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CERTAIN SIMPSON-TYPE INEQUALITIES FOR TWICE-DIFFERENTIABLE FUNCTIONS BY CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. In this paper, an equality is established by twice-differentiable convex functions with respect to the conformable fractional integrals. Moreover, several Simpson-type inequalities are presented for the case of twice-differentiable convex functions via conformable fractional integrals by using the established equality. Furthermore, our results are provided by using special cases of obtained theorems.

1. Introduction & preliminaries

The theory of convexity plays a interesting role in many areas of research. This theory offers us with a powerful tool for solving sundry problems that appear in applied and pure mathematics. In recent years, the concept of convexity has been generalized and improved in many directions.

Definition 1.1. [1] Let I denote an interval of real numbers. Then, a function $f : I \to \mathbb{R}$ is said to be convex, if

$$f\left(tx+(1-t)\,y\right) \leq tf\left(x\right)+(1-t)\,f\left(y\right)$$

is valid $\forall x, y \in I$ and $\forall t \in [0,1]$.

Sarikaya et al. [2] introduced Simpson-type inequality for the case of twice-differentiable convex function, and they used the following lemma to prove the main equalities and inequalities.

Lemma 1.2 (See [2]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice-differentiable function on I^o such that $f'' \in L_1[a,b]$, where a,b $\in I$ with $a < b$. Then, we have the following equality*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx$$
$$=(b-a)^2\int_0^{\frac{1}{2}}\frac{t}{2}\left(\frac{1}{3}-t\right)f''\left(tb+(1-t)\,a\right)dt+\int_{\frac{1}{2}}^1(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)f''\left(tb+(1-t)\,a\right)dt$$

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$$= \frac{(b-a)^2}{48} \int_0^1 \left(-1+4t-3t^2\right) \left[f''\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right) + f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right].$$

Sarikaya et al. [2] obtain several inequalities of Simpson-type based on convexity. They also established the following Simpson-type inequality.

Theorem 1.3 (See [2]). *Assume that the assumptions of Lemma 1.2 hold. If $|f''|$ is convex on $[a,b]$, then the following inequality*

$$\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{162}\left[|f''(a)| + |f''(b)| \right]$$

is valid.

Theorem 1.4 (See [3]). *Let us consider that the conditions of Lemma 1.2 hold. If $|f''|^q$ is convex on $[a,b]$, then the following inequality holds:*

$$\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leq \frac{(b-a)^2}{48}\left(\int_0^1 \left|1-4t+3t^2\right|^p dt\right)^{\frac{1}{p}} \left[\left(\frac{|f''(a)|^q + 3|f''(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|f''(a)|^q + |f''(b)|^q}{4}\right)^{\frac{1}{q}} \right]$$
$$\leq \frac{(b-a)^2}{48}\left(4\int_0^1 \left|1-4t+3t^2\right|^p dt\right)^{\frac{1}{p}} \left[|f''(a)| + |f''(b)| \right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.5 (See [2, 4]). *Suppose that the assumptions of Lemma 1.2 hold. If $|f''|^q$ is convex on $[a,b]$, then the following inequality holds:*

$$\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$
$$\leq \frac{(b-a)^2}{162}\left[\left(\frac{59|f''(a)|^q + 133|f''(b)|^q}{3\times 2^6}\right)^{\frac{1}{q}} + \left(\frac{133|f''(a)|^q + 59|f''(b)|^q}{3\times 2^6}\right)^{\frac{1}{q}} \right].$$

It is known that fractional analysis can be considered as a generalization of classical analysis. Fractional analysis has been investigated by some researchers and they have studied the fractional derivatives and integrals in variant ways with several notations. Although the expressions of these generalized definitions can be transformed into each other, but have variant physical meanings. The popularity of this field continues to increase very strongly in resent years (see [5, 6]). Fractional derivatives are also used to model a wide range of mathematical biology, as well as physics, and engineering problems [7–9]. It is well known that the first fractional integral operator is the Riemann-Liouville fractional integral operator. Using only the derivative's fundamental limit formulation, a newly well-behaved straightforward fractional derivative known as the conformable derivative is developed in paper [10]. Furthermore, several significant requirements that can't be fulfilled by the Riemann-Liouville and Caputo definitions are fulfilled by the conformable derivative. By the way, Abdelhakim [11] shows that the conformable approach in [10] can't yield good results when compared to the Caputo definition for specific functions. This flaw in the conformable definition is avoided by sundry extensions of the conformable approach [12, 13].

The basic definitions of Riemann-Liouville integrals and conformable integrals, which are used throughout the paper, are given as follows:

The *gamma function*, *beta function*, and *incomplete beta function* are defined

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt,$$

$$\mathcal{B}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$

and

$$\mathscr{B}(x,y,r) := \int_0^r t^{x-1}(1-t)^{y-1} dt,$$

respectively for $x, y \in \mathbb{R}$. Kilbas et al. [14] presented fractional integrals, also called *Riemann-Liouville integrals* as follows:

Definition 1.6. [14] The Riemann-Liouville integrals $J_{a+}^{\beta} f(x)$ and $J_{b-}^{\beta} f(x)$ of order $\beta > 0$ are given by

$$J_{a+}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t) dt, \quad x > a \tag{1}$$

and

$$J_{b-}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b (t-x)^{\beta-1} f(t) dt, \quad x < b, \tag{2}$$

respectively. Here, $f \in L_1[a,b]$ and Γ denotes the Gamma function. The Riemann-Liouville integrals coincides with the classical integrals in the case $\beta = 1$.

The fractional version of Simpson-type inequalities for the case of twice-differentiable functions was proved in [3] as follows:

Lemma 1.7 (See [3]). *If $f : [a,b] \to \mathbb{R}$ is an absolutely continuous function on (a,b) so that $f'' \in L_1[a,b]$ with $a < b$, then the following equality holds:*

$$\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[J_{b-}^{\beta} f\left(\frac{a+b}{2}\right) + J_{a+}^{\beta} f\left(\frac{a+b}{2}\right)\right]$$
$$= \frac{(b-a)^2}{8(\beta+1)} \int_0^1 \left(\frac{1-2\beta}{3} + \frac{2(\beta+1)}{3} t - t^{\beta+1}\right)$$
$$\times \left[f''\left(\frac{1+t}{2} b + \frac{1-t}{2} a\right) + f''\left(\frac{1+t}{2} a + \frac{1-t}{2} b\right)\right] dt.$$

Theorem 1.8 (See [3]). *Let us note that the assumptions of Lemma 1.7 are valid. Let us also note that the mapping $|f''|$ is convex on $[a,b]$. Then, we have the following inequality*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[J_{b-}^{\beta} f\left(\frac{a+b}{2}\right) + J_{a+}^{\beta} f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{(b-a)^2}{8(\beta+1)}\Omega_1(\beta)\left[\left|f''(a)\right|+\left|f''(b)\right|\right],$$

where Ω_1 is defined by

$$\Omega_1(\beta)=\int_0^1\left|\frac{1-2\beta}{3}+\frac{2(\beta+1)}{3}t-t^{\beta+1}\right|dt.$$

Theorem 1.9 (See [3]). *Let us consider that the assumptions of Lemma 1.7 hold. If the mapping $|f''|^q$, $q>1$ is convex on $[a,b]$, then we have the following inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[J_{b-}^{\beta}f\left(\frac{a+b}{2}\right)+J_{a+}^{\beta}f\left(\frac{a+b}{2}\right)\right]\right|\\
&\leq \frac{(b-a)^2}{8(\beta+1)}\left(\int_0^1\left|\frac{1-2\beta}{3}+\frac{2(\beta+1)}{3}t-t^{\beta+1}\right|^p dt\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right],\\
&\leq \frac{(b-a)^2}{8(\beta+1)}\left(4\int_0^1\left|\frac{1-2\beta}{3}+\frac{2(\beta+1)}{3}t-t^{\beta+1}\right|^p dt\right)^{\frac{1}{p}}\left[\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]^{\frac{1}{q}}.\end{aligned}$$

Here, $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.10 (See [3]). *Suppose that the assumptions of Lemma 1.7 hold. If the mapping $|f''|^q$, $q\geq 1$ is convex on $[a,b]$, then we have the following inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[J_{b-}^{\beta}f\left(\frac{a+b}{2}\right)+J_{a+}^{\beta}f\left(\frac{a+b}{2}\right)\right]\right|\\
&\leq \frac{(b-a)^2}{8(\beta+1)}\left(\Omega_1(\beta)\right)^{1-\frac{1}{q}}\left\{\left(\frac{(\Omega_1(\beta)+\Omega_2(\beta))\left|f''(b)\right|^q+(\Omega_1(\beta)-\Omega_2(\beta))\left|f''(a)\right|^q}{2}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{(\Omega_1(\beta)+\Omega_2(\beta))\left|f''(a)\right|^q+(\Omega_1(\beta)-\Omega_2(\beta))\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}\right\},\end{aligned}$$

where Ω_1 is defined as in Theorem 1.8 and Ω_2 is defined by

$$\Omega_2(\beta)=\int_0^1\left|\frac{1-2\beta}{3}+\frac{2(\beta+1)}{3}t-t^{\beta+1}\right|tdt.$$

Remark 1.11. For classical integrals,

(i) If we choose $\beta=1$, then Lemma 1.7 coincides with Lemma 1.2.
(ii) Let us consider $\beta=1$. Then, Theorem 1.8 becomes to Theorem 1.3.
(iii) For $\beta=1$, Theorem 1.9 leads to Theorem 1.4.
(iv) Considering $\beta=1$, then Theorem 1.10 reduces to Theorem 1.5.

In paper [15], Jarad et al. established the following fractional conformable integral operators. They also derived certain characteristics and relationships between these operators and several other fractional operators in the literature. The fractional conformable integral operators are defined by as follows.

DEFINITION 1.12. [15] The fractional conformable integral operator ${}_{a}^{\beta}J^{\alpha}f(x)$ and ${}^{\beta}J_{b}^{\alpha}f(x)$ of order $\beta \in \mathbb{R}^{+}$ and $\alpha \in (0,1]$ are given by

$$
{}^{\beta}\mathcal{J}_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\beta)}\int_{a}^{x}\left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{(t-a)^{1-\alpha}}dt, \qquad t>a, \tag{3}
$$

and

$$
{}^{\beta}\mathcal{J}_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\beta)}\int_{x}^{b}\left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{(b-t)^{1-\alpha}}dt, \qquad t<b, \tag{4}
$$

respectively for $f \in L_1[a,b]$.

Consider that the fractional integral in (3) becomes to the Riemann-Liouville fractional integral in (1) if we choose $\alpha = 1$. Moreover, the fractional integral in (4) is equal to the Riemann-Liouville fractional integral in (2) if we take $\alpha = 1$. It is referred the reader to [16–19] for a better understanding of fractional integral inequalities.

This article is organized according to the following plan: In section 2, an equality will be established for the case of twice-differentiable functions by the conformable fractional integrals. Furthermore, we will also show that the newly established equalities are the generalization of the existing Simpson-type inequalities. Finally, summary and concluding remarks are presented in Section 3.

2. Main results

LEMMA 2.1. *Note that $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function on (a,b) so that $f''\in L_1[a,b]$ with $a<b$. Then, the following equality holds:*

$$
\begin{aligned}
&\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\\
&=\frac{(b-a)^{2}\alpha^{\beta}}{4}\left[I_1+I_2\right],
\end{aligned} \tag{5}
$$

where

$$
\begin{cases}
I_1=\int\limits_0^1\left(\int\limits_t^1\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right)f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt,\\[2ex]
I_2=\int\limits_0^1\left(\int\limits_t^1\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right)f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt.
\end{cases}
$$

Proof. Let us use the facts of the fundamental rules of integration by parts. Then, it yields

$$
I_1=\int\limits_0^1\left(\int\limits_t^1\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right)f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt
$$

$$
\begin{aligned}
&= \frac{2}{b-a}\left(\int_{t}^{1}\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right)f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\Bigg|_{0}^{1}\\
&\quad -\frac{2}{b-a}\int_{0}^{1}\left[\frac{1}{2}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\\
&= -\frac{2}{b-a}\left(\int_{0}^{1}\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right)f'\left(\frac{a+b}{2}\right)\\
&\quad -\frac{2}{b-a}\Bigg\{\frac{2}{b-a}\left[\frac{1}{2}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\Bigg|_{0}^{1}\\
&\quad -\frac{\beta}{b-a}\int_{0}^{1}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\Bigg\}\\
&= -\frac{2}{b-a}\left(\int_{0}^{1}\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right)f'\left(\frac{a+b}{2}\right)\\
&\quad -\frac{2}{(b-a)^{2}3\alpha^{\beta}}\left(2f\left(\frac{a+b}{2}\right)+f(b)\right)\\
&\quad +\frac{2\beta}{(b-a)^{2}}\int_{\frac{a+b}{2}}^{b}\left(\frac{1-\left(\frac{2}{b-a}\right)^{\alpha}(b-x)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\frac{2}{b-a}\right)^{\alpha}\frac{f(x)}{(b-x)^{1-\alpha}}dx\\
&= -\frac{2}{b-a}\left(\int_{0}^{1}\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right)f'\left(\frac{a+b}{2}\right)\\
&\quad -\frac{2}{(b-a)^{2}3\alpha^{\beta}}\left(2f\left(\frac{a+b}{2}\right)+f(b)\right)\\
&\quad +\left(\frac{2}{b-a}\right)^{\alpha\beta+1}\frac{\Gamma(\beta+1)}{(b-a)}\frac{1}{\Gamma(\beta)}\int_{\frac{a+b}{2}}^{b}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(b-x)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(x)}{(b-x)^{1-\alpha}}dx.
\end{aligned}
$$

Hence, we obtain

$$
I_1 = -\frac{2}{b-a}\left(\int_{0}^{1}\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right)f'\left(\frac{a+b}{2}\right)
$$

(6)

$$
-\frac{2}{(b-a)^{2}3\alpha^{\beta}}\left(2f\left(\frac{a+b}{2}\right)+f(b)\right)+\left(\frac{2}{b-a}\right)^{\alpha\beta+1}\frac{\Gamma(\beta+1)}{(b-a)}\,{}^{\beta}\mathcal{J}_{b-}^{\alpha}f\left(\frac{a+b}{2}\right).
$$

Then, similar to foregoing process, we have

$$I_2 = \frac{2}{b-a}\left(\int_0^1 \left[\frac{1}{3\alpha^\beta} - \frac{1}{2}\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right] ds\right) f'\left(\frac{a+b}{2}\right)$$

(7)

$$-\frac{2}{(b-a)^2 3\alpha^\beta}\left(2f\left(\frac{a+b}{2}\right) + f(a)\right) + \left(\frac{2}{b-a}\right)^{\alpha\beta+1} \frac{\Gamma(\beta+1)}{(b-a)}\, {}^\beta\mathcal{J}^\alpha_{a+} f\left(\frac{a+b}{2}\right).$$

If we substitute equalities (6) and (7) and multiply $\frac{(b-a)^2\alpha^\beta}{4}$ simultaneously, then we can easily have

$$\frac{(b-a)^2\alpha^\beta}{4}\left[I_1 + I_2\right] = \frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{b-} f\left(\frac{a+b}{2}\right) + {}^\beta\mathcal{J}^\alpha_{a+} f\left(\frac{a+b}{2}\right)\right]$$

$$-\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

This ends the proof of Lemma 2.1. □

REMARK 2.2. In Lemma 2.1, we have the equalities as follows:

(i) If we assign $\alpha = 1$ in (5), then Lemma 2.1 equals to Lemma 1.7.

(ii) Let us note $\alpha = 1$ and $\beta = 1$ in (5). Then, Lemma 2.1 becomes to Lemma 1.2.

THEOREM 2.3. *Consider that $f : [a,b] \to \mathbb{R}$ is a differentiable function on (a,b) so that $f'' \in L_1[a,b]$. If $|f''|$ is convex on $[a,b]$, then the following inequality*

$$\left|\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{b-} f\left(\frac{a+b}{2}\right) + {}^\beta\mathcal{J}^\alpha_{a+} f\left(\frac{a+b}{2}\right)\right] - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right|$$

$$\leq \frac{(b-a)^2\alpha^\beta}{4}\varphi_1(\alpha,\beta)\left[|f''(a)| + |f''(b)|\right]$$

(8)

is valid. Here,

$$\varphi_1(\alpha,\beta) = \int_0^1 \left|\int_t^1 \left[\frac{1}{3\alpha^\beta} - \frac{1}{2}\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right] ds\right| dt.$$

Proof. If we take the modules of both sides of (5), then we obtain

$$\left|\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{b-} f\left(\frac{a+b}{2}\right) + {}^\beta\mathcal{J}^\alpha_{a+} f\left(\frac{a+b}{2}\right)\right] - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right|$$

$$\leq \frac{(b-a)^2\alpha^\beta}{4}\left\{\int_0^1 \left|\int_t^1 \left[\frac{1}{3\alpha^\beta} - \frac{1}{2}\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right] ds\right| \left|f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right| dt\right.$$

$$\left. + \int_0^1 \left|\int_t^1 \left[\frac{1}{3\alpha^\beta} - \frac{1}{2}\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right] ds\right| \left|f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right| dt\right\}.$$

(9)

It is known that $|f''|$ is convex on $[a,b]$. Then, it yields

$$\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$
$$\leq\frac{(b-a)^{2}\alpha^{\beta}}{4}\left\{\int_{0}^{1}\left|\int_{t}^{1}\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|\left[\frac{1-t}{2}\left|f''(a)\right|+\frac{1+t}{2}\left|f''(b)\right|\right]dt\right.$$
$$\left.+\int_{0}^{1}\left|\int_{t}^{1}\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|\left[\frac{1+t}{2}\left|f''(a)\right|+\frac{1-t}{2}\left|f''(b)\right|\right]dt\right\}.$$

Hence, the proof of Theorem 2.3 is completed. □

Remark 2.4. In Theorem 2.3, we get the inequalities as follows:

(i) If it is chosen $\alpha=1$ in (8), then Theorem 2.3 reduces to Theorem 1.8.

(ii) For $\alpha=1$ and $\beta=1$ in (8), then Theorem 2.3 is equal to Theorem 1.3.

Theorem 2.5. *If $f:[a,b]\to\mathbb{R}$ is a differentiable function on (a,b) such that $f''\in L_1([a,b])$ and $|f''|^q$ is convex on $[a,b]$ with $q>1$, then the following inequality holds:*

$$\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$
$$\leq\frac{(b-a)^{2}\alpha^{\beta}}{4}\left(\psi^{\beta}_{\alpha}(p)\right)^{\frac{1}{p}}\left[\left(\frac{|f''(a)|^{q}+3|f''(b)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f''(a)|^{q}+|f''(b)|^{q}}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{(b-a)^{2}\alpha^{\beta}}{4}\left(4\psi^{\beta}_{\alpha}(p)\right)^{\frac{1}{p}}\left[|f''(a)|^{q}+|f''(b)|^{q}\right]. \tag{10}$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\psi^{\beta}_{\alpha}(p)=\int_{0}^{1}\left|\int_{t}^{1}\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|^{p}dt.$$

Proof. Let us apply Hölder inequality in (9). Then, it yields

$$\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$
$$\leq\frac{(b-a)^{2}\alpha^{\beta}}{4}\left(\int_{0}^{1}\left|\int_{t}^{1}\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|^{p}dt\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_{0}^{1}\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{0}^{1}\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right].$$

From the fact that $|f''|^q$ is convex on $[a,b]$, we have

$$\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$

$$\leq \frac{(b-a)^2\alpha^\beta}{4}\left(\int_0^1\left|\int_t^1\left[\frac{1}{3\alpha^\beta}-\frac{1}{2}\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_0^1\left(\frac{1-t}{2}\left|f''(a)\right|^q+\frac{1+t}{2}\left|f''(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left(\frac{1+t}{2}\left|f''(a)\right|^q+\frac{1-t}{2}\left|f''(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right]$$

$$=\frac{(b-a)^2\alpha^\beta}{4}\left(\int_0^1\left|\int_t^1\left[\frac{1}{3\alpha^\beta}-\frac{1}{2}\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\left[\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right].$$

Consider $\eta_1=\left|f''(a)\right|^q$, $\varrho_1=3\left|f''(b)\right|^q$, $\eta_2=3\left|f''(a)\right|^q$, and $\varrho_2=\left|f''(b)\right|^q$. If we apply the inequality $\sum_{k=1}^n(\eta_k+\varrho_k)^s\leq\sum_{k=1}^n\eta_k^s+\sum_{k=1}^n\varrho_k^s$ with $0\leq s<1$, then the proof of Theorem 2.5 is finished simultaneously. □

REMARK 2.6. In Theorem 2.5, we have the inequalities as follows:

(i) If we take $\alpha=1$ in (10), then Theorem 2.5 coincides with to Theorem 1.9.
(ii) Let us consider $\alpha=1$ and $\beta=1$ in (10). Then, Theorem 2.5 leads to Theorem 1.4.

THEOREM 2.7. *Let $f:[a,b]\to\mathbb{R}$ denote a differentiable function on (a,b) such that $f''\in L_1([a,b])$ and $|f''|^q$ be convex on $[a,b]$ with $q\geq 1$. Then, the following inequality holds:*

$$\left|\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^\beta\mathcal{J}_{b-}^\alpha f\left(\frac{a+b}{2}\right)+{}^\beta\mathcal{J}_{a+}^\alpha f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$

$$\leq\frac{(b-a)^2\alpha^\beta}{4}\left(\varphi_1(\alpha,\beta)\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\frac{(\varphi_1(\alpha,\beta)-\varphi_2(\alpha,\beta))}{2}\left|f''(a)\right|^q+\frac{(\varphi_1(\alpha,\beta)+\varphi_2(\alpha,\beta))}{2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right.$$

(11)

$$\left.+\left(\frac{(\varphi_1(\alpha,\beta)+\varphi_2(\alpha,\beta))}{2}\left|f''(a)\right|^q+\frac{(\varphi_1(\alpha,\beta)-\varphi_2(\alpha,\beta))}{2}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right],$$

where

$$\varphi_2(\alpha,\beta)=\int_0^1 t\left|\int_t^1\left[\frac{1}{3\alpha^\beta}-\frac{1}{2}\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta\right]ds\right|dt.$$

Proof. Applying the power-mean inequality in (9), we have

$$\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$

$$\leq\frac{(b-a)^{2}\alpha^{\beta}}{4}\left(\int_{0}^{1}\left|\int_{t}^{1}\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_{0}^{1}\left|\int_{t}^{1}\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{0}^{1}\left|\int_{t}^{1}\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right].$$

It is known that $|f''|^{q}$ is convex on $[a,b]$. Then, we have

$$\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\mathcal{J}_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$

$$\leq\frac{(b-a)^{2}\alpha^{\beta}}{4}\left(\int_{0}^{\frac{1}{2}}\left|\int_{0}^{t}\left[\left(\frac{1-(1-2s)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right]ds\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_{0}^{1}\left|\int_{t}^{1}\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|\left(\frac{1-t}{2}\left|f''(a)\right|^{q}+\frac{1+t}{2}\left|f''(b)\right|^{q}\right)dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{0}^{1}\left|\int_{t}^{1}\left[\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|\left(\frac{1+t}{2}\left|f''(a)\right|^{q}+\frac{1-t}{2}\left|f''(b)\right|^{q}\right)dt\right)^{\frac{1}{q}}\right]$$

$$=\frac{(b-a)^{2}\alpha^{\beta}}{4}\left(\varphi_{1}(\alpha,\beta)\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\frac{(\varphi_{1}(\alpha,\beta)-\varphi_{2}(\alpha,\beta))}{2}\left|f''(a)\right|^{q}+\frac{(\varphi_{1}(\alpha,\beta)+\varphi_{2}(\alpha,\beta))}{2}\left|f''(b)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(\varphi_{1}(\alpha,\beta)+\varphi_{2}(\alpha,\beta))}{2}\left|f''(a)\right|^{q}+\frac{(\varphi_{1}(\alpha,\beta)-\varphi_{2}(\alpha,\beta))}{2}\left|f''(b)\right|^{q}\right)^{\frac{1}{q}}\right],$$

which completes the proof of Theorem 2.7. □

Remark 2.8. In Theorem 2.7, we obtain the inequalities as follows:

(i) Consider $\alpha=1$ in (11). Then, Theorem 2.7 reduces to Theorem 1.10.

(ii) If we select $\alpha=1$ and $\beta=1$ in (11), then Theorem 2.7 is equal to Theorem 1.5.

3. Summary & concluding remarks

In the present paper, we have established an equality for the case of twice-differentiable convex functions by using the conformable fractional integrals. In addition to this, sundry Simpson-type inequalities are proved with respect to twice-differentiable functions. Moreover, several important inequalities are acquire with taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Furthermore, we derive our results by using special cases of obtained theorems.

We hope that the ideas and techniques of this paper will inspire to mathematicians working in this field. With the techniques used in the obtained inequalities, various types of fractional integrals can be used to obtain new inequalities in the future. In addition, new inequalities can be acquired by considering different order derivatives of the functions. Furthermore, one can obtain sundry Simpson-type inequalities for the case of convex functions by using quantum calculus.

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