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ON THE NUMBER OF ZEROS OF BICOMPLEX ENTIRE FUNCTIONS

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Abstract. This paper portrays the results on bicomplex entire functions that are concerned with the positioning of zeros of Eneström-Kakeya type. Moreover, some examples are provided to validate our results.

1. Introduction

The classical Eneström-Kakeya theorem gives the lower and upper bounds on the zeros of a complex polynomial with positive coefficients. Several generalisations of this power tool of determining zeros have been obtained over the years (see [4], [5] , [13]). Recently, Carney et. al [2] have extended it to quaternionic polynomials which lack the commutativity in general. Our prior motive is to establish the analogs results of this theorem for bicomplex entire functions which are of quite importance in analytic number theory and the theory of error correcting codes. In addition, bicomplex entire functions have applications in mathematical analysis, mathematical physics, and engineering . They provide a framework for studying problems involving multidimensional complex variables and can be used to model systems with bicomplex-valued variables.

2. Background

Bicomplex algebra was introduced by Segre [12], who became inspired by the work of Irish mathematician William Hamilton on quaternions and is actually a generalisation of complex numbers. The set of bicomplex numbers is denoted by $\mathbb{BC}$ and are generally represented as $\mathbb{BC} = \{Z = z_1 + jz_2 \mid z_1, z_2 \in \mathbb{C}\}$, where $\mathbb{C}$ is the set of complex numbers with the imaginary unit i, and where i and $j \neq i$ are commuting imaginary units. Therefore they form a commutative integral domain but not a field, as there exists infinite number of zero divisiors of the form $Z = \lambda(1 \pm ij)$, where $\lambda \in \mathbb{C}\backslash\{0\}$. We start with a property of bicomplex numbers which has no analogs for complex numbers and called as idempotent representation.

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2.1. Idempotent representation of bicomplex numbers. Consider the bicomplex numbers

$$e = \frac{1+ij}{2} \text{ and } e^{\dagger} = \frac{1-ij}{2}$$

then

$$e.e^{\dagger} = 0 \text{ (i.e., each of them is a zero divisior).}$$

Also

$$e^2 = e \text{ and } \left(e^{\dagger}\right)^2 = e^{\dagger} \text{(i.e., they are idempotents).}$$

Now for any $Z = z_1 + z_2 \in \mathbb{BC}$ we have

$$\begin{aligned} Z = z_1 + jz_2 &= \frac{z_1 - iz_2 + z_1 + iz_2}{2} + j\frac{z_2 + iz_1 + z_2 - iz_1}{2} \\ &= \frac{z_1 - iz_2}{2} + \frac{z_1 + iz_2}{2} + ij\frac{z_1 - iz_2}{2} - ij\frac{z_1 + iz_2}{2} \\ &= (z_1 - iz_2)\frac{1+ij}{2} + (z_1 + iz_2)\frac{1-ij}{2}, \end{aligned}$$

that is,

$$Z = \alpha_1 e + \alpha_2 e^{\dagger}, \tag{1}$$

where $\alpha_1 = z_1 - iz_2$ and $\alpha_2 = z_1 + iz_2$ are complex numbers. Formula (1) is called the idempotent representation of the bicomplex number Z and is a unique one. From the idempotent representation of any bicomplex number $Z = z_1 + jz_2$ as $Z = (z_1 - iz_2)\, e + (z_1 + iz_2)\, e^{\dagger}$, we get the idea of defining two spaces $\mathbb{A} = \{z_1 - iz_2 : z_1, z_2 \in \mathbb{C}\}$ and $\overline{\mathbb{A}} = \{z_1 + iz_2 : z_1, z_2 \in \mathbb{C}\}$, known as auxiliary complex spaces. Though $\mathbb{A}$ and $\overline{\mathbb{A}}$ contain same elements as in $\mathbb{C}$ but this convenient notation are used for special representation of elements in the sense that each $Z = z_1 + jz_2 = (z_1 - iz_2)\, e + \left(z_1 + iz_2 e^{\dagger}\right) \in \mathbb{BC}$ associates the points $(z_1 - iz_2) \in \mathbb{A}$ and $(z_1 + iz_2) \in \overline{\mathbb{A}}$. Also to each point $(z_1 - iz_2, z_1 + iz_2) \in \mathbb{A} \times \overline{\mathbb{A}}$, there is a unique point in $\mathbb{BC}$.

2.2. Cartesian Product. The cartesian set in $\mathbb{BC}$ determined by $X_1 \subset \mathbb{A}$ and $X_2 \subset \overline{\mathbb{A}}$ is defined as $X_1 \times_e X_2 := \left\{z_1 + jz_2 \in \mathbb{BC} : z_1 + jz_2 = \omega_1 e + \omega_2 e^{\dagger}, (\omega_1, \omega_2) \in X_1 \times X_2\right\}$.

2.3. The Euclidean norm of a bicomplex number. The Euclidean norm of a bicomplex number $Z = z_1 + jz_2 = \alpha_1 e + \alpha_2 e^{\dagger} \in \mathbb{BC}$ is a function $\|\| : \mathbb{BC} \to \mathbb{R}^{+}$denoted by $\|Z\|$ and given by:

$$\|Z\| = \sqrt{|z_1|^2 + |z_2|^2} = \frac{1}{\sqrt{2}}\sqrt{|\alpha_1|^2 + |\alpha_2|^2}.$$

It is easy to verify that for any two bicomplex numbers Z and W,

$$\|Z.W\| \leq \sqrt{2}\, \|Z\|.\|W\|.$$

2.4. Open discus in $\mathbb{BC}$. An open discus $D(a; r_1, r_2)$ with centre $a = a_1e + a_2e^{\dagger}$ and associated radii $r_1 > 0, r_2 > 0$ is defined as

$$\begin{aligned} D(a; r_1, r_2) &= B(a_1, r_1) \times_e B(a_2, r_2) \\ &= \left\{\omega_1 e + \omega_2 e^{\dagger} \in \mathbb{BC} : |\omega_1 - a_1| < r_1, |\omega_2 - a_2| < r_2\right\}, \end{aligned}$$

where $B(z, r)$ represent open ball with centre z and radius r.

2.5. Closed discus in $\mathbb{BC}$. A closed discus $\bar{D}(a; r_1, r_2)$ with centre $a = a_1e + a_2e^{\dagger}$ and associated radii $r_1 > 0, r_2 > 0$ is defined as

$$\begin{aligned} D(a; r_1, r_2) &= \bar{B}(a_1, r_1) \times_e \bar{B}(a_2, r_2) \\ &= \left\{\omega_1 e + \omega_2 e^{\dagger} \in \mathbb{BC} : |\omega_1 - a_1| \le r_1, |\omega_2 - a_2| \le r_2\right\}, \end{aligned}$$

where $\bar{B}(z, r)$ represent closed ball with centre z and radius r.

Geometrically, $\bar{D}(a; r_1, r_2)$ represents a double cylinder (duocylinder) in 4 dimensional Euclidean space.

2.6. Entire function in $\mathbb{BC}$. A bicomplex entire function $f(Z)$ can be represented by an everywhere convergent power series as $f(Z) = \sum_{s=0}^{\infty} A_s Z^s$, where A_s are bicomplex numbers.

2.7. Bicomplex Polynomial. Let $p(Z) = \sum_{s=0}^{n} A_s Z^s$ be a bicomplex polynomial of degree n, then p has the idempotent representation as

$$p(Z) = \sum_{s=0}^{n} (\alpha_s \zeta_1^s)\, e + \sum_{s=0}^{n} (\beta_s \zeta_2^s)\, e^{\dagger} = p_1(\zeta_1)\, e + p_2(\zeta_2)\, e^{\dagger},$$

where p_1 and p_2 are considered as complex polynomials in one variable. To have more detailed information on bicomplex numbers, refer to the book " Bicomplex Holomorphic Functions: The Algebra, Geometry and Analysis of Bicomplex Numbers" by M. Elena Luna-Elizarrarás et. al [7].

The study of location of zeros of bicomplex entie functions and in particular bicomplex polynomials is not as easy as complex polynomials. There exists bicomplex polynomials which have no zero in $\mathbb{BC}$ or having infinite number of zeros. For that reason, let us first state the analogue of the fundamental theorem of algebra for bicomplex polynomials [8].

THEOREM 2.1. *Consider a bicomplex polynomial $p(Z) = \sum_{s=0}^{n} A_s Z^s$. If all the coefficients A_s with the exception of the free term $A_0 = \gamma_0 e + \delta_0 e^{\dagger}$ are complex multiples of e (respectively of $e^{\dagger}$), but A_0 has $\delta_0 \neq 0$ (respectively $\gamma_0 \neq 0$), then $p(Z)$ has no roots. In all other cases, $p(Z)$ has at least one root.*

EXAMPLE 2.1: Consider the bicomplex polynomial

$$(1 + ij)Z^2 + 2 = 0.$$

This equation can be to written in idempotent form as:

$$2eZ^2 + 2(e + e^{\dagger}) = 0.$$

This gives,

$$(Z^2+1)e = 0,$$

$$e^{\dagger} = 0,$$

which has no roots.
Note: If $p(Z)$ is a monic bicomplex polynomial of degree n, then $p(Z)$ has exactly n^2 zeros.

So, finding the number of zeros of bicomplex entire functions in particular of polynomials is not an easy task and with that motivation, we shall give some results that predict the location of zeros of these functions of some special type. For that, we require the following lemmas:

3. Lemmas

The first two lemmas are by G. B. Price [11].

Lemma 1. *Let* $A = A_1e + A_2e^{\dagger} = \left\{\zeta_1 e + \zeta_2 e^{\dagger} : \zeta_1 \in A_1, \zeta_2 \in A_2\right\}$ *be a domain in* $\mathbb{BC}$*. A bicomplex function* $F = F_1e + F_2e^{\dagger} : A \longrightarrow \mathbb{BC}$ *is said to be holomorphic if and only if both the component functions* F_1 *and* F_2 *are complex holomorphic in* A_1 *and* A_2 *respectively.*

Lemma 2. *Let* F *be a bicomplex holomorphic function defined in a domain* $A = A_1e + A_2e^{\dagger} = \left\{\zeta_1 e + \zeta_2 e^{\dagger} : \zeta_1 \in A_1, \zeta_2 \in A_2\right\}$ *such that* $F(Z) = F_1(\zeta_1)\, e + F_2(\zeta_2)\, e^{\dagger}$*, for all* $z = \zeta_1 e + \zeta_2 e^{\dagger} \in A$*. Then,* $F(Z)$ *has a zero in* A *if and only if* $F_1(\zeta_1)$ *and* $F_2(\zeta_2)$ *both have zero at* ζ_1 *in* A_1 *and at* ζ_2 *in* A_2 *respectively.*

The next lemma is credited to Hermann Amandus Schwarz and popularly known as Schwarz lemma.

Lemma 3. *If a complex function* $h(z)$ *is holomorphic in* $|z| \leq R$ *such that* $h(0) = 0$ *and* $|h(z)| \leq M$ *for* $|z| = R$*, then*

$$|h(z)| \leq \frac{M|z|}{R}.$$

Finally, we require the following lemma which is due to Govil and Rahman [4].

Lemma 4. *If for some real* β

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2, \quad a_j \neq 0$$

and for some positive real numbers t_1 *and* t_2*,* $t_1 |a_j| \geq t_2 |a_{j-1}|$*, then*

$$|t_1 a_j - t_2 a_{j-1}| \leq \big(t_1 |a_j| - t_2|a_{j-1}|\big) \cos\alpha + \big(t_1|a_j| + t_2|a_{j-1}|\big) \sin\alpha.$$

4. Main Results

In this section, we present the following two main results:

THEOREM 4.1. *Let* $f(Z) = \sum_{s=0}^{\infty} A_s Z^s$ *be a bicomplex entire function with real coefficients and for some* $k_i \geq 1$,

$$k_0 A_0 \geq k_1 A_1 \geq \ldots \geq k_{r-1} A_{r-1} \geq A_r \geq \ldots \quad .$$

Then $f(Z)$ *does not vanish in the open discus* $D(0; t_0, t_0)$, *where*

$$t_0 = \frac{|A_0|}{k_0 A_0 + 2 \sum_{m=0}^{r-1} |A_m (k_m - 1)|}.$$

Proof. By using the fact $e + e^{\dagger} = 1$ and by the idempotent representation of a bicomplex number Z, $f(Z)$ can be expressed as

$$\begin{aligned} f(Z) &= \sum_{s=0}^{\infty} \left(A_s e + A_s e^{\dagger}\right) \left(\zeta_1 e + \zeta_2 e^{\dagger}\right)^s \\ &= \sum_{s=0}^{\infty} \left(A_s e + A_s e^{\dagger}\right) \left(\zeta_1^s e + \zeta_2^s e^{\dagger}\right) \\ &= \sum_{s=0}^{\infty} A_s \zeta_1^s e + \sum_{s=0}^{\infty} A_s \zeta_2^s e^{\dagger} \\ &= h_1(\zeta_1) e + h_2(\zeta_2) e^{\dagger}. \end{aligned}$$

As $f(Z)$ being entire is analytic in the closed discus $\bar{D}(0; 1, 1) \subset \mathbb{BC}$, therefore by Lemma 1, $h_1(\zeta_1)$ and $h_2(\zeta_2)$ both are holomorphic respectively in $X_1 = \{\zeta_1 \in A_1 : |\zeta_1| \leq 1\} \subset \mathbb{C}$ and $X_2 = \{\zeta_2 \in A_2 : |\zeta_2| \leq 1\} \subset \mathbb{C}$ where $A_1 = \{z_1 - iz_2 : z_1, z_2 \in \mathbb{C}\}$ and $A_2 = \{z_1 + iz_2 : z_1, z_2 \in \mathbb{C}\}$.

Now consider the function

$$\begin{aligned} F(\zeta_1) &= (\zeta_1 - 1) h_1(\zeta_1) \\ &= -A_0 + (A_0 - A_1)\zeta_1 + (A_1 - A_2)\zeta_1^2 + \ldots + (A_{r-2} - A_{r-1})\zeta_1^{r-1} + \\ &\quad (A_{r-1} - A_r)\zeta_1^r + \ldots \\ &= -A_0 + (k_0 A_0 - k_1 A_1 + A_0(1 - k_0) + A_1(k_1 - 1))\zeta_1 \\ &\quad + (k_1 A_1 - k_2 A_2 + A_1(1 - k_1) + A_2(k_2 - 1))\zeta_1^2 + \ldots + \\ &\quad (k_{r-2} A_{r-2} - k_{r-1} A_{r-1} + A_{r-2}(1 - k_{r-2}) + A_{r-1}(k_{r-1} - 1))\zeta_1^{r-1} + \\ &\quad (k_{r-1} A_{r-1} - A_r + A_{r-1}(1 - k_{r-1}))\zeta_1^r + \ldots \\ &= -A_0 + G(\zeta_1) \quad \text{where} \quad G(\zeta_1) = (k_0 A_0 - k_1 A_1 + A_0(1 - k_0) + A_1(k_1 - 1))\zeta_1 \\ &\quad + (k_1 A_1 - k_2 A_2 + A_1(1 - k_1) + A_2(k_2 - 1))\zeta_1^2 + \ldots + (k_{r-2} A_{r-2} \\ &\quad k_{r-1} A_{r-1} + A_{r-2}(1 - k_{r-2}) + A_{r-1}(k_{r-1} - 1))\zeta_1^{r-1} + \\ &\quad (k_{r-1} A_{r-1} - A_r + A_{r-1}(1 - k_{r-1}))\zeta_1^r + (A_r - A_{r+1})\zeta_1^{r+1} + \ldots \quad . \end{aligned}$$

For $|\zeta_1| = 1$, we have

$$\begin{aligned} |G(\zeta_1)| &= \big|(k_0A_0 - k_1A_1 + A_0(1-k_0) + A_1(k_1-1))\zeta_1 + (k_1A_1 - k_2A_2 + \\ &\quad A_1(1-k_1) + A_2(k_2-1))\zeta_1^2 + \ldots + (k_{r-2}A_{r-2} - k_{r-1}A_{r-1} + \\ &\quad A_{r-2}(1-k_{r-2}) + A_{r-1}(k_{r-1}-1))\zeta_1^{r-1} + (k_{r-1}A_{r-1} - A_r \\ &\quad +A_{r-1}(1-k_{r-1}))\zeta_1^r + (A_r - A_{r+1})\zeta_1^{r+1} + \ldots\big| \\ &\le k_0A_0 + 2\sum_{m=0}^{r-1} |A_m(k_m-1)| = S(\text{say}). \end{aligned}$$

Since $G(\zeta_1)$ is holomorphic in $|\zeta_1| \le 1, G(0) = 0$ and $|G(\zeta_1)| \le S$ for $|\zeta_1| = 1$, therefore in view of Lemma 3, we have

$$|G(\zeta_1)| \le \frac{S\,|\zeta_1|}{1}, \quad \text{for } |\zeta_1| \le 1.$$

Now for $|\zeta_1| < 1$, we get

$$\begin{aligned} |F(\zeta_1)| &\ge |-A_0 + G(\zeta_1)| \\ &\ge |A_0| - |G(\zeta_1)| \\ &\ge |A_0| - S\,|\zeta_1| > 0 \text{ if } |\zeta_1| < \frac{|A_0|}{S} = t_0 \text{ (say)}. \end{aligned}$$

This implies that $|h_1(\zeta_1)| > 0$ for $|\zeta_1| < t_0$.

Similarly, it can be shown that $|h_2(\zeta_2)| > 0$ for $|\zeta_2| < t_0$.

This shows that the functions $h_1(\zeta_1)$ and $h_2(\zeta_2)$ do not vanish respectively in $A_1' = \{\zeta_1 \in X_1 : |\zeta_1| < t_0\}$ and $A_2' = \{\zeta_2 \in X_2 : |\zeta_2| < t_0\}$.
Hence in view of Lemma 2, $f(Z) = h_1(\zeta_1)\,e + h_2(\zeta_2)\,e^\dagger$ do not vanish in $A_1'e + A_2'e^\dagger = D(0; t_0, t_0)$. That proves the theorem. □

The following example ensures the validity of Theorem 4.1.

Example 2.2 Let $f(Z) = e^Z - \frac{5}{6}$. Then $f(Z) = \frac{1}{6} + Z + \frac{Z^2}{2!} + \frac{Z^3}{3!} + \ldots$. Let $k_0 = 6$ and $k_i = 1$ for all $i = 1, 2, 3, \ldots$. Therefore by Theorem 4.1, $f(Z)$ does not vanish in $D(0, t_0, t_0)$, where $t_0 = \frac{\frac{1}{6}}{1+\frac{5}{3}} = \frac{1}{16}$.

Counter Example: Consider $f(Z) = \frac{5}{6} - e^Z$. Then, $f(Z) = \frac{-1}{6} - Z - \frac{Z^2}{2!} - \frac{Z^3}{3!} - \ldots$. Choose $k_0 = 12$ and $k_i = 1$ for all $i = 1, 2, 3, \ldots$ Clearly, $k_0A_0 < k_1A_1 < k_2A_2 < \ldots$. Therefore, all the zeros of $f(Z)$ lie in the discus $D(0; t_0, t_0)$, where $t_0 = \frac{1/6}{-2+11/3} = \frac{1}{10}$.

Theorem 4.2. *Let $f(Z) = \sum_{s=0}^{\infty} A_s Z^s$ be a bicomplex entire function with real coefficients such that*

$$|A_0| \ge |A_1| \ge \ldots \ge |A_{r-1}| \ge A_r \ge A_{r+1} \ldots \quad .$$

Then $f(Z)$ does not vanish in the open discus $D(0; t_0, t_0)$ where

$$t_0 = \frac{|A_0|}{|A_0| \cos\alpha + (|A_{r-1}| - |\, A_0 \,|) \sin\alpha + 2 \sum_{s=0}^{r-2} |A_s| \sin\alpha + M_p},$$

and

$$M_p = \sum_{p=r-1}^{\infty} |A_p - A_{p+1}|\,.$$

Proof.

$$\begin{aligned} f(Z) &= \sum_{s=0}^{\infty} \left(A_s e + A_s e^{\dagger}\right) \left(\zeta_1 e + \zeta_2 e^{\dagger}\right)^s \\ &= \sum_{s=0}^{\infty} \left(A_s e + A_s e^{\dagger}\right) \left(\zeta_1^s e + \zeta_2^s e^{\dagger}\right) \\ &= \sum_{s=0}^{\infty} A_s \zeta_1^s e + \sum_{s=0}^{\infty} A_s \zeta_2^s e^{\dagger} \\ &= h_1(\zeta_1)\, e + h_2(\zeta_2)\, e^{\dagger}. \end{aligned}$$

Since $f(Z)$ is analytic in any closed discus; in particular $\bar{D}(0;1,1) \subset \mathbb{BC}$, which implies the respective functions $h_1(\zeta_1)$ and $h_2(\zeta_2)$ are analytic in the complex regions $X_1 = \{\zeta_1 \in A_1 : |\zeta_1| \leq 1\} \subset \mathbb{C}$ and $X_2 = \{\zeta_2 \in A_2 : |\zeta_2| \leq 1\} \subset \mathbb{C}$ where $A_1 = \{z_1 - iz_2 : z_1, z_2 \in \mathbb{C}\}$ and $A_2 = \{z_1 + iz_2 : z_1, z_2 \in \mathbb{C}\}$.

Now consider the function

$$\begin{aligned} F(\zeta_1) &= (\zeta_1 - 1)\, h_1(\zeta_1) \\ &= -A_0 + (A_0 - A_1)\,\zeta_1 + (A_1 - A_2)\,\zeta_1^2 + \ldots + (A_{r-2} - A_{r-1})\,\zeta_1^{r-1} + \\ &\quad (A_{r-1} - A_r)\,\zeta_1^r + \ldots \\ &= -A_0 + G(\zeta_1) \text{ where } G(\zeta_1) = (A_0 - A_1)\,\zeta_1 + (A_1 - A_2)\,\zeta_1^2 + \ldots + \\ &\quad (A_{r-2} - A_{r-1})\,\zeta_1^{r-1} + (A_{r-1} - A_r)\,\zeta_1^r + (A_r - A_{r+1})\,\zeta_1^{r+1} + \ldots \quad . \end{aligned}$$

For $|\zeta_1| = 1$, we have

$$\begin{aligned} |G(\zeta_1)| &= |\, (A_0 - A_1)\,\zeta_1 + (A_1 - A_2)\,\zeta_1^2 + \ldots + (A_{r-2} - A_{r-1})\,\zeta_1^{r-1} + \\ &\quad (A_{r-1} - A_r)\,\zeta_1^r + (A_r - A_{r+1})\,\zeta_1^{r+1} + \ldots \,| \\ &\leq |A_0 - A_1| + |A_1 - A_2| + \ldots + |A_{r-2} - A_{r-1}| + |A_{r-1} - A_r| \\ &\quad + |A_r - A_{r+1}| + \ldots \quad . \end{aligned}$$

Applying Lemma 4, it immediately yields

$$\begin{aligned} |G(\zeta_1)| &\leq |A_0| \cos\alpha + (|A_0| + |A_{r-1}|) \sin\alpha - |A_{r-1}| \cos\alpha + 2 \sum_{s=1}^{r-2} |A_s| \sin\alpha \\ &\quad + |A_{r-1} - A_r| + |A_r - A_{r+1}| + \ldots \\ &\leq |A_0| \cos\alpha + (|A_{r-1}| - |A_0|) \sin\alpha + 2 \sum_{s=0}^{r-2} |A_s| \sin\alpha + M_p = M \text{ (say)} \\ &\quad \text{where } M_p = \sum_{p=r-1}^{\infty} |A_p - A_{p+1}|. \end{aligned}$$

Since $G(\zeta_1)$ is holomorphic in $|\zeta_1| \leq 1, G(0) = 0$ and $|G(\zeta_1)| \leq M$ for $|\zeta_1| = 1$, therefore by Lemma 3, we have

$$|G(\zeta_1)| \leq \frac{M|\zeta_1|}{1}, \text{ for } |\zeta_1| \leq 1.$$

Now for $|\zeta_1| < 1$,we get

$$\begin{aligned} |F(\zeta_1)| &\geq |-A_0 + G(\zeta_1)| \\ &\geq |A_0| - |G(\zeta_1)| \\ &\geq |A_0| - M|\zeta_1| > 0 \quad \text{if} \quad |\zeta_1| < \frac{|A_0|}{M} = t_0 \text{ (say)}. \end{aligned}$$

This shows that $|h_1(\zeta_1)| > 0$ for $|\zeta_1| < t_0$.

Likewise, we can shown that $|h_2(\zeta_2)| > 0$ for $|\zeta_2| < t_0$.

This implies that the functions $h_1(\zeta_1)$ and $h_2(\zeta_2)$ do not vanish respectively in $A_1' = \{\zeta_1 \in X_1 : |\zeta_1| < t_0\}$ and $A_2' = \{\zeta_2 \in X_2 : |\zeta_2| < t_0\}$.

Hence in view of Lemma 2, $f(Z) = h_1(\zeta_1)e + h_2(\zeta_2)e^{\dagger}$ do not vanish in $A_1'e + A_2'e^{\dagger} = D(0; t_0, t_0)$. which is our desired result. □

Example 2.3 Let $f(Z) = e^Z + Z^2 - 3Z - 4$. Then $f(Z) = -3 - 2Z + 3\frac{Z^2}{2!} + \frac{Z^3}{3!} + \ldots$. Clearly $|A_0| \geq |A_1| \geq |A_2| \geq |A_3|, \ldots$. Hence by Theorem 4.2, $f(Z)$ does not vanish in $D(0, t_0, t_0)$ where $t_0 = \frac{3}{3\cos\alpha - \sin\alpha + \sum_{p=1}^{\infty} |A_p - A_{p+1}|}$.

Counter Example: Consider $f(Z) = 4Z^2 + 3Z + 1$. Then, clearly $|A_0| < |A_1| < |A_2|$. Therefore, all the zeros of $f(Z)$ lie within or on $D(0; t_0, t_0)$, where $t_0 = \frac{1}{\cos\alpha + 11\sin\alpha + 4}$.

5. Conclusion:

This paper concludes with a summary of the key findings and contributions of the research, highlighting the importance of bicomplex polynomials, aims to contribute to the existing body of knowledge, stimulate further research, and foster new applications of bicomplex polynomials in various scientific and engineering domains.

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