Korean J. Math. **31** (2023), No. 2, pp. 153–160
http://dx.doi.org/10.11568/kjm.2023.31.2.153

# ON THE GROWTH OF POLYNOMIALS

Rubia Akhter, B. A. Zargar, and M. H. Gulzar

Abstract. In this paper, we study the growth of polynomials $P(z)$ of degree $n$ defined by $P(z) = z^s(a_0 + \sum_{j=t}^{n-s} a_j z^j), \quad t \geq 1, \quad 0 \leq s \leq n-1$ which do not vanish in the disk $|z| \leq k, \quad k \geq 1$ except for the $s$-fold zeros at origin. Our result generalises and refines many results known in the literature.

## 1. Introduction

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n$ and let $||P|| = \max_{|z|=1} |P(z)|, M(P,R) = \max_{|z|=R} |P(z)|$. For a polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree $n$, it is well known and is a simple consequence of the *Maximum Modulus Principle* [9] that for $R \geq 1$,

$$M(P,R) \leq R^n ||P||. \tag{1.1}$$

The result is best possible with equality holding for $P(z) = \lambda z^n, \quad \lambda$ being a complex number. Since the extremal polynomial $P(z) = \lambda z^n$ in (1.1) has all its zeros at the origin, it should be possible to improve upon the bound in (1.1) for the polynomials not vanishing at the origin. This fact was observed by Ankeny and Rivilin [1], who proved that if a polynomial $P(z)$ has no zeros in $|z| < 1$, then for $R \geq 1$,

$$M(P,R) \leq \left(\frac{R^n + 1}{2}\right) ||P||. \tag{1.2}$$

Inequality (1.2) becomes equality for $P(z) = \lambda + \mu z^n$ where $|\lambda| = |\mu|$. As a refinement of (1.2), Govil [6] show that if a polynomial $P(z)$ has no zero in $|z| < 1$, then for $R \geq 1$,

$$\begin{aligned} M(P,R) \leq & \left(\frac{R^n + 1}{2}\right) ||P|| - \frac{n}{2}\left(\frac{||P||^2 - 4|a_n|^2}{||P||}\right) \\ & \times \left\{\frac{(R-1)||P||}{||P|| + 2|a_n|} - \ln\left(1 + \frac{(R-1)||P||}{||P|| + 2|a_n|}\right)\right\}. \end{aligned} \tag{1.3}$$

Received December 27, 2022. Revised April 27, 2023. Accepted April 27, 2023.
2010 Mathematics Subject Classification: 15A18, 30A10, 30C15.
Key words and phrases: Polynomial, Growth, S-fold zeros.
The research is financially supported by NBHM, Government of India, under the research project 02011/36/2017/R&D-II.

The above inequality becomes equality for the polynomial $P(z) = \lambda + \mu z^n$, where $|\lambda| = |\mu|$.

Dalal and Govil [2] used a recurrence relation and sharpened the bound in (1.3) as follows:

THEOREM 1.1. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$, *is a polynomial of degree* $n$ *which does not vanish in* $|z| < 1$, *then for* $R \geq 1$ *and* $N$, $1 \leq N \leq n$

$$M(P,R) \leq \left(\frac{R^n+1}{2}\right)||P|| - \frac{n||P||}{2}\left(1 - \frac{2|a_n|}{||P||}\right)h(N), \tag{1.4}$$

*where*

$$h(N) = \left(\frac{R^N-1}{N}\right) + \sum_{\nu=1}^{N-1}\left(\frac{R^{N-\nu}-1}{N-\nu}\right)(-1)^{\nu}\left(1+\frac{2|a_n|}{||P||}\right)\left(\frac{2|a_n|}{||P||}\right)^{\nu-1}$$
$$(-1)^N\left(1+\frac{2|a_n|}{||P||}\right)\left(\frac{2|a_n|}{||P||}\right)^{N-1}\ln\left(1+\frac{(R-1)||P||}{||P||+2|a_n|}\right),$$

*for* $N \geq 1$ *and* $h(0) = 0$.

Gardner, Govil and Musukula [3] used the coefficients of the polynomial $P(z)$ and proved the following generalisation and refinement of (1.3).

THEOREM 1.2. *If* $P(z) = a_0 + \sum_{j=t}^{n} a_j z^j$, $t \geq 1$, $a_t \neq 0$, *is a polynomial of degree* $n$ *and* $P(z)$ *does not vanish in* $|z| < k$, $k \geq 1$, *then for* $R \geq 1$,

(1.5)
$$M(P,R) \leq \left(\frac{R^n+S_1}{1+S_1}\right)||P|| - \left(\frac{R^n-1}{1+S_1}\right)m - \frac{n}{1+S_1}\left(\frac{(||P||-m)^2-(1+S_1)^2|a_n|}{||P||-m}\right)$$
$$\left\{\frac{(R-1)(||P||-m)}{(||P||-m)+(1+S_1)|a_n|} - \ln\left(1+\frac{(R-1)(||P||-m)}{(||P||-m)+(1+S_1)|a_n|}\right)\right\}$$

*where*

$$S_1 = k^{t+1}\left\{\frac{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|-m}k^{t-1}+1}{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|-m}k^{t+1}+1}\right\}. \tag{1.6}$$

Several papers and research monographs have been written on this subject (See, for example [3], [4], [5]- [7]).

## 2. Main Result

In this paper, we prove the following result which generalises and sharpens the Theorem 1.1 due to Dalal and Govil and Theorem 1.2 due to Gardner, Govil and Musukula.

THEOREM 2.1. *If* $P(z) = z^s(a_0 + \sum_{j=t}^{n-s} a_j z^j)$, $t \geq 1$, $0 \leq s \leq n-1$ *is a polynomial of degree* $n$ *having no zero in* $|z| \leq k$, $k \geq 1$ *except* $s$*-fold zeros at the origin, then* *for* $R \geq 1$

$$\begin{aligned} M(P,R) \leq & \left(\frac{s\varLambda_t(R^n-1)+n(R^n+\varLambda_t)}{n(1+\varLambda_t)}\right)||P|| - \left(\frac{R^n-1}{n}\right)\left(\frac{n-s}{k^s(1+\varLambda_t)}\right)m \\ & - \left(\frac{k^s(n+s\varLambda_t)||P||-(n-s)m-nk^s|a_{n-s}|(1+\varLambda_t)}{k^s(1+\varLambda_t)}\right)h(N) \end{aligned} \tag{2.1}$$

*where*

(2.2)

$$\begin{aligned} h(N) = & \left(\frac{R^N-1}{N}\right) + \sum_{\nu=1}^{N-1}\left(\frac{R^{N-\nu}-1}{N-\nu}\right)(-1)^\nu\left(\frac{nk^s|a_{n-s}|(1+\varLambda_t)}{k^s(n+s\varLambda_t)||P||-(n-s)m}+1\right) \\ & \times\left(\frac{nk^s|a_{n-s}|(1+\varLambda_t)}{k^s(n+s\varLambda_t)||P||-(n-s)m}\right)^{\nu-1} + (-1)^N\left(\frac{nk^s|a_{n-s}|(1+\varLambda_t)}{k^s(n+s\varLambda_t)||P||-(n-s)m}+1\right) \\ & \times\left(\frac{nk^s|a_{n-s}|(1+\varLambda_t)}{k^s(n+s\varLambda_t)||P||-(n-s)m}\right)^{N-1} \\ & \times\ln\left(1+\frac{(R-1)(k^s(n+s\varLambda_t)||P||-(n-s)m)}{k^s(n+s\varLambda_t)||P||-(n-s)m+nk^s|a_{n-s}|(1+\varLambda_t)}\right) \end{aligned}$$

*and* $\varLambda_t$ *is defined in* (3.1).

Remark 2.1. *From Lemma 3.5, we have* $h(1) \leq h(N)$. *Using this in* (2.1), *we get*

$$\begin{aligned} M(P,R) \leq & \left(\frac{s\varLambda_t(R^n-1)+n(R^n+\varLambda_t)}{n(1+\varLambda_t)}\right)||P|| - \left(\frac{R^n-1}{n}\right)\left(\frac{n-s}{k^s(1+\varLambda_t)}\right)m \\ & - \left(\frac{k^s(n+s\varLambda_t)||P||-(n-s)m-nk^s|a_{n-s}|(1+\varLambda_t)}{k^s(1+\varLambda_t)}\right)h(1) \end{aligned} \tag{2.3}$$

*and from* (2.2)

$$\begin{aligned} h(1) = & (R-1) - \left(\frac{nk^s|a_{n-s}|(1+\varLambda_t)}{k^s(n+s\varLambda_t)||P||-(n-s)m}+1\right) \\ & \times\ln\left(1+\frac{(R-1)(k^s(n+s\varLambda_t)||P||-(n-s)m)}{k^s(n+s\varLambda_t)||P||-(n-s)m+nk^s|a_{n-s}|(1+\varLambda_t)}\right) \end{aligned}$$

*where* $\varLambda_t$ *is defined in* (3.1).

Using Lemma 3.6 in (2.3) and substitute the value of $h(1)$, we get the following result

Corollary 2.1. *If* $P(z) = z^s(a_0 + \sum_{j=t}^{n-s} a_j z^j)$, $t \geq 1$, $0 \leq s \leq n-1$ *is a polynomial of degree* $n$ *having no zero in* $|z| \leq k$, $k \geq 1$ *except* $s$*-fold zeros at the* *origin, then for* $R \geq 1$

(2.4)

$$M(P,R) \leq \left(\frac{s\Lambda_t(R^n-1)+n(R^n+\Lambda_t)}{n(1+\Lambda_t)}\right)||P|| - \left(\frac{R^n-1}{n}\right)\left(\frac{n-s}{k^s(1+\Lambda_t)}\right)m$$
$$-\frac{1}{k^s(1+\Lambda_t)}\left(\frac{(k^s(n+s\Lambda_t)||P||-(n-s)m)^2-(nk^s|a_{n-s}|(1+\Lambda_t))^2}{(k^s(n+s\Lambda_t)||P||-(n-s)m)}\right)$$
$$\times\left[\left(\frac{(R-1)(k^s(n+s\Lambda_t)||P||-(n-s)m)}{k^s(n+s\Lambda_t)||P||-(n-s)m-nk^s|a_{n-s}|(1+\Lambda_t)}\right)\right.$$
$$\left.-\ln\left(1+\frac{(R-1)(k^s(n+s\Lambda_t)||P||-(n-s)m)}{k^s(n+s\Lambda_t)||P||-(n-s)m+nk^s|a_{n-s}|(1+\Lambda_t)}\right)\right]$$

*where* $\Lambda_t$ *is defined in* (3.1).

## 3. Lemmas

For the proof of the theorem, we need the following lemmas. The first lemma is due to E. Khojastehnezhed and M. Bidkham [8]

Lemma 3.1. *If* $P(z) = z^s(a_0 + \sum_{\nu=t}^{n-s} a_\nu z^\nu)$, $\quad t \geq 1, \quad 0 \leq s \leq n-1$ *is a polynomial of degree* $n$ *having no zero in* $|z| \leq k, \quad k \geq 1$ *except* $s$*-fold zeros at the origin, then*

$$||P'|| \leq \frac{n+s\Lambda_t}{1+\Lambda_t}||P|| - \frac{n-s}{k^s(1+\Lambda_t)}m$$

*where*

$$\Lambda_t = k^{t+1}\left\{\frac{\left(\frac{t}{n-s}\right)\frac{k^s|a_t|}{k^s|a_0|-m}k^{t-1}+1}{\left(\frac{t}{n-s}\right)\frac{k^s|a_t|}{k^s|a_0|-m}k^{t+1}+1}\right\} \tag{3.1}$$

*and* $m = \min_{|z|=k}|P(z)|$.

Lemma 3.2. *If* $P(z)$ *is a polynomial of degree* $n$*, then for* $|z| = R \geq 1$,

$$|P(z)| \leq R^n\left\{1 - \frac{(||P||-|a_n|)(R-1)}{|a_n|+R||P||}\right\}||P||.$$

This lemma is due to Govil [6].

Lemma 3.3. *The function*

$$\left\{1-\frac{(x-|a_n|)(r-1)}{|a_n|+rx}\right\}x$$

*is an increasing function of* $x$ *for* $x > 0$.

We omit the proof of the above lemma as it follows easily by using derivative test.

LEMMA 3.4. *Let* $h(N) = \int\limits_1^R \frac{(r-1)r^{N-1}}{r+a} dr$ *for* $N \geq 1$. *Then*

$$h(N) = \left(\frac{R^N - 1}{N}\right) + \sum_{\nu=1}^{N-1} \left(\frac{R^{N-\nu} - 1}{N-\nu}\right) (-1)^\nu (a+1) a^{\nu-1}$$
$$+ (-1)^N (a+1) a^{N-1} \ln\left(\frac{R+a}{1+a}\right).$$

LEMMA 3.5. *The function* $h(N)$ *defined in Lemma 3.4 is a non-negative increasing function of* $N$ *for* $N \geq 1$.

Lemmas 3.4 and 3.5 are due to Dalal and Govil [2].

LEMMA 3.6. *If* $P(z) = z^s(a_0 + \sum\limits_{\nu=t}^{n-s} a_\nu z^\nu)$, $\quad t \geq 1, \quad 0 \leq s \leq n-1$ *is a polynomial of degree* $n$ *having no zero in* $|z| \leq k, \quad k \geq 1$ *except* $s$*-fold zeros at the origin, then*

$$n|a_{n-s}| \leq \frac{n + s\Lambda_t}{1 + \Lambda_t} ||P|| - \frac{n-s}{k^s(1+\Lambda_t)} m$$

*where* $\Lambda_t$ *is defined in* (3.1) *and* $m = \min\limits_{|z|=k} |P(z)$.

## 4. Proof Of Theorem

**Proof of Theorem.** For each $\theta, \quad 0 \leq \theta \leq 2\pi \quad$ and $\quad R \geq 1$, we have

$$P(Re^{\iota\theta}) - P(e^{\iota\theta}) = \int\limits_1^R e^{\iota\theta} P'(re^\iota\theta) dr.$$

which implies

$$|P(Re^{\iota\theta}) - P(e^{\iota\theta})| \leq \int\limits_1^R |P'(re^\iota\theta)| dr$$

Now applying Lemma 3.2 to the polynomial $P'(z)$ which is of degree $n-1$, we get

$$|P(Re^{\iota\theta}) - P(e^{\iota\theta})| \leq \int\limits_1^R r^{n-1} \left\{1 - \frac{(||P'|| - n|a_{n-s}|)(r-1)}{n|a_{n-s}| + r||P'||}\right\} ||P'|| dr. \tag{4.1}$$

Since by Lemma 3.3, the integrand in (4.1) is an increasing function of $||P'||$, hence applying inequality of Lemma 3.1 in (4.1)

$$
\begin{aligned}
&|P(Re^{\iota\theta}) - P(e^{\iota\theta})| \\
&\leq \int_1^R r^{n-1} \left\{ 1 - \frac{\left( \frac{n+s\Lambda_t}{1+\Lambda_t}||P|| - \frac{n-s}{k^s(1+\Lambda_t)}m - n|a_{n-s}| \right)(r-1)}{n|a_{n-s}| + r\left( \frac{n+s\Lambda_t}{1+\Lambda_t}||P|| - \frac{n-s}{k^s(1+\Lambda_t)}m \right)} \right\} \\
&\times \left( \frac{n+s\Lambda_t}{1+\Lambda_t}||P|| - \frac{n-s}{k^s(1+\Lambda_t)}m \right) dr \\
&= \left( \frac{n+s\Lambda_t}{1+\Lambda_t}||P|| - \frac{n-s}{k^s(1+\Lambda_t)}m \right) \int_1^R r^{n-1} dr - \left( \frac{n+s\Lambda_t}{1+\Lambda_t}||P|| - \frac{n-s}{k^s(1+\Lambda_t)}m \right) \\
&\times \int_1^R r^{n-1} \left( \frac{k^s(n+s\Lambda_t)||P|| - (n-s)m - nk^s|a_{n-s}|(1+\Lambda_t)}{nk^s|a_{n-s}|(1+\Lambda_t) + r(k^s(n+s\Lambda_t)||P|| - (n-s)m)} \right)(r-1)dr \\
&= \frac{(R^n-1)}{n} \left( \frac{n+s\Lambda_t}{1+\Lambda_t}||P|| - \frac{n-s}{k^s(1+\Lambda_t)}m \right) - \left( \frac{k^s(n+s\Lambda_t)||P|| - (n-s)m}{k^s(1+\Lambda_t)} \right) \\
&\times \int_1^R r^{n-1} \left( \frac{k^s(n+s\Lambda_t)||P|| - (n-s)m - nk^s|a_{n-s}|(1+\Lambda_t)}{nk^s|a_{n-s}|(1+\Lambda_t) + r(k^s(n+s\Lambda_t)||P|| - (n-s)m)} \right)(r-1)dr \\
&= \frac{(R^n-1)}{n} \left( \frac{n+s\Lambda_t}{1+\Lambda_t}||P|| - \frac{n-s}{k^s(1+\Lambda_t)}m \right) - \left( \frac{k^s(n+s\Lambda_t)||P|| - (n-s)m}{k^s(1+\Lambda_t)} \right) \\
&\times \int_1^R r^{n-1} \left( \frac{1 - \frac{nk^s|a_{n-s}|(1+\Lambda_t)}{k^s(n+s\Lambda_t)||P|| - (n-s)m}}{r + \frac{nk^s|a_{n-s}|(1+\Lambda_t)}{k^s(n+s\Lambda_t)||P|| - (n-s)m}} \right)(r-1)dr
\end{aligned}
$$

That is

(4.2)

$$
\begin{aligned}
|P(Re^{\iota\theta}) - P(e^{\iota\theta})| \leq \frac{(R^n-1)}{n} \frac{n+s\Lambda_t}{1+\Lambda_t}||P|| - \left( \frac{n-s}{k^s(1+\Lambda_t)}m \right) \\
- \left( \frac{(k^s(n+s\Lambda_t)||P|| - (n-s)m)(1-a)}{k^s(1+\Lambda_t)} \right) \int_1^R \frac{(r-1)r^{n-1}}{r+a} dr
\end{aligned}
$$

where $a = \frac{nk^s|a_{n-s}|(1+\Lambda_t)}{k^s(n+s\Lambda_t)||P|| - (n-s)m}$. By Lemma 3.5, the integral $\int_1^R \frac{(r-1)r^{n-1}}{r+a} dr$ is a non-negative increasing function of $N$ for $1 \leq N \leq n$, therefore for every $N$, for $1 \leq N \leq n$, we have

$$
\int_1^R \frac{(r-1)r^{N-1}}{r+a} dr \leq \int_1^R \frac{(r-1)r^{n-1}}{r+a} dr
$$

The above inequality is equivalent to

$$|P(Re^{\iota\theta}) - P(e^{\iota\theta})| \leq \frac{(R^n-1)}{n}\left(\frac{n+s\Lambda_t}{1+\Lambda_t}||P|| - \frac{n-s}{k^s(1+\Lambda_t)}m\right) - \left(\frac{(k^s(n+s\Lambda_t)||P|| - (n-s)m)(1-a)}{k^s(1+\Lambda_t)}\right)h(N) \tag{4.3}$$

where $h(N) = \int\limits_1^R \frac{(r-1)r^{N-1}}{r+a}dr$ by Lemma 3.4. Now substituting the value of '$a$' and using the obvious inequality

$$|P(Re^{\iota\theta})| \leq |P(Re^{\iota\theta}) - P(e^{\iota\theta})| + |P(e^{\iota\theta})|$$

in (4.3), we get for $0 \leq \theta \leq 2\pi$ and $R \geq 1$,

$$|P(Re^{\iota\theta})| \leq \left(\frac{s\Lambda_t(R^n-1) + n(R^n+\Lambda_t)}{n(1+\Lambda_t)}\right)||P|| - \left(\frac{R^n-1}{n}\right)\left(\frac{n-s}{k^s(1+\Lambda_t)}\right)m - \left(\frac{k^s(n+s\Lambda_t)||P|| - (n-s)m - nk^s|a_{n-s}|(1+\Lambda_t)}{k^s(1+\Lambda_t)}\right)h(N). \tag{4.4}$$

The inequality (4.4) is equivalent to inequality (2.2) and this completes the proof of Theorem 2.1. □

## References


[1] N. C. Ankeny and T. J. Rivilin, *On a Theorem of S. Bernstein*, Pacific J. Math. **5** (1955), 849–862.

[2] A. Dalal and N. K. Govil, *On sharpening of a Theorem of Ankeny and Rivilin*, Anal. Theory Appl. **36** (2018), 225–234.

[3] R. B. Gardner, N. K. Govil and S. R. Musukula, *Rate of growth of polynomials not vanishing inside a circle*, J. Ineq. Pure Appl. Math. **6** (2005), Art 53.

[4] R. B. Gardner, N. K. Govil and A. Weems, *Growth of polynomial not vanishing inside a circle*, Int. J. Pure Appl. Math., 13(2004), 487-494.

[5] N. K. Govil, *On the maximum modulus of polynomials not vanishing inside the unit circle*, Approx. Theory Appl. **5** (1989), 79–82.

[6] N. K. Govil, *On the growth of polynomials*, J. Ineq. Appl. **7** (2002), 623–631.

[7] N. K. Govil, M. A. Qazi and Q. I. Rahman, *Inequalities describing the growth of polynomialsnot vanishing in a disk of prescribed radius*, Math. Inequal. Appl. **6** (2003), 491–498.

[8] E. Khojastehnezhad and M. Bidkham, *Generalisation of some inequalities for the polar derivative of polynomials with restricted zeros*, TWMS J. App. Eng. Math. **9** (2019), 485–492.

[9] M. Riesz, *Über einen Satz des Herrn Serge Bernstein*, Acta. Math. **40** (1916), 337–347.

**Rubia Akhter**
Department of Mathematics, Kashmir University,
Srinagar-190006, India
*E-mail*: rubiaakhter039@gmail.com

**B. A. Zargar**
Department of Mathematics, Kashmir University,
Srinagar-190006, India
*E-mail*: bazargar@gmail.com

**M. H. Gulzar**
Department of Mathematics, Kashmir University,
Srinagar-190006, India
*E-mail*: gulzarmh@gmail.com