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ON THE BOUNDS OF THE EIGENVALUES OF MATRIX POLYNOMIALS

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Abstract. Let $P(z) := \sum_{j=0}^{n} A_j z^j,\ A_j \in \mathbb{C}^{m\times m}, 0 \leq j \leq n$ be a matrix polynomial of degree n, such that

$$A_n \geq A_{n-1} \geq \ldots \geq A_0 \geq 0,\ A_n > 0.$$

Then the eigenvalues of $P(z)$ lie in the closed unit disk.

This theorem proved by Dirr and Wimmer [IEEE Trans. Automat. Control **52**(2007), 2151-2153] is infact a matrix extension of a famous and elegant result on the distribution of zeros of polynomials known as Eneström-Kakeya theorem. In this paper, we prove a more general result which inter alia includes the above result as a special case. We also prove an improvement of a result due to Lê, Du, Nguyên [Oper. Matrices, **13**(2019), 937-954] besides a matrix extention of a result proved by Mohammad [Amer. Math. Monthly, vol.74, No.3, March 1967].

1. Introduction and statement of results

Let $\mathbb{C}^{m\times m}$ be the set of all $m \times m$ matrices with entries from the field $\mathbb{C}$. By a matrix polynomial we mean a function $P : \mathbb{C} \to \mathbb{C}^{m\times m}$ defined by

$$P(z) := \sum_{j=0}^{n} A_j z^j,\ A_j \in \mathbb{C}^{m\times m}. \tag{1}$$

If $A_n \neq 0$, then $P(z)$ is said to be a matrix polynomial of degree n. If $A_n = I$, where I is the identity matrix, then the matrix polynomial $P(z)$ is called monic. We say λ is an eigenvalue of $P(z)$ if there exists $u \in \mathbb{C}^m \setminus \{0\}$ such that $P(\lambda)u = 0$. In this case **u** is said to be an eigenvector of $P(z)$.

For matrices $A, B \in \mathbb{C}^{m\times m}$, we write $A \geq 0$ and $A > 0$ if A is positive semidefinite and positive definite, respectively. $A \geq B$ and $A > B$ mean $A - B \geq 0$ and $A - B > 0$, respectively.

We denote by $\lambda_{max}(A)$ and $\lambda_{min}(A)$ the maximum and minimum eigenvalues of a Hermitian matrix A respectively. Also the spectral radius denoted by $\rho(A)$ of a

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matrix A is defined by

$$\rho(A) = \max\left\{|\lambda| \ ; \ \lambda \text{ is an eigenvalue of A}\right\}. \tag{2}$$

Dirr and Wimmer [3] proved the following result concerning the bounds on the eigenvalues of matrix polynomials.

THEOREM 1.1. *Let* $P(z) := \sum_{j=0}^{n} A_j z^j$, $A_j \in \mathbb{C}^{m\times m}$, $0 \leq j \leq n$ *be a matrix polynomial of degree n such that*

$$A_n \geq A_{n-1} \geq \ldots \geq A_0 \geq 0, \ A_n > 0. \tag{3}$$

Then the eigenvalues of $P(z)$ *lie in the closed unit disk* $|\lambda| \leq 1$.

The Eneström-Kakeya theorem [4, 7] is a special case of Theorem 1.1 if we put $m = 1$. Note that conclusion of Theorem 1.1 is also true if we replace relation symbol " $>$ " by " $\geq$ " in (3).

In this paper we first obtain the following generalization of Theorem 1.1.

THEOREM 1.2. *Let* $P(z) := \sum_{j=0}^{n} A_j z^j$ *be a matrix polynomial such that* $A_j \in \mathbb{C}^{m\times m}, 0 \leq j \leq n$, *are positive-definite. If the scalars* $t_1 > t_2 \geq 0$ *can be found such that*

$$t_1 t_2 A_j + (t_1 - t_2)A_{j-1} - A_{j-2} \geq 0, \ j = 1, 2, \cdots, n+1, \tag{4}$$

where $A_{-1} = A_{n+1} = 0$. *Then the eigenvalues of* $P(z)$ *lie in the closed disk*

$$|\lambda| \leq t_1. \tag{5}$$

For $t_1 = 1, t_2 = 0$, Theorem 1.2 reduces to Theorem 1.1. Moreover a result due to Aziz and Mohammad [2] is a special case of Theorem 1.2 if we put $m = 1$.

On applying Theorem 1.2, to the matrix polynomial $Q(z) = z^n P\left(\frac{1}{z}\right)$, we get the following:

COROLLARY 1.3. *Let* $P(z) := \sum_{j=0}^{n} A_j z^j$ *be a matrix polynomial such that* $A_j \in \mathbb{C}^{m\times m}, 0 \leq j \leq n$ *are positive-definite. If* $t_1 > t_2 \geq 0$ *can be found such that*

$$t_1 t_2 A_j + (t_1 - t_2)A_{j+1} - A_{j+2} \geq 0, \ j = -1, 0, \cdots, n-1, \tag{6}$$

where $A_{-1} = A_{n+1} = 0$. *Then the eigenvalues of* $P(z)$ *lie in the region*

$$|\lambda| \geq \frac{1}{t_1}. \tag{7}$$

On combining Theorem 1.2 and Corollary 1.3 and making $t_2 = 0$, a result due to Lê, Du and Nguyên [8, Theorem 2.6] follows immediately .

We next prove the following improvement of a result due to Lê, Du and Nguyên [8, Theorem 2.3].

THEOREM 1.4. *Let* $P(z) := \sum_{j=0}^{n} A_j z^j$, *be a matrix polynomial such that* $A_j \in \mathbb{C}^{m\times m}, 0 \le j \le n$ *satisfy*

$$A_n \ge A_{n-1} \ge \ldots \ge A_0 \ge 0,\ A_n > 0. \tag{8}$$

Then the eigenvalues of $P(z)$ *lie in the annular region*

$$\frac{\lambda_{\min}(A_0)}{\lambda_{\max}(2A_n - A_0)} \le |\lambda| \le 1. \tag{9}$$

The bound obtained is sharp and equality holds for $P(z) = \sum_{j=0}^{n} I z^j$.

A result due to Gardner and Govil [5] is a special case of Theorem 1.4, if we put in particular $m = 1$. Also note that if $A_0 > 0$, then the lower bound given by Theorem 1.4 is always better than that obtained in [8, Theorem 2.3].
Finally we obtain the following result.

THEOREM 1.5. *Let* $P(z) := \sum_{j=0}^{n-1} A_j z^j + I z^n$, $A_j \in \mathbb{C}^{m\times m}, 0 \le j \le n$ *be a monic matrix polynomial. Denote*

$$L_p = n^{\frac{1}{q}} \left(\sum_{j=0}^{n-1} \|A_j\|^p \right)^{\frac{1}{p}}, \tag{10}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\|\cdot\|$ *is the subordinate matrix norm.*
Then the eigenvalues of $P(z)$ *lie in the closed disk*

$$|\lambda| \le \max\left(L_p, L_p^{\frac{1}{n}} \right). \tag{11}$$

The bound obtained is sharp and equality holds for $P(z) = I z^n - \frac{1}{n}\sum_{j=0}^{n-1} I z^j$.

A result due to Mohammad [10] is a special case of Theorem 1.5, if we put $m = 1$. Letting $q \to \infty$ in Theorem 1.5 we get the following.

COROLLARY 1.6. *The eigenvalues of* $P(z) := \sum_{j=0}^{n-1} A_j z^j + I z^n, A_j \in \mathbb{C}^{m\times m}, 0 \le j \le n-1$ *lie in the closed disk*

$$|\lambda| \le \max(L_1, L_1^{\frac{1}{n}}), \tag{12}$$

where

$$L_1 = \sum_{j=0}^{n-1} \|A_j\|. \tag{13}$$

In the special case when $\|\cdot\|$ is the subordinate norm $\|\cdot\|_2$ defined by $\|A\|_2 := \max_{\mathbf{u}^*\mathbf{u}=1} \sqrt{(A\mathbf{u})^*(A\mathbf{u})},\ A \in \mathbb{C}^{m\times m}$, then for a Hermitian matrix A, $\|A\|_2 = \rho(A)$. In this context we have the following.

Corollary 1.7. *Let* $P(z) := \sum_{j=0}^{n-1} A_j z^j + Iz^n$ *be a matrix polynomial such that* $A_j \in \mathbb{C}^{m\times m}, 0 \le j \le n-1$ *are Hermitian matrices. Denote*

$$L'_p = n^{\frac{1}{q}} \left(\sum_{j=0}^{n-1} (\rho(A_j))^p \right)^{\frac{1}{p}}, \ \frac{1}{p} + \frac{1}{q} = 1. \tag{14}$$

Then the eigenvalues of $P(z)$ *satisfy*

$$|\lambda| \le \max(L'_p, {L'_p}^{\frac{1}{n}}). \tag{15}$$

Letting $q \to \infty$ in Corollary 1.7 we get the following.

Corollary 1.8. *Let* $P(z) := \sum_{j=0}^{n-1} A_j z^j + Iz^n$ *be a matrix polynomial* $A_j \in \mathbb{C}^{m\times m}, 0 \le j \le n-1$ *are Hermitian. Then the eigenvalues of P(z) lie in the closed disk*

$$|\lambda| \le \max(L'_1, {L'_1}^{\frac{1}{n}}), \tag{16}$$

where

$$L'_1 = \sum_{j=0}^{n-1} \rho(A_j). \tag{17}$$

2. Lemma and proofs of theorems

For the proof of these theorems we need the following lemma (for reference see. [6]).

Lemma 2.1. *Let* $M \in \mathbb{C}^{m\times m}$ *be a Hermitian matrix, then*

$$\lambda_{\min}(M) = \min_{u\in\mathbb{C}^m, \boldsymbol{u}^*\boldsymbol{u}=1} \{\boldsymbol{u}^* M \boldsymbol{u}\} \tag{18}$$

and

$$\lambda_{\max}(M) = \max_{u\in C^m, \boldsymbol{u}^*\boldsymbol{u}=1} \{\boldsymbol{u}^* M \boldsymbol{u}\}. \tag{19}$$

Proof of Theorem 1.2*:* Let λ be an eigenvalue of $P(z)$ and $\mathbf{u}$ be the corresponding eigenvector. Define

$$\begin{aligned} P_{\mathbf{u}}(z) =& \mathbf{u}^* P(z) \mathbf{u} \\ =& \sum_{j=0}^{n} \mathbf{u}^* A_j \mathbf{u} z^j. \end{aligned}$$

Since $\mathbf{u}^* A_j \mathbf{u} \in \mathbb{C}$, therefore $\mathbf{u}^* P(z)\mathbf{u}$ is a polynomial with complex coefficients. Since

$$t_1 t_2 A_j + (t_1 - t_2) A_{j-1} - A_{j-2} \ge 0, \ j = 1, 2, \ldots, n+1,$$

we obtain

$$\mathbf{u}^* \left(t_1 t_2 A_j + (t_1 - t_2) A_{j-1} - A_{j-2} \right) \mathbf{u} \ge 0,$$

i.e.,

$$t_1 t_2 \mathbf{u}^* A_j \mathbf{u} + (t_1 - t_2) \mathbf{u}^* A_{j-1} \mathbf{u} - \mathbf{u}^* A_{j-2} \mathbf{u} \ge 0. \tag{20}$$

Define

$$\begin{aligned} G_{\mathbf{u}}(z) &= (t_2+z)(t_1-z)P_{\mathbf{u}}(z) \\ &= \mathbf{u}^*\left(\left(t_1t_2+(t_1-t_2)z-z^2\right)\sum_{j=0}^{n}A_jz^j\right)\mathbf{u} \\ &= \mathbf{u}^*\left(\sum_{j=0}^{n+2}\left(t_1t_2A_j+(t_1-t_2)A_{j-1}-A_{j-2}\right)z^j\right)\mathbf{u}, \end{aligned}$$

where

$$A_{-2}=A_{-1}=A_{n+1}=A_{n+2}=0.$$

Let

$$\begin{aligned} H_{\mathbf{u}}(z) &= z^{n+2}G_{\mathbf{u}}\left(\frac{1}{z}\right) \\ &= \mathbf{u}^*\left(\sum_{j=0}^{n+2}\left(t_1t_2A_j+(t_1-t_2)A_{j-1}-A_{j-2}\right)z^{n-j+2}\right)\mathbf{u} \\ &= -\mathbf{u}^*A_n\mathbf{u}+K_{\mathbf{u}}(z), \end{aligned} \tag{21}$$

where

$$K_{\mathbf{u}}(z)=\mathbf{u}^*\left(\sum_{j=1}^{n+2}\left(t_1t_2A_{n-j+2}+(t_1-t_2)A_{n-j+1}-A_{n-j}\right)z^j\right)\mathbf{u}.$$

Then, by (20),

$$\begin{aligned} \max_{|z|=\frac{1}{t_1}}|K_u(z)| &\le \sum_{j=1}^{n+2}\left|\mathbf{u}^*\left(t_1t_2A_{n-j+2}+(t_1-t_2)A_{n-j+1}-A_{n-j}\right)\mathbf{u}\right|\frac{1}{t_1^j} \\ &= \sum_{j=1}^{n+2}\mathbf{u}^*\left(t_1t_2A_{n-j+2}+(t_1-t_2)A_{n-j+1}-A_{n-j}\right)\mathbf{u}\frac{1}{t_1^j} \\ &= \mathbf{u}^*A_n\mathbf{u}. \end{aligned}$$

Since $K_{\mathbf{u}}(z)$ is a polynomial with complex coefficients, by Schwarz's lemma (for ref. see [1]) we have for $|z|\le\dfrac{1}{t_1}$

$$|K_{\mathbf{u}}(z)|\le(\mathbf{u}^*A_n\mathbf{u})t_1|z|.$$

Therefore from (21), we have for $|z|\le\dfrac{1}{t_1}$

$$\begin{aligned} |H_{\mathbf{u}}(z)| &\ge |\mathbf{u}^*A_n\mathbf{u}|-|\mathbf{u}^*A_n\mathbf{u}zt_1| \\ &= \mathbf{u}^*A_n\mathbf{u}(1-|z|t_1). \end{aligned}$$

Thus we get for $|z|<\dfrac{1}{t_1}$

$$|H_{\mathbf{u}}(z)|>0.$$

Consequently the zeros of $H_{\mathbf{u}}(z)$ lie in $|z|\ge\dfrac{1}{t_1}$ and thus that of $G_{\mathbf{u}}(z)$ lie in $|z|\le t_1$. Therefore all the zeros of $P_{\mathbf{u}}(z)$ lie in the closed disk $|z|\le t_1$. Since λ is a zero of

$P_{\mathbf{u}}(z)$, therefore $|\lambda| \leq t_1$. That is the eigenvalues of $P(z)$ lie in the closed disk

$$|\lambda| \leq t_1.$$

This proves the theorem. □

Proof of Theorem 1.4*:* For the proof of the upper bound (see [3, Theorem 2.1]). To prove the lower bound, let λ be an eigenvalue of $P(z)$ and $\mathbf{u}$ be the corresponding unit eigenvector. Define

$$\begin{aligned} P_{\mathbf{u}}(z) =& \mathbf{u}^* P(z)\mathbf{u} \\ =& \sum_{j=0}^{n} \mathbf{u}^* A_j \mathbf{u} z^j. \end{aligned} \tag{22}$$

From (8), it follows that

$$\mathbf{u}^* A_n \mathbf{u} \geq \mathbf{u}^* A_{n-1} \mathbf{u} \geq \ldots \geq \mathbf{u}^* A_0 \mathbf{u} \geq 0. \tag{23}$$

Define

$$\begin{aligned} G_{\mathbf{u}}(z) &= (1-z)P_{\mathbf{u}}(z) \\ &= (1-z)\sum_{j=0}^{n} \mathbf{u}^* A_j \mathbf{u} z^j \\ &= \mathbf{u}^* A_0 \mathbf{u} + \mathbf{u}^* \left(\sum_{j=1}^{n} (A_j - A_{j-1}) z^j - A_n z^{n+1} \right) \mathbf{u} \\ &= \mathbf{u}^* A_0 \mathbf{u} + H_{\mathbf{u}}(z), \end{aligned}$$

where

$$H_{\mathbf{u}}(z) = \mathbf{u}^* \left(\sum_{j=1}^{n} (A_j - A_{j-1}) z^j - A_n z^{n+1} \right) \mathbf{u}.$$

Thus for $|z| = 1$, we have on using (23)

$$\begin{aligned} |H_{\mathbf{u}}(z)| &= \left| \mathbf{u}^* \left(\sum_{j=1}^{n} (A_j - A_{j-1}) z^j - A_n z^{n+1} \right) \mathbf{u} \right| \\ &\leq \sum_{j=1}^{n} (\mathbf{u}^* A_j \mathbf{u} - \mathbf{u}^* A_{j-1} \mathbf{u}) + \mathbf{u}^* A_n \mathbf{u} \\ &= \mathbf{u}^*(2A_n - A_0)\mathbf{u}. \end{aligned}$$

Now $H_{\mathbf{u}}(z)$ is a polynomial with complex coefficients, therefore by Schwarz's lemma (for ref. see [1]) we have for $|z| \leq 1$

$$|H_{\mathbf{u}}(z)| \leq \mathbf{u}^*(2A_n - A_0)\mathbf{u}|z|.$$

Thus we have for $|z| \leq 1$

$$\begin{aligned} |G_{\mathbf{u}}(z)| &\geq |\mathbf{u}^* A_0 \mathbf{u}| - |\mathbf{u}^*(2A_n - A_0)\mathbf{u} z| \\ &= \mathbf{u}^* A_0 \mathbf{u} - \mathbf{u}^*(2A_n - A_0)\mathbf{u}|z|. \end{aligned}$$

Notice that $\dfrac{\mathbf{u}^* A_0 \mathbf{u}}{\mathbf{u}^*(2A_n - A_0)\mathbf{u}} \leq 1$. So that if $|z| < \dfrac{\mathbf{u}^* A_0 \mathbf{u}}{\mathbf{u}^*(2A_n - A_0)\mathbf{u}}$, then $G_{\mathbf{u}}(z) \neq 0$ and in turn $P_{\mathbf{u}}(z) \neq 0$. Therefore, the zeros of $P_{\mathbf{u}}(z)$ lie in the region

$$|z| \geq \frac{\mathbf{u}^* A_0 \mathbf{u}}{2\mathbf{u}^* A_n \mathbf{u} - \mathbf{u}^* A_0 \mathbf{u}}.$$

Since λ is a zero of $P_{\mathbf{u}}(z)$, therefore

$$|\lambda| \geq \frac{\mathbf{u}^* A_0 \mathbf{u}}{2\mathbf{u}^* A_n \mathbf{u} - \mathbf{u}^* A_0 \mathbf{u}}. \tag{24}$$

This gives on using Lemma 2.1

$$|\lambda| \geq \frac{\lambda_{\min}(A_0)}{\lambda_{\max}(2A_n - A_0)}. \tag{25}$$

This proves the theorem completely. □

Proof of Theorem 1.5*:* Let $\mathbf{u}$ be a unit vector. Then

$$\begin{aligned}
\|P(z)\mathbf{u}\| &= \left\| \mathbf{u} z^n + \sum_{j=0}^{n-1} A_j \mathbf{u} z^j \right\| \\
&\geq \|\mathbf{u} z^n\| - \sum_{j=0}^{n-1} \|A_j \mathbf{u} z^j\| \\
&\geq |z^n| - \sum_{j=0}^{n-1} \|A_j\| |z|^j. \qquad (26)
\end{aligned}$$

Thus by Holder's inequality, we have

$$\begin{aligned}
\|P(z)\mathbf{u}\| &\geq |z|^n - n^{\frac{1}{q}} \left(\sum_{j=0}^{n-1} \left(\|A_j\| |z|^j \right)^p \right)^{\frac{1}{p}} \\
&= |z|^n \left(1 - n^{\frac{1}{q}} \left(\sum_{j=0}^{n-1} \left(\frac{\|A_j\|}{|z|^{(n-j)}} \right)^p \right)^{\frac{1}{p}} \right). \qquad (27)
\end{aligned}$$

Let $|z| > \max(1, L_p)$. Then

$$\begin{aligned}
\|P(z)\mathbf{u}\| &\geq |z|^n \left(1 - \frac{n^{\frac{1}{q}}}{|z|} \left(\sum_{j=0}^{n-1} \|A_j\|^p \right)^{\frac{1}{p}} \right) \\
&= |z|^n \left(1 - \frac{L_p}{|z|} \right) \\
&> 0.
\end{aligned}$$

Therefore each eigenvalue of $P(z)$ lies in the closed disk

$$|\lambda| \leq \max\{1, L_p\}. \tag{28}$$

The result follows from (28), if $L_p \geq 1$. If however $L_p < 1$, then from (28), it follows

$$|\lambda| \leq 1.$$

Assume $L_p^{\frac{1}{n}} < |z| \leq 1$, then from (27) we have

$$\begin{aligned} \|P(z)\mathbf{u}\| &\geq |z|^n \left(1 - \frac{n^{\frac{1}{q}}}{|z|^n} \left(\sum_{j=0}^{n-1} \|A_j\|^p\right)^{\frac{1}{p}}\right) \\ &= |z|^n \left(1 - \frac{L_p}{|z|^n}\right) \\ &> 0. \end{aligned}$$

Thus in this case the result also follows and hence the theorem is proved completely. □

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