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MEASURE INDUCED BY THE PARTITION OF THE GENERAL REGION

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Abstract. In this paper we first consider the partition of the general region made by the monotonically increasing and continuous function and then obtain the measure from the partition of the region. The results obtained here is a little bit different from the previous results in [1, 2, 3] and finally we discuss the difference.

1. Introduction

Kitagawa [6] introduced the Wiener space of functions of two variables which is the collection of the continuous functions $f(x, y)$ on the unit square $[0, 1] \times [0, 1]$ satisfying $f(x, y) = 0$ for $xy = 0$. Yeh [8] treated the integration on this space for the more general function and made a firm logical foundation of this space, called a Yeh-Wiener space.

In [9], Yeh introdeced the conditional Wiener integral for real valued conditioning function and evaluated it using the inversion formulae. Chang, Ahn and the first author [4] treated the conditional Yeh-Wiener integral for real valued conditioning function. Furthermore, Park and Skoug [7] considered the conditional Yeh-Wiener integral for vector valued conditioning function and evaluated it using the simple formula.

Yeh-Wiener measure space ([4, 6, 7, 8]) comes from the rectangle made by the constant function. But, the modified and the generalized measure

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space ([1, 2, 3]) was obtained by the general region rather than the rectangle. They used a strictly decreasing and a monotonically decreasing function to obtain the region.

In this paper we use the monotonically increasing and continuous function to get a general region and obtain the partition of the region. And we also make a measure from the partition of the generalized region. The results obtained here is a little bit different from the results in [1, 2, 3] and finally we discuss the difference.

2. Partition of the general region

Let g be a monotonically increasing and continuous function on $[a, b]$ with $g(a) \geq 0$. Then g can be sectionally constant or strictly increasing on $[a, b]$.

Let $\{s_0, s_1, \ldots, s_{k+1}\}$ be a partition of $[a, b]$ satisfying

(2.1)
- (i) $a = s_0 < s_1 < \cdots < s_k < s_{k+1} = b$.
- (ii) g is either constant or strictly increasing on $I_i \equiv [s_{i-1}, s_i]$.
- (iii) g is not constant or strictly increasing on two consecutive intervals I_i and I_{i+1}.

Then the partition points s_i depends on the function g. Let $\{x_0, x_1, \ldots, x_m\}$ be a partition of $[a, b]$ including the points in (2.1) and satisfying

$$\begin{aligned} a = x_0 < x_1 < \cdots < x_{l_1} = s_1 < x_{l_1+1} < \cdots \\ < x_{l_1+l_2} = s_2 < \cdots < x_{l_1+\cdots+l_k} = s_k < \cdots \\ < x_m = s_{k+1} = b, \end{aligned} \tag{2.2}$$

where $l_1 + \cdots + l_{k+1} = m$ and $l_i \geq 1$ for $i = 1, 2, \ldots, k+1$. The notation (2.2) is similar but slightly different from the notation used in [5]. In [5], the notation was used to consider a Feynman's time ordered operational calculus. For notational convenience, let

$$\widehat{k} = \begin{cases} k+1, & \text{if } k \text{ is odd} \\ k, & \text{if } k \text{ is even.} \end{cases} \tag{2.3}$$

There are two cases for which g is constant or strictly increasing on I_1. We first assume that g is constant on I_1. Then, from the partition

$\{x_0, x_1, \ldots, x_m\}$ satisfying the conditions (2.1) and (2.2), we can make a partition $\{y_0, y_1, \ldots, y_n\}$ of $[0, g(b)]$ satisfying

$$\text{(2.4)} \quad \begin{aligned} &\text{(i)} \quad 0 = y_0 < y_1 < \cdots < y_p = g(a) < \cdots < y_n = g(b) \text{ for } p \geq 0. \\ &\text{(ii)} \quad y_{p+l_2+\cdots+l_{2i-2}} = g(x) \text{ on } I_{2i-1} \text{ for } i = 1, 2, \ldots, \frac{1}{2}\widehat{(k+1)}. \\ &\text{(iii)} \quad y_{p+l_2+l_4+\cdots+l_{2i-2}+q} = g(x_{l_1+l_2+\cdots+l_{2i-1}+q}) \\ &\qquad \text{for } i = 1, 2, \ldots, \frac{1}{2}\widehat{k}, \text{ and } 1 \leq q \leq l_{2i}. \end{aligned}$$

Let $\{t_n\}$ be a sequence satisfying

$$\text{(2.5)} \quad \begin{aligned} &0 = t_0 < y_p = t_1 < y_{p+l_2} = t_2 < y_{p+l_2+l_4} = t_3 < \cdots \\ &\quad < y_{p+l_2+\cdots+l_{\widehat{k}}} = y_n. \end{aligned}$$

Then $y_n = t_{1+\widehat{k}/2}$ and we have

$$\text{(2.6)} \quad \begin{aligned} &0 = t_0 < y_1 < \cdots < y_p = t_1 < y_{p+1} < \cdots \\ &\quad < y_{p+l_2} = t_2 < \cdots < y_{p+l_2+l_4\cdots+l_{\widehat{k}}} = y_n, \end{aligned}$$

where $y_p = g(a)$ and $y_n = g(b)$.

We summarize the above consideration in the following theorem.

THEOREM 2.1. *Let Ω_g be the region given by*

$$\text{(2.7)} \quad \Omega_g = \{(x, y) : a \leq x \leq b, 0 \leq y \leq g(x)\},$$

where g is monotonically increasing and continuous on $[a, b]$ with $g(a) \geq 0$ and g is constant on I_1 in (2.1). Then we can obtain the partition Λ of the region Ω_g by

$$\text{(2.8)} \quad \begin{aligned} \Lambda = \{(x_i, y_{j_i}) : i = 0, 1, \ldots, m, \text{ and } j_i = 0, 1, \ldots, q \\ \text{such that } y_q = g(x_i)\}, \end{aligned}$$

where $x_0, x_1, \ldots, x_m$ are given by (2.2) and $y_0, y_1, \ldots, y_n$ are given by (2.4).

In the partition Λ of the above theorem, the points

$$(s_0, t_1), (s_1, t_1), (s_2, t_2), \ldots, (s_{k+1}, t_{1+\widehat{k}/2})$$

of the region Ω_g play an important role to understand the shape of the monotonically increasing and continuous function g.

REMARK 2.2. *For the region Ω_g in Theorem 2.1, let $C(\Omega_g)$ denote the space of all real valued continuous functions f on Ω_g satisfying $f(x,0) = f(a,y) = 0$ for all $a \le x \le b$ and $0 \le y \le g(a)$. Then the special space $C(\Omega_g)$ with $k = 0$ and $g(a) > 0$ in (2.1) is the Yeh-Wiener measure space.*

REMARK 2.3. *Let g be a strictly increasing function on I_1 in (2.1). Then, for the partition $\{x_0, x_1, \dots, x_m\}$ of $[a,b]$ satisfying the condition (2.2), we have the corresponding partition $\{y_0, y_1, \dots, y_n\}$ of $[0, g(b)]$ which satisfies*

$$(2.9) \quad \begin{aligned} &\text{(i)}\ \ 0 = y_0 < y_1 < \cdots < y_p = g(a) < \cdots < y_n = g(b) \text{ for } p \ge 0 \\ &\text{(ii)}\ \ y_{p+l_1+l_3+\cdots+l_{2i-1}} = g(x) \text{ on } I_{2i} \text{ for } i = 1, 2, \dots, \frac{1}{2}\widehat{k} \\ &\text{(iii)}\ \ y_{p+l_1+l_3+\cdots+l_{2i-1}+q} = g(x_{l_1+l_2+\cdots+l_{2i}+q}) \\ &\qquad \text{for } i = 0, 1, \dots, \frac{1}{2}\widehat{(k-1)} \text{ and } 0 \le q \le l_{2i+1}, \end{aligned}$$

where $l_0 = l_{-1} = 0$ and $\widehat{0} = \widehat{-1} = 0$.

3. Measure of the cylinder set

Let Λ be the partition of the general region Ω_g in Theorem 2.1 which has the even number k in (2.1). Let N be the number of elements in the partition Λ excluding the points on the lines $x = a$ and $y = 0$, and let X_Λ be a random vector from $C(\Omega_g)$ to $\mathbb{R}^N$ defined by

$$(3.1) \quad \begin{aligned} X_\Lambda(f) = (f(x_i, y_{j_i}) : i = 1, \dots, m \text{ and } j_i = 1, \dots, q \\ \text{such that } y_q = g(x_i)\}. \end{aligned}$$

Let J be the cylinder set of the type

$$(3.2) \quad J = \{f \in C(\Omega_g) : X_\Lambda(f) \in B\}$$

for B in $\mathcal{B}^N$, the Borel σ-algebra of $\mathbb{R}^N$.

To make a measure of J, we first use the partition Λ of the region Ω_g and consider

$$\begin{aligned}A(\Lambda) = {}& [(x_1 - a)\cdots(s_1 - x_{l_1-1})]^p \\ & [(x_{l_1+1} - s_1)^{p+1}\cdots(s_2 - x_{l_1+l_2-1})^{p+l_2}]\cdots \\ & [(x_{L_{k-1}+1} - s_{k-1})^{E_{k-2}+1}\cdots(s_k - x_{L_k-1})^{E_k}] \\ & [(x_{L_k+1} - s_k)\cdots(b - x_{m-1})]^n \\ & [y_1(y_2 - y_1)\cdots(t_1 - y_{p-1})]^m \\ & [(y_{p+1} - t_1)^{m-l_1}\cdots(y_{p+l_2} - y_{p+l_2-1})^{m-l_1-l_2+1}]\cdots \\ & [(y_{E_{k-2}+1} - y_{E_{k-2}})^{l_k+l_{k+1}}\cdots(y_n - y_{n-1})^{l_{k+1}+1}],\end{aligned} \tag{3.3}$$

where $L_i = l_1+l_2+\cdots+l_i$ for $i = 1, 2, \ldots, k$, and $E_i = p+l_2+l_4+\cdots+l_i$ for $i = 2, 4, \ldots, k$. And if we let $\Delta_i x = x_i - x_{i-1}$, $\Delta_j y = y_j - y_{j-1}$ and $\Delta_{ij} u = u_{i,j} - u_{i,j-1} - u_{i-1,j} + u_{i-1,j-1}$, then we have

(3.4)

$$\begin{aligned}B(\Lambda, \vec{u}) = {}& \sum_{i=1}^{l_1}\sum_{j=1}^{p}\frac{(\Delta_{ij}u)^2}{2\Delta_i x\Delta_j y} + \sum_{i=l_1+1}^{l_1+l_2}\sum_{j=1}^{p-l_1+i}\frac{(\Delta_{ij}u)^2}{2\Delta_i x\Delta_j y} + \cdots \\ & + \sum_{i=l_1+\cdots+l_{k-1}+1}^{l_1+\cdots+l_k}\ \sum_{j=1}^{p+l_2+\cdots+l_{k-2}-l_1-\cdots-l_{k-1}+i}\frac{(\Delta_{ij}u)^2}{2\Delta_i x\Delta_j y} \\ & + \sum_{i=l_1+\cdots+l_k+1}^{m}\sum_{j=1}^{n}\frac{(\Delta_{ij}u)^2}{2\Delta_i x\Delta_j y} \\ = {}& \sum_{i=1}^{l_1}\sum_{j=1}^{p}\frac{(\Delta_{ij}u)^2}{2\Delta_i x\Delta_j y} + \sum_{i=l_1+1}^{L_2}\sum_{j=1}^{p-l_1+i}\frac{(\Delta_{ij}u)^2}{2\Delta_i x\Delta_j y} + \cdots \\ & + \sum_{i=L_{k-1}+1}^{L_k}\ \sum_{j=1}^{E_{k-2}-L_{k-1}+i}\frac{(\Delta_{ij}u)^2}{2\Delta_i x\Delta_j y} + \sum_{i=L_k+1}^{m}\sum_{j=1}^{n}\frac{(\Delta_{ij}u)^2}{2\Delta_i x\Delta_j y}.\end{aligned}$$

Expressions (3.3) and (3.4) are complicate to understand. Hence we will illustrate Λ, $A(\Lambda)$ and $B(\Lambda, \vec{u})$ by an example.

EXAMPLE 3.1. *Let g be a monotonically increasing and continuous function defined on $[a, b]$ such that $g(x) = t_1$ for $a \le x \le s_1$, g is linear on $[s_1, s_2]$ and $g(x) = t_2$ for $s_2 \le x \le b$, where $a = s_0 < s_1 < s_2 < s_3 = b$*

and $0 \le t_1 < t_2$*. Let* $\{x_0, x_1, \ldots, x_5\}$ *be a partition of* $[a, b]$ *such that*

$$a = x_0 < x_1 < x_2 = s_1 < x_3 < x_4 = s_2 < x_5 = s_3 = b$$

and let $\{y_0, y_1, \ldots, y_4\}$ *be a partition of* $[0, t_2]$ *such that*

$$0 = y_0 < y_1 < y_2 = t_1 = g(a) < y_3 < y_4 = t_2 = g(b).$$

Then

$$\Lambda = \{(x_0, y_0), (x_0, y_1), (x_0, y_2), (x_1, y_0), \ldots, (x_5, y_4)\}.$$

Note that Λ *has* 23 *elements and* $N = 15$*. In this case*

$$A(\Lambda) = (x_1 - a)^2 (x_2 - x_1)^2 (x_3 - x_2)^3 (x_4 - x_3)^4 (x_5 - x_4)^4$$
$$(y_1 - y_0)^5 (y_2 - y_1)^5 (y_3 - y_2)^3 (y_4 - y_3)^2$$

and

$$B(\Lambda, \vec{u}) = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{(\Delta_{ij} u)^2}{2\Delta_i x \Delta_j y} + \sum_{i=3}^{4} \sum_{j=1}^{i} \frac{(\Delta_{ij} u)^2}{2\Delta_i x \Delta_j y} + \sum_{j=1}^{4} \frac{(\Delta_{5j} u)^2}{2\Delta_5 x \Delta_j y},$$

where $u_{i,0} = u_{0,j} = 0$ *for* $i = 0, 1, \ldots, 5$ *and* $j = 0, 1, 2$*, and* $u_{2,3} = u_{3,4} = 0$.

THEOREM 3.2. *Let* Λ *be the partition of the region* Ω_g *in Theorem* 2.1 *with an even number* k *in* (2.1)*, and let* J *be the cylinder set given by* (3.2)*. Then we can make a measure* $\tilde{m}$ *of a set* J *by*

$$\tilde{m}(J) = \int_B W(\Lambda, \vec{u})\, d\vec{u} \tag{3.5}$$

where

$$W(\Lambda, \vec{u}) = \{(2\pi)^N A(\Lambda)\}^{-1/2} \exp\{-B(\Lambda, \vec{u})\} \tag{3.6}$$

for $\vec{u}$ *in* $\mathbb{R}^N$*, and* $A(\Lambda)$ *and* $B(\Lambda, \vec{u})$ *are given by* (3.3) *and* (3.4)*, respectively.*

Let $\mathcal{J}$ be the collection of subsets of type J. Then it can be shown that $\mathcal{J}$ is a semi-algebra of subsets of $C(\Omega_g)$ and the set function $\tilde{m}$ is a measure defined on $\mathcal{J}$ and the factor $W(\Lambda, \vec{u})$ is chosen to make $\tilde{m}(C(\Omega)) = 1$. The measure $\tilde{m}$ can be extended to a measure on the Carathedory extension of interval class $\mathcal{J}$ in the usual way.

In [3], for a monotonically decreasing function g, Chang and Ahn used a partition of the region Ω_g to define a probability measure on $C(\Omega_g)$. At first, we tried a similar method as in [3] for a monotonically increasing function g, that is, we define $A(\Lambda)$ and $B(\Lambda, \vec{u})$ by the same way as in equation (2.7) of [3]. In this case, for the same partition Λ as in Example 3.1, $A(\Lambda)$ and $B(\Lambda, \vec{u})$ are given by

$$
\begin{aligned}
A(\Lambda) = {} & (x_1 - a)^2 (x_2 - x_1)^2 (x_3 - x_2)^2 (x_4 - x_3)^3 (x_5 - x_4)^4 \\
& (y_1 - y_0)^5 (y_2 - y_1)^5 (y_3 - y_2)^2 (y_4 - y_3)
\end{aligned}
$$

and

$$
B(\Lambda, \vec{u}) = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{(\Delta_{ij} u)^2}{2\Delta_i x \Delta_j y} + \sum_{i=3}^{4} \sum_{j=1}^{i-1} \frac{(\Delta_{ij} u)^2}{2\Delta_i x \Delta_j y} + \sum_{j=1}^{4} \frac{(\Delta_{5j} u)^2}{2\Delta_5 x \Delta_j y},
$$

that is, the exponent in $A(\Lambda)$ here is different from $A(\Lambda)$ in Example 3.1 and we do not use the variables $u_{2,3}$ and $u_{3,4}$ in $B(\Lambda, \vec{u})$. But such an attempt was unsuccessful as one can see in the following example.

EXAMPLE 3.3. *Let g be a monotonically increasing function defined on the interval $[0, 3]$ as follows.*

$$
g(x) = \begin{cases} 1, & 0 \le x \le 1 \\ x, & 1 \le x \le 2 \\ 2, & 2 \le x \le 3 \end{cases}
$$

Let the partition of $[0, 3]$ to be $s_0 = 0, s_1 = 1, s_2 = 2$ and $s_3 = 3$. Then $t_0 = 0, t_1 = 1$ and $t_2 = 2$. Let Λ be the following partition of Ω_g.

$$
\Lambda = \{(0,0), (0,1), (1,0), (1,1), (2,0), (2,1), (2,2), (3,0), (3,1), (3,2)\}
$$

Then equation (2.7) of [3] suggests $A(\Lambda)$ and $B(\Lambda, \vec{u})$ as

$$
A(\Lambda) = s_1 (s_2 - s_1)(s_3 - s_2)^2 t_1^3 (t_2 - t_1) = 1
$$

and

$$
\begin{aligned}
B(\Lambda, \vec{u}) &= \frac{1}{2}[(\Delta_{11} u)^2 + (\Delta_{21} u)^2 + (\Delta_{31} u)^2 + (\Delta_{32} u)^2] \\
&= \frac{1}{2}[u_{1,1}^2 + (u_{2,1} - u_{1,1})^2 + (u_{3,1} - u_{2,1})^2 \\
&\quad + (u_{3,2} - u_{3,1} - u_{2,2} + u_{2,1})^2],
\end{aligned}
$$

since $u_{i,0} = u_{0,j} = 0$ *for* $i = 0, 1, 2, 3$ *and* $j = 0, 1$. *Note that* $N = 5$, *and so by (3.5) the measure* $\tilde{m}(C(\Omega_g))$ *can be ginen as follows.*

$$\begin{aligned}\tilde{m}(C(\Omega_g)) &= \int_{\mathbb{R}^5} W(\Lambda, \vec{u})\, d\vec{u} \\ &= (2\pi)^{-5/2} \int_{\mathbb{R}^5} \exp\Big\{-\frac{1}{2}[u_{1,1}^2 + (u_{2,1} - u_{1,1})^2 \\ &\quad + (u_{3,1} - u_{2,1})^2 + (u_{3,2} - u_{3,1} - u_{2,2} + u_{2,1})^2]\Big\}\, d\vec{u},\end{aligned}$$

where $d\vec{u} = du_{1,1}\, du_{2,1}\, du_{2,2}\, du_{3,1}\, du_{3,2}$. *Now evaluating the last integral, we have*

$$\tilde{m}(C(\Omega_g)) = (2\pi)^{-1/2} \int_{\mathbb{R}} 1\, du_{2,2} = \infty.$$

That is, a similar method as in [3] *for a monotonically increasing function* g *fails to construct a measure.*

Hence, in this paper, we modified $A(\Lambda)$ and $B(\Lambda, \vec{u})$ for a monotonically increasing function g as in (3.3) and (3.4), respectively. In the following example, we will show that our construction gives a probability measure on Ω_g for the same function g and partition Λ as in Example 3.3.

EXAMPLE 3.4. *Let* g *and* Λ *be the same as in Example 3.3. Then* $A(\Lambda)$ *and* $B(\Lambda, \vec{u})$ *in (3.3) and (3.4), respectively, can be expressed as*

$$A(\Lambda) = s_1(s_2 - s_1)^2(s_3 - s_2)^2 t_1^3 (t_2 - t_1)^2 = 1$$

and

$$\begin{aligned}B(\Lambda, \vec{u}) &= \frac{1}{2}[(\Delta_{1,1}u)^2 + (\Delta_{2,1}u)^2 + (\Delta_{2,2}u)^2 + (\Delta_{3,1}u)^2 + (\Delta_{3,2}u)^2] \\ &= \frac{1}{2}[u_{1,1}^2 + (u_{2,1} - u_{1,1})^2 + (u_{2,2} - u_{2,1} + u_{1,1})^2 \\ &\quad + (u_{3,1} - u_{2,1})^2 + (u_{3,2} - u_{3,1} - u_{2,2} + u_{2,1})^2],\end{aligned}$$

since $u_{i,0} = u_{0,j} = 0$ *for* $i = 0, 1, 2, 3$ *and* $j = 0, 1$, *and* $u_{1,2} = 0$. *Hence by (3.5) the measure* $\tilde{m}(C(\Omega_g))$ *can be evaluated as*

$$\begin{aligned}\tilde{m}(C(\Omega_g)) &= \int_{\mathbb{R}^5} W(\Lambda, \vec{u})\, d\vec{u} \\ &= (2\pi)^{-5/2} \int_{\mathbb{R}^5} \exp\Big\{-\frac{1}{2}[u_{1,1}^2 + (u_{2,1} - u_{1,1})^2 \\ &\quad + (u_{2,2} - u_{2,1} + u_{1,1})^2 + (u_{3,1} - u_{2,1})^2 \\ &\quad + (u_{3,2} - u_{3,1} - u_{2,2} + u_{2,1})^2]\Big\}\, d\vec{u} \\ &= 1.\end{aligned}$$

That is, we can construct a probability measure on $C(\Omega_g)$ *for a monotonically increasing function* g.

Remark 3.5. *In* [3], *the region was made by the monotonically decreasing function. As we can see in Example 3.3 and Example 3.4,* $A(\Lambda)$ *and* $B(\Lambda, \vec{u})$ *in (3.3) and (3.4) are quite different to make a measure if the function* g *is monotonically increasing or monotonically decreasing.*

Remark 3.6. *In this paper we use an even number* k, *but we can also obtain the similar result for an odd number* k *without any problem. Furthermore, we can consider a monotonically increasing function which is strictly increasing on* I_1 *using Remark 2.3.*

Remark 3.7. *The results of this paper will be useful to investigate the generalized conditional Yeh-Wiener integral.*

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