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STABILITY AND SOLUTION OF TWO FUNCTIONAL EQUATIONS IN UNITAL ALGEBRAS

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Abstract. In this paper, we consider two functional equations:

$$\begin{aligned} &h(\mathcal{F}(x,y,z)+2x+y+z)+h(xy+z)+yh(x)+yh(z) \\ &= h(\mathcal{F}(x,y,z)+2x+y)+h(xy)+yh(x+z)+2h(z), \end{aligned} \tag{1}$$

$$\begin{aligned} &h(\mathcal{F}(x,y,z)-y+z+2e)+2h(x+y)+h(xy+z)+yh(x)+yh(z) \\ &= h(\mathcal{F}(x,y,z)-y+2e)+2h(x+y+z)+h(xy)+yh(x+z), \end{aligned} \tag{2}$$

without any regularity assumption for all x,y,z in a unital algebra A, where $\mathcal{F}: A^3 \to A$ is defined by

$$\mathcal{F}(x,y,z) := h(x+y+z) - h(x+y) - h(z)$$

for all $x,y,z \in A$. Also, we find general solutions of these equations in unital algebras. Finally, we prove the Hyers-Ulam stability of (1) and (2) in unital Banach algebras.

1. Introduction

The stability problem of functional equations started from a question of Ulam [24] in 1940, concerning the stability of group homomorphisms. In 1941, Hyers [10] gave an answer to the question of Ulam in the context of Banach spaces in the case of additive mappings, that was an important step toward more solutions in this field.

The theory of stability is an important branch of the qualitative theory of differential equations. During the last decades, many interesting results have been investigated on different types differential equations and system of additive functional equations (for more details, see [11], [13], [15], [16], [21], [22]).

The method provided by Hyers [10] which produces the additive function will be called a direct method. This method is the most important and useful tool to study the stability of different functional equations.

During the last two decades, a number of articles and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability

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to a number of functional equations and mappings, for example, Cauchy-Jensen mappings, k-additive mappings, invariant means, multiplicative mappings, bounded nth differences, generalized orthogonality mappings, differential equations, and Navier-Stokes equations (see [9], [12], [14], [18], [19]).

In recent years, the stability of different (others functional, differential and integral) equations and other subjects has been intensively studied (see [2], [3], [4], [5], [7], [20], [23]).

Let X and Y denote vector spaces. A mapping $Q : X \to Y$ is said to be quadratic if it satisfies

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in X$.

To obtain a Jordan and von Neumann type characterization theorem for the quasi-inner-product spaces, Drygas [6] considered the functional equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y)$$

for all $x, y \in \mathbb{R}$. However, the general solution of this functional equation was given by Ebanks, Kannappan and Sahoo [8] as

$$f(x) = Q(x) + A(x)$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and $Q : \mathbb{R} \to \mathbb{R}$ is a quadratic function.

Lemma 1.1. *[25, Proposition 1] Let X be a real vector space and Y be a Banach space. If a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \le \delta$$

for some $\delta > 0$ and for all $x, y \in X$, then there exist a unique additive mapping $\mathcal{D} : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x) - \mathcal{D}(x)\| \le \frac{3}{2}\delta$$

for all $x \in X$.

The main goal of this paper is to present the general solutions of (1) and (2) without any regularity condition and prove the Hyers-Ulam stability of (1) and (2) in unital Banach algebras.

Throughout this paper, let A be a unital algebra with unit e.

2. General solutions of (1) and (2) in unital algebras

In this section, we stablish the general solutions of (1) and (2) without any regularity condition.

Theorem 2.1. *Let A be a unital algebra. A mapping $h : A \to A$ satisfies the functional equation* (1) *for all $x, y, z \in A$ if and only if h is additive.*

Proof. It is clear that the additivity implies (1).

We assume that h satisfies (1). Then we shall show that h is additive. Suppose that h is any mapping satisfying (1). Then setting $y = 3e$ and $z = 0$ in (1), we obtain $h(0) = 0$.

Letting $y = e$ in (1), we get

$$h\left[\mathcal{F}(x,e,z)+2x+e+z\right]=h\left[\mathcal{F}(x,e,z)+2x+e\right]+h(z) \tag{3}$$

for all $x, z \in A$.

Letting $x = -e$ in (3), we get

$$h(z-e)=h(z)+h(-e) \tag{4}$$

for all $z \in A$.

Letting $x = e$ in (1), we obtain

$$\begin{aligned} &h\left[\mathcal{F}(e,y,z)+2e+y+z\right]+h(y+z)+yh(e)+yh(z)\\ &=h\left[\mathcal{F}(e,y,z)+2e+y\right]+h(y)+yh(e+z)+2h(z) \end{aligned} \tag{5}$$

for all $y, z \in A$.

Letting $y = 0$ and replacing x by $x + e$ in (1), we obtain

$$h\left[\mathcal{F}(x+e,0,z)+2x+2e+z\right]=h\left[\mathcal{F}(x+e,0,z)+2x+2e\right]+h(z)$$

for all $x, z \in A$. Therefore,

$$h\left[\mathcal{F}(x,e,z)+2x+2e+z\right]=h\left[\mathcal{F}(x,e,z)+2x+2e\right]+h(z) \tag{6}$$

for all $x, z \in A$, since $\mathcal{F}(x+e,0,z)=\mathcal{F}(x,e,z)$.

Since $h(z-e)=h(z)+h(-e)$ and $\mathcal{F}(e,-e,z)=0$, letting $y=-e$ in (5), we have

$$h(z+e)=h(z)+h(e) \tag{7}$$

for all $z \in A$.

Hence, by (5) and (7), we obtain

$$h\left[\mathcal{F}(e,y,z)+2e+y+z\right]+h(y+z)=h\left[\mathcal{F}(e,y,z)+2e+y\right]+h(y)+2h(z) \tag{8}$$

for all $y, z \in A$.

Let $y_0=\mathcal{F}(x,e,z)+2x+e$. By (3) and (6), we get

$$h(y_0+z)=h(y_0)+h(z) \tag{9}$$

and

$$h(y_0+e+z)=h(y_0+e)+h(z). \tag{10}$$

Accordingly, by (8), if both

$$h(y+z)=h(y)+h(z)$$

and

$$h(y+e+z)=h(y+e)+h(z)$$

(equivalently, $\mathcal{F}(e,y,z)=0$) hold, then

$$h(y+2e+z)=h(y+2e)+h(z)$$

for all $y, z \in A$. Thus, it follows from (9) and (10) that

$$h(y_0+2e+z)=h(y_0+2e)+h(z),$$

where $y_0=\mathcal{F}(x,e,z)+2x+e$. That is,

$$h\left[\mathcal{F}(x,e,z)+2x+3e+z\right]=h\left[\mathcal{F}(x,e,z)+2x+3e\right]+h(z) \tag{11}$$

for all $x, z \in A$.

Replacing x by $x-2e$ in (11), we obtain

$$h\left[\mathcal{F}(x-2e,e,z)+2x-e+z\right]=h\left[\mathcal{F}(x-2e,e,z)+2x-e\right]+h(z)$$

for all $x,z\in A$. Since $\mathcal{F}(x-2e,e,z)=\mathcal{F}(x,-e,z)$ for all $x,z\in A$,

$$h\left[\mathcal{F}(x,-e,z)+2x-e+z\right]=h\left[\mathcal{F}(x,-e,z)+2x-e\right]+h(z) \tag{12}$$

for all $x,z\in A$.

On the other hand, setting $y=-e$ in (1), we get

$$\begin{aligned} &h\left[\mathcal{F}(x,-e,z)+2x-e+z\right]+h(-x+z)-h(x)\\ &=h\left[\mathcal{F}(x,-e,z)+2x-e\right]+h(-x)-h(x+z)+3h(z) \end{aligned} \tag{13}$$

for all $x,z\in A$.

It follows from (12) and (13) that

$$h(-x+z)-h(-x)-h(z)=-h(x+z)+h(x)+h(z)$$

for all $x,z\in A$. This means that

$$\mathcal{F}(-x,-y,z)=-\mathcal{F}(x,y,z) \tag{14}$$

for all $x,y,z\in A$.

Finally, replacing y by $-y$ and letting $x=-e$ in (1) and using (4), we get

$$\begin{aligned} &h\left[\mathcal{F}(-e,-y,z)-y+z-2e\right]+h(y+z)\\ &=h\left[\mathcal{F}(-e,-y,z)-y-2e\right]+h(y)+2h(z) \end{aligned} \tag{15}$$

for all $y,z\in A$, since $h(z-e)=h(z)+h(-e)$.

It follows from (14) and (15) that

$$h\left[-\mathcal{F}(e,y,z)-y+z-2e\right]+h(y+z)=h\left[-\mathcal{F}(e,y,z)-y-2e\right]+h(y)+2h(z) \tag{16}$$

for all $y,z\in A$.

Also, we have

$$\begin{aligned} &\mathcal{F}[-\mathcal{F}(e,y,z)-y-2e,0,z]\\ &=h\left[-\mathcal{F}(e,y,z)-y+z-2e\right]-h\left[-\mathcal{F}(e,y,z)-y-2e\right]-h(z) \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\mathcal{F}[\mathcal{F}(e,y,z)+y+2e,0,z]\\ &=h\left[\mathcal{F}(e,y,z)+y+z+2e\right]-h\left[\mathcal{F}(e,y,z)+y+2e\right]-h(z) \end{aligned} \tag{18}$$

for all $y,z\in A$. By (14), (17) and (18), we conclude that

$$\begin{aligned} &h\left[-\mathcal{F}(e,y,z)-y+z-2e\right]-h\left[-\mathcal{F}(e,y,z)-y-2e\right]-h(z)\\ &=\mathcal{F}[-\mathcal{F}(e,y,z)-y-2e,0,z]\\ &=-\mathcal{F}[\mathcal{F}(e,y,z)+y+2e,0,z]\\ &=-h\left[\mathcal{F}(e,y,z)+y+z+2e\right]+h\left[\mathcal{F}(e,y,z)+y+2e\right]+h(z) \end{aligned} \tag{19}$$

for all $y,z\in A$.

It follows from (16) and (19) that

$$h\left[\mathcal{F}(e,y,z)+2e+y+z\right]+h(y)=h\left[\mathcal{F}(e,y,z)+2e+y\right]+h(y+z) \tag{20}$$

for all $z,y\in A$.

It follows from (8) and (20) that $h(y+z)=h(y)+h(z)$ for all $z,y\in A$. $\square$

LEMMA 2.2. [17] *Let A be a complex Banach algebra and $f : A \to A$ be an additive mapping such that $f(\lambda x) = \lambda f(x)$ for all $\lambda \in \mathbb{T}^1$ and all $x \in A$, then f is $\mathbb{C}$-linear.*

THEOREM 2.3. *Let A be a unital algebra. Then a mapping $h : A \to A$ is $\mathbb{C}$-linear if and only if*

$$\begin{aligned} &h\left[\lambda(\mathcal{F}(x,y,z) + 2x + y + z)\right] + \lambda h(xy + z) + \lambda y h(x) + y h(\lambda z) \\ &= h\left[\lambda(\mathcal{F}(x,y,z) + 2x + y)\right] + \lambda h(xy) + \lambda y h(x + z) + 2h(\lambda z) \end{aligned} \tag{21}$$

for all $x, y, z \in A$ and all $\lambda \in \mathbb{T}^1$.

Proof. It is clear that the $\mathbb{C}$-linearity implies (21).

Assume that h satisfies (21). Then we shall show that h is $\mathbb{C}$-linear. It is clear that, for $\lambda = 1$, by Theorem 2.1, we conclude that h is additive. Suppose that h is any mapping satisfying (21). Then setting $x = -2e$ and $y = 2e$ in (21), we obtain

$$\begin{aligned} &h(\lambda(z - 2e)) + \lambda h(z - 4e) + 2\lambda h(-2e) + 2h(\lambda z) \\ &= h(-2\lambda e) + \lambda h(-4e) + 2\lambda h(z - 2e) + 2h(\lambda z) \end{aligned}$$

for all $z \in A$ and all $\lambda \in \mathbb{T}^1$.

Since h is additive, we have

$$h(\lambda z) = \lambda h(z)$$

for all $z \in A$ and all $\lambda \in \mathbb{T}^1$. Hence by Lemma 2.2, h is $\mathbb{C}$-linear. □

THEOREM 2.4. *Let A be a unital algebra. A mapping $h : A \to A$ satisfies the functional equation* (2) *for all $x, y, z \in A$ if and only if h is additive.*

Proof. It is clear that additivity implies (2). Assume that h satisfies (2). Then we shall show that h is additive. Suppose that h is any mapping satisfying (2). Then setting $y = e$ and $z = 0$ in (2), we obtain $h(0) = 0$.

Letting $y = e$ in (2), we have

$$h\left[\mathcal{F}(x,e,z) + z + e\right] + 2h(x + e) + h(z) = h\left[\mathcal{F}(x,e,z) + e\right] + 2h(x + z + e)$$

for all $x, z \in A$. Hence

$$h\left[\mathcal{F}(x,e,z) + z + e\right] - h\left[\mathcal{F}(x,e,z) + e\right] = 2h(x + z + e) - 2h(x + e) - h(z) \tag{22}$$

for all $x, z \in A$.

Letting $x = 0$ in (2), we get

$$h\left[\mathcal{F}(0,y,z) - y + z + 2e\right] + 2h(y) + h(z) = h\left[\mathcal{F}(0,y,z) - y + 2e\right] + 2h(y + z) \tag{23}$$

for all $y, z \in A$, since $h(0) = 0$.

Replacing y by $\mathcal{F}(x,e,z) + e$ in (23), we obtain

$$\begin{aligned} &h\left[\mathcal{F}(0, \mathcal{F}(x,e,z) + e, z) - \mathcal{F}(x,e,z) + z + e\right] + 2h(\mathcal{F}(x,e,z) + e) + h(z) \\ &= h\left[\mathcal{F}(0, \mathcal{F}(x,e,z) + e, z) - \mathcal{F}(x,e,z) + e\right] + 2h(\mathcal{F}(x,e,z) + e + z) \end{aligned} \tag{24}$$

for all $x, z \in A$. Since

$$\mathcal{F}(0, \mathcal{F}(x,e,z) + e, z) = h[\mathcal{F}(x,e,z) + z + e] - h[\mathcal{F}(x,e,z) + e] - h(z),$$

(24) yields

$$
\begin{aligned}
&h\big(h[\mathcal{F}(x,e,z)+z+e]-h[\mathcal{F}(x,e,z)+e]-h(z)-\mathcal{F}(x,e,z)+z+e)\\
&+2h[\mathcal{F}(x,e,z)+e]+h(z)\\
=&h\big(h[\mathcal{F}(x,e,z)+z+e]-h[\mathcal{F}(x,e,z)+e]-h(z)-\mathcal{F}(x,e,z)+e\big)\\
&+2h[\mathcal{F}(x,e,z)+z+e] \qquad (25)
\end{aligned}
$$

for all $x, z \in A$.

Thus, it follows from (22) and (25) that

$$
\begin{aligned}
&h\big(2h(x+z+e)-2h(x+e)-h(z)-h(z)-\mathcal{F}(x,e,z)+z+e)\\
&+2h[\mathcal{F}(x,e,z)+e]+h(z)\\
=&h\big(2h(x+z+e)-2h(x+e)-h(z)-h(z)-\mathcal{F}(x,e,z)+e\big)\\
&+2h[\mathcal{F}(x,e,z)+z+e]
\end{aligned}
$$

for all $x, z \in A$. Hence

$$
\begin{aligned}
&h\big[h(x+z+e)-h(x+e)-h(z)+z+e\big]\\
&+2h[h(x+z+e)-h(x+e)-h(z)+e]+h(z)\\
&=h\big[h(x+z+e)-h(x+e)-h(z)+e\big]\\
&+2h[h(x+z+e)-h(x+e)-h(z)+z+e]
\end{aligned}
$$

for all $x, z \in A$, since $\mathcal{F}(x,e,z) = h(x+e+z) - h(x+e) - h(z)$. Therefore,

$$
\begin{aligned}
&h\left[h(x+z+e)-h(x+e)-h(z)+z+e\right]\\
&=h\left[h(x+z+e)-h(x+e)-h(z)+e\right]+h(z)
\end{aligned}
$$

for all $x, z \in A$. That is,

$$
h\left[\mathcal{F}(x,e,z)+z+e\right]-h\left[\mathcal{F}(x,e,z)+e\right]=h(z) \qquad (26)
$$

for all $x, z \in A$.

Thus, it follows from (22) and (26) that

$$
2h(x+z+e)-2h(x+e)-h(z)=h(z)
$$

for all $x, z \in A$. So

$$
h(x+z+e)=h(x+e)+h(z)
$$

for all $x, z \in A$ and thus h is additive. □

Theorem 2.5. *Let A be a unital algebra. Then a mapping $h : A \to A$ is $\mathbb{C}$-linear if and only if*

$$
\begin{aligned}
&h\left[\lambda(\mathcal{F}(x,y,z)-y+z+2e)\right]+2\lambda h(x+y)+\lambda h(xy+z)+yh(\lambda x)+yh(\lambda z)\\
&=h\left[\lambda(\mathcal{F}(x,y,z)-y+2e)\right]+2\lambda h(x+y+z)+\lambda h(xy)+yh(\lambda(x+z)) \qquad (27)
\end{aligned}
$$

for all $x, y, z \in A$ and all $\lambda \in \mathbb{T}^1$.

Proof. It is clear that the $\mathbb{C}$-linearity implies (27).

Assume that h satisfies (27). Then we shall show that h is $\mathbb{C}$-linear. It is clear that, for $\lambda = 1$, by Theorem 2.4, we conclude that h is additive. Suppose that h is any mapping satisfying (21). Then setting $x = -2e$ and $y = 2e$ in (27), we obtain

$$\begin{aligned} &h(\lambda z) + \lambda h(z - 4e) + 2h(-2\lambda e) + 2h(\lambda z) \\ &= 2\lambda h(z) + \lambda h(-4e) + 2h(\lambda(z - 2e)) \end{aligned}$$

for all $z \in A$ and all $\lambda \in \mathbb{T}^1$.

Since h is additive, we have

$$h(\lambda z) = \lambda h(z)$$

for all $z \in A$ and all $\lambda \in \mathbb{T}^1$. Hence by Lemma 2.2 h is $\mathbb{C}$-linear. □

3. Stability of (1) and (2)

In this section, we prove the Hyers-Ulam stability of the functional equations (1) and (2) in unital Banach algebras.

For a given mapping $h : A \to A$, we define

$$\begin{aligned} \Delta h(x, y, z) :=& h[\mathcal{F}(x, y, z) + 2x + y + z] + h(xy + z) + yh(x) + yh(z) \\ &- h[\mathcal{F}(x, y, z) + 2x + y] - h(xy) - yh(x + z) - 2h(z) \end{aligned}$$

and

$$\begin{aligned} \Gamma h(x, y, z) :=& h[\mathcal{F}(x, y, z) - y + z + 2e] + 2h(x + y) + h(xy + z) + yh(x) + yh(z) \\ &- h[\mathcal{F}(x, y, z) - y + 2e] - 2h(x + y + z) - h(xy) - yh(x + z) \end{aligned}$$

for all $x, y, z \in A$.

Theorem 3.1. *Let A be a unital Banach algebra and $h : A \to A$ be a mapping satisfying*

$$\begin{gathered} \mathcal{F}(x, y, z) = -2x - y, \\ \|\Delta h(x, y, z)\| \le \delta \end{gathered} \tag{28}$$

for all $x, y, z \in A$. Then there exist a unique additive mapping $\mathcal{D} : A \to A$ and a unique quadratic mapping $Q : A \to A$ such that

$$\|h(x) - Q(x) - \mathcal{D}(x)\| \le \frac{25}{2}\delta$$

for all $x \in A$.

Proof. Letting $x = y = z = 0$ in (28), we get $\|h(0)\| \le \frac{\delta}{2}$. Letting $y = -e$ in (28), we obtain

$$\|h(z + x) + h(z - x) - 2h(z) - h(x) - h(-x) - h(0)\| \le \delta.$$

Since $\|h(0)\| \le \frac{\delta}{2}$,

$$\|h(z + x) + h(z - x) - 2h(z) - h(x) - h(-x)\| \le \frac{3}{2}\delta \tag{29}$$

for all $x, z \in A$. By Lemma 1.1 there exist a unique additive mapping $\mathcal{D} : A \to A$ and a unique quadratic mapping $Q : A \to A$ such that

$$\|h(x) - Q(x) - \mathcal{D}(x)\| \le \frac{9}{4}\delta$$

for all $x \in A$. $\square$

THEOREM 3.2. *Let A be a unital Banach algebra and $h : A \to A$ be an odd mapping satisfying* (28). *Then there exists a unique additive mapping $\mathcal{D} : A \to A$*

$$\|h(x) - \mathcal{D}(x)\| \leq 3\delta$$

for all $x \in A$.

Proof. Since $h(-x) = -h(x)$ for all $x \in A$, with the use of (29) we have

$$\|h(z + x) + h(z - x) - 2h(z)\| \leq \frac{3}{2}\delta$$

for all $x, z \in A$.

Taking $z = x$ in the above inequlity, we gain

$$\|h(2z) - 2h(z)\| \leq \frac{3}{2}\delta$$

for all $z \in A$, since $h(0) = 0$.

Thus

$$\|h(z + x) + h(z - x) - h(2z)\| \leq \|h(z + x) + h(z - x) - 2h(z)\| + \|h(2z) - 2h(z)\| \leq 3\delta$$

for all $x, z \in A$. By Hyers' Theorem [10], there exists a unique additive mapping $\mathcal{D} : A \to A$ such that

$$\|h(x) - \mathcal{D}(x)\| \leq 3\delta$$

for all $x \in A$. $\square$

THEOREM 3.3. *Let A be a unital Banach algebra and $h : A \to A$ be an even mapping satisfying* (28). *Then there exists a unique quadratic mapping $Q : A \to A$*

$$\|h(x) - Q(x)\| \leq \frac{2}{3}\delta$$

for all $x \in A$.

Proof. Since $h(-x) = h(x)$ for all $x \in A$, with the use of (29) we have

$$\|h(z + x) + h(z - x) - 2h(z) - 2h(x)\| \leq \frac{3}{2}\delta$$

for all $x, z \in A$. By [1, Theorem 1], there exists a unique quadratic mapping $Q : A \to A$ such that

$$\|h(x) - Q(x)\| \leq \frac{1}{2}\delta + \frac{1}{3}\|h(0)\|$$

for all $x \in A$. On the other hand, letting $x = y = z = 0$ in (28), we get $\|h(0)\| \leq \frac{\delta}{2}$. Therefore,

$$\|h(x) - Q(x)\| \leq \frac{2}{3}\delta$$

for all $x \in A$. $\square$

THEOREM 3.4. *Let A be a unital Banach algebra and $h : A \to A$ be a mapping satisfying*

$$\begin{aligned} \mathcal{F}(x, y, z) &= y - 2e, \\ \|\Gamma h(x, y, z)\| &\leq \delta \end{aligned} \tag{30}$$

for all $x, y, z \in A$. Then there exists a unique additive mapping $\mathcal{D} : A \to A$ such that

$$\|h(x) - \mathcal{D}(x)\| \leq 2 + \frac{\delta}{2}$$

for all $x \in A$.

Proof. Letting $x = y = z = 0$ in (28), we get $h(0) = 2e$. Letting $y = 0$ in (30), we obtain

$$\|2h(x + z) - 2h(x) - 2h(z) - 2h(0)\| \leq \delta.$$

Since $\|h(0)\| = 2$,

$$\|h(x + z) - h(x) - h(z)\| \leq 2 + \frac{\delta}{2}$$

for all $x, z \in A$. By Hyers' Theorem [10], there exists a unique additive mapping $\mathcal{D} : A \to A$ such that

$$\|h(x) - \mathcal{D}(x)\| \leq 2 + \frac{\delta}{2}$$

for all $x \in A$. □

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