyperbolicity of Ω as we mentioned at the end of Section 2.1. Therefore $\mathfrak{aff}(\Omega)$ is a purely real subalgebra of $\dim_{\mathbb{R}} = 2$.

Let $\mathcal{V}, \mathcal{W} \in \mathfrak{aff}(\mathbb{C})$ form a basis of $\mathfrak{aff}(\Omega)$. Then the vector fields $\mathcal{V}$ and $\mathcal{W}$ can be written as

$$\mathcal{V} = (\alpha z + \beta)\frac{\partial}{\partial z}\ , \quad \mathcal{W} = (\mu z + \nu)\frac{\partial}{\partial z}$$

for some $\alpha, \beta, \mu, \nu \in \mathbb{C}$ because they are infinitesimal affine transformations of $\mathbb{C}$. Let us consider

$$[\mathcal{V}, \mathcal{W}] = \mu(\alpha z + \beta)\frac{\partial}{\partial z} - \alpha(\mu z + \nu)\frac{\partial}{\partial z} = (\beta\mu - \alpha\nu)\frac{\partial}{\partial z} \in \mathfrak{aff}(\Omega)\ .$$

If $\beta\mu - \alpha\nu = 0$, then $\mathcal{W}$ is a complex multiple of $\mathcal{V}$. This means that $\mathfrak{aff}(\mathbb{C})$ is a complex algebra. Therefore the complex number $a = \beta\mu - \alpha\nu$ is not zero and

$$[\mathcal{V}, \mathcal{W}] = a\frac{\partial}{\partial z} = a\mathcal{T} \in \mathfrak{aff}(\Omega)$$

Now we get the translation

$$\mathcal{T}_{sa}(z) = z + sa \quad (s \in \mathbb{R})$$

as an affine automorphism of Ω.

If γ is a loop (a closed curve) in Ω, then the subset $D = \bigcup_{s\in\mathbb{R}} \mathcal{T}_{sa}(\gamma)$ of Ω is a closed strip (a closed region bounded by two parallel lines), so D is simply connected. Therefore γ can be contracted to a point in $D \subset \Omega$, i.e. Ω is simply connected. By the Riemann mapping theorem, Ω is biholomorphic to the unit disc Δ.

Step 2. Now let us consider the hyperbolic metric ds^2_Ω of Ω and write it by

$$ds^2_\Omega = \lambda_\Omega\, |dz|^2\ .$$

An automorphism f of Ω is an isometry with respect to ds^2_Ω by Ahlfors' Schwarz lemma, so it follows that $f^* ds^2_\Omega = ds^2_\Omega$, i.e.

$$(\lambda_\Omega \circ f)\, |f'|^2 = \lambda_\Omega$$

where f' is the complex derivative of the holomorphic function f. If f is an affine transformation, then $f' \equiv b$ for some $b \neq 0$; thus the canonical potential

$$\varphi_\Omega = \frac{1}{2}\log\lambda_\Omega$$

satisfies

$$\varphi_\Omega \circ f + \log|b| = \varphi_\Omega\ . \tag{11}$$

Differentiating it, we get $f^*(\partial\varphi_\Omega) = \partial\varphi_\Omega$, so

$$\|\partial\varphi_\Omega\|_\Omega\, (f(p)) = \|f^*(\partial\varphi_\Omega)\|_\Omega\, (p) = \|\partial\varphi_\Omega\|_\Omega\, (p)$$

for any $p \in \Omega$ because f is isometric. Since Ω is affine homogeneous, we get $\|\partial\varphi_\Omega\|_\Omega$ is identically constant.

For the affine transformation $\mathcal{T}_{sa}$ of Ω, its complex derivative $\mathcal{T}'_{sa}$ is identically 1. Thus Equation (11) implies that

$$\varphi_\Omega \circ \mathcal{T}_{sa} = \varphi_\Omega \; .$$

This implies that

$$(\operatorname{Re} a\mathcal{T})\varphi_\Omega = \left.\frac{d}{ds}\right|_{s=0} \varphi_\Omega \circ \mathcal{T}_{sa} = \left.\frac{d}{ds}\right|_{s=0} \varphi_\Omega \equiv 0 \; . \tag{12}$$

Step 3. For a biholomorphism $G : \Omega \to \Delta$, let us define $\psi : \Delta \to \mathbb{R}$ by

$$\psi = (G^{-1})^* \varphi_\Omega = \varphi_\Omega \circ G^{-1} = \frac{1}{2}\log(\lambda_\Omega \circ G^{-1}) \; .$$

Since G is isometric from (Ω, ds^2_Ω) to (Δ, ds^2_Δ) where $ds^2_\Delta = \lambda_\Delta \, |dz|^2$ is the Poincaré metric in (6), we have $G^* ds^2_\Delta = ds^2_\Omega$ which is written by

$$(\lambda_\Delta \circ G)\left|\frac{\partial G}{\partial z}\right|^2 = \lambda_\Omega \quad \text{and} \quad \lambda_\Delta = (\lambda_\Omega \circ G^{-1})\left|\frac{\partial G^{-1}}{\partial \zeta}\right|^2$$

where z and ζ are coordinate functions on Ω and Δ, respectively. From the fact that $(1/2)\log\lambda_\Delta$ is the potential of ds^2_Δ, it follows that

$$\lambda_\Delta = \frac{1}{2}\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\log\lambda_\Delta = \frac{1}{2}\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\log(\lambda_\Omega \circ G^{-1}) = \frac{\partial^2\psi}{\partial\zeta\partial\bar{\zeta}} \; .$$

This means that ψ is also a potential of the Poincaré metric. Moreover $\|\partial\psi\|_\Delta$ is constant because

$$\|\partial\psi\|_\Delta\,(p) = \left\|(G^{-1})^*\partial\varphi_\Omega\right\|_\Delta(p) = \|\partial\varphi_\Omega\|_\Omega\,(G^{-1}(p))$$

for any $p \in \Delta$. As we mentioned in Section 2.4,

1. the constant for $\|\partial\psi\|_\Delta$ is 1 by Kai-Ohsawa [5],
2. $\psi = \varphi_0 \circ \mathcal{R}_\theta + c$ for some $c \in \mathbb{R}$ and for φ_0 of (8) by Theorem 2.1.

Replacing G by $\mathcal{R}_\theta \circ G$, we may let $\psi = \varphi_0 + c$. Since $(\operatorname{Re} a\mathcal{T})\varphi_\Omega \equiv 0$ from (12), the complete holomorphic vector field $\mathcal{V} = G_*(a\mathcal{T})$ satisfies

$$\begin{aligned}(\operatorname{Re}\mathcal{V})\varphi_0 = (\operatorname{Re}\mathcal{V})\psi = (\mathcal{V} + \overline{\mathcal{V}})\psi = \Big(G_*(a\mathcal{T}) + \overline{G_*(a\mathcal{T})}\Big)\,\psi \\ = (a\mathcal{T} + \overline{a\mathcal{T}})G^*\psi = (a\mathcal{T} + \overline{a\mathcal{T}})\varphi_\Omega \equiv 0 \; .\end{aligned}$$

By Theorem 2.2,

$$G_*(a\mathcal{T}) = \mathcal{V} = t\mathcal{P} \tag{13}$$

for some $t \neq 0$.

Step 4. Let $F : \mathbf{H} \to \Delta$ be the Cayley transform given in (9) and let

$$H = G^{-1} \circ F : \mathbf{H} \to \Omega$$

which is a biholomorphism. From (10) and (13),

$$F_*\mathcal{T} = \mathcal{P} \; , \quad (G^{-1})_*\mathcal{P} = \frac{a}{t}\mathcal{T} \; , \quad \text{so} \quad H_*\mathcal{T} = \frac{a}{t}\mathcal{T}$$

where a and t are nonzero constants. The last equation can be written by

$$\frac{\partial H}{\partial z} = \frac{a}{t} \; ;$$

thus $H(z) = (a/t)z + c$ for some $c \in \mathbb{C}$. This completes the proof. $\square$

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