Korean J. Math. **31** (2023), No. 3, pp. 323–337
https://dx.doi.org/10.11568/kjm.2023.31.3.323

INTEGRATION OF BICOMPLEX VALUED FUNCTION ALONG HYPERBOLIC CURVE

Chinmay Ghosh* and Soumen Mondal

Abstract. In this paper, we have defined bicomplex valued functions of bounded variations and rectifiable hyperbolic path. We have studied the integration of product-type bicomplex valued functions on rectifiable hyperbolic path. Also we have established bicomplex analogue of the Fundamental Theorem of Calculus for hyperbolic line integral.

1. Introduction

In 1883, Hamilton [9] discovered four dimensional quaternions to extend the complex number system to more than two dimensions. Quaternions have algebraic properties of real and complex numbers except the commutativity of multiplication. In 1892, C. Segre [14] introduced another four dimensional generalization of complex numbers, called bicomplex numbers. Unlike quaternions the set of bicomplex numbers form a commutative ring having divisors of zero. Following Segre, many mathematicians developed the theory of functions on bicomplex numbers. The theory of bicomplex variables is presented systematically in the book [12] of G. B. Price. There is an interesting subset of the set of bicomplex numbers, called the set of hyperbolic numbers. G. B. Price has not given much focus on hyperbolic numbers. In [15] we get a geometrical view of the hyperbolic numbers.

In 2016, A. S. Balankin et al. [2] introduced the concept of hyperbolic intervals and the hyperbolic length of the hyperbolic interval. In 2019, J. Bory-Rayes et al. [3] introduced the integration of product-type functions over hyperbolic curves using the concept of limit over a filter base. In 2022, M. E. Luna-Elizarrarás [10] defined the partition of a hyperbolic interval and introduced the integration of functions of hyperbolic variable.

In this paper, we have studied the integration of product-type bicomplex function over hyperbolic path in a different way. Our results are presented in the line of the book [5].

Received February 4, 2023. Revised August 30, 2023. Accepted August 30, 2023.
2010 Mathematics Subject Classification: 30E20, 30G35, 32A30.
Key words and phrases: Bounded variation, Partitions, Rectifiable hyperbolic path, Hyperbolic interval, Product-type bicomplex functions.
$*$ Corresponding author.

2. Basic definitions

We denote the set of real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$ respectively. We may think three imaginary numbers $\mathbf{i}_1, \mathbf{i}_2$ and $\mathbf{j}$ governed by the rules

$$\mathbf{i}_1^2 = -1, \mathbf{i}_2^2 = -1, \mathbf{j}^2 = 1$$

$$\begin{aligned} \mathbf{i}_1\mathbf{i}_2 &= \mathbf{i}_2\mathbf{i}_1 = \mathbf{j} \\ \mathbf{i}_1\mathbf{j} &= \mathbf{j}\mathbf{i}_1 = -\mathbf{i}_2 \\ \mathbf{i}_2\mathbf{j} &= \mathbf{j}\mathbf{i}_2 = -\mathbf{i}_1. \end{aligned}$$

Then we have two complex planes $\mathbb{C}(\mathbf{i}_1) = \{x + \mathbf{i}_1 y : x, y \in \mathbb{R}\}$ and $\mathbb{C}(\mathbf{i}_2) = \{x + \mathbf{i}_2 y : x, y \in \mathbb{R}\}$, both of which are identical to $\mathbb{C}$. Bicomplex numbers([1], [13]) are defined as $\zeta = z_1 + \mathbf{i}_2 z_2$ for $z_1, z_2 \in \mathbb{C}(\mathbf{i}_1)$. The set of all bicomplex numbers is denoted by $\mathbb{BC}$. In particular if $z_1 = x, z_2 = \mathbf{i}_1 y$ where $x, y \in \mathbb{R}$ we get $\zeta = x + \mathbf{j}y$ and these type of numbers are called hyperbolic numbers or duplex numbers. The set of all hyperbolic numbers is denoted by $\mathbb{D}$. For $(z_1 + \mathbf{i}_2 z_2), (w_1 + \mathbf{i}_2 w_2) \in \mathbb{BC}$, the addition and multiplication are definde as

$$\begin{aligned} (z_1 + \mathbf{i}_2 z_2) + (w_1 + \mathbf{i}_2 w_2) &= (z_1 + w_1) + \mathbf{i}_2 (z_2 + w_2) \\ (z_1 + \mathbf{i}_2 z_2)(w_1 + \mathbf{i}_2 w_2) &= (z_1 w_1 - z_2 w_2) + \mathbf{i}_2 (z_1 w_2 + z_2 w_1). \end{aligned}$$

With these operations $\mathbb{BC}$ forms a commutative ring with zero divisors. The elements $z_1 + \mathbf{i}_2 z_2 \in \mathbb{BC}$ such that $z_1^2 + z_2^2 = 0$ are the zero divisors. We denote the set of nonzero zero divisors in $\mathbb{BC}$ by $\mathfrak{O}$ whereas $\mathfrak{O}_0 = \mathfrak{O} \cup \{0\}$. On the other hand let us denote the set of nonzero zero divisors in $\mathbb{D}$ by $\mathbb{O}$ whereas $\mathbb{O}_0 = \mathbb{O} \cup \{0\}$.The interesting property of a bicomplex number is its idempotent representation. Setting $\mathbf{e}_1 = \frac{1+\mathbf{j}}{2}$ and $\mathbf{e}_2 = \frac{1-\mathbf{j}}{2}$, we get

$$z_1 + \mathbf{i}_2 z_2 = (z_1 - \mathbf{i}_1 z_2)\mathbf{e}_1 + (z_1 + \mathbf{i}_1 z_2)\mathbf{e}_2.$$

Many calculations become easier in this representation.

The set of nonnegative hyperbolic numbers is

$$\mathbb{D}^+ = \{\nu_1 \mathbf{e}_1 + \nu_2 \mathbf{e}_2 : \nu_1, \nu_2 \geq 0\}.$$

A hyperbolic number ζ is said to be (strictly) positive if $\zeta \in \mathbb{D}^+ \setminus \{0\}$.The set of nonnegative hyperbolic numbers is also defined as

$$\mathbb{D}^+ = \left\{x + y\mathbf{k} : x^2 - y^2 \geq 0, x \geq 0\right\}.$$

On the realization of $\mathbb{D}^+$, M.E. Luna-Elizarraras et.al. [11] defined a partial order relation on $\mathbb{D}$. For two hyperbolic numbers ζ_1, ζ_2 the relation $\preceq_{\mathbb{D}}$ is defined as

$$\zeta_1 \preceq_{\mathbb{D}} \zeta_2 \text{ if and only if } \zeta_2 - \zeta_1 \in \mathbb{D}^+.$$

One can check that this relation is reflexive, transitive and antisymmetric. Therefore $\preceq_{\mathbb{D}}$ is a partial order relation on $\mathbb{D}$. This partial order relation $\preceq_{\mathbb{D}}$ on $\mathbb{D}$ is an extension of the total order relation $\leq$ on $\mathbb{R}$. We say $\zeta_1 \prec_{\mathbb{D}} \zeta_2$ if $\zeta_1 \preceq_{\mathbb{D}} \zeta_2$ but $\zeta_1 \neq \zeta_2$. Also we say $\zeta_2 \succeq_{\mathbb{D}} \zeta_1$ if $\zeta_1 \preceq_{\mathbb{D}} \zeta_2$ and $\zeta_2 \succ_{\mathbb{D}} \zeta_1$ if $\zeta_1 \prec_{\mathbb{D}} \zeta_2$.

Definition 2.1. [11] For any hyperbolic number $\zeta = \nu_1 \mathbf{e}_1 + \nu_2 \mathbf{e}_2$, the $\mathbb{D}-$modulus of ζ is defined by

$$|\zeta|_{\mathbb{D}} = |\nu_1 \mathbf{e}_1 + \nu_2 \mathbf{e}_2|_{\mathbb{D}} = |\nu_1| \mathbf{e}_1 + |\nu_2| \mathbf{e}_2 \in \mathbb{D}^+$$

where $|\nu_1|$ and $|\nu_2|$ are the usual modulus of real numbers.

Definition 2.2. [11] A subset A of $\mathbb{D}$ is said to be $\mathbb{D}-$ bounded if there exists $M \in \mathbb{D}^+$ such that $|\zeta|_{\mathbb{D}} \preceq_{\mathbb{D}} M$ for any $\zeta \in A$.

Set

$$\begin{aligned} A_1 &= \{x \in \mathbb{R} : \exists\, y \in \mathbb{R},\ x\mathbf{e}_1 + y\mathbf{e}_2 \in A\}, \\ A_2 &= \{y \in \mathbb{R} : \exists\, x \in \mathbb{R},\ x\mathbf{e}_1 + y\mathbf{e}_2 \in A\}. \end{aligned}$$

If A is $\mathbb{D}-$bounded then A_1 and A_2 are bounded subset of $\mathbb{R}$.

Definition 2.3. [11] For a $\mathbb{D}-$ bounded subset A of $\mathbb{D}$, the supremum of A with respect to the $\mathbb{D}-$ modulus is defined by

$$\sup_{\mathbb{D}} A = \sup A_1\mathbf{e}_1 + \sup A_2\mathbf{e}_2.$$

Definition 2.4. [11] A sequence of hyperbolic numbers $\{\zeta_n\}_{n\geq 1}$ is said to be convergent to $\zeta \in \mathbb{D}$ if for $\varepsilon \in \mathbb{D}^+ \setminus \{0\}$ there exists $k \in \mathbb{N}$ such that

$$|\zeta_n - \zeta|_{\mathbb{D}} \prec_{\mathbb{D}} \varepsilon.$$

Then we write

$$\lim_{n\to\infty} \zeta_n = \zeta.$$

Definition 2.5. [11] A sequence of hyperbolic numbers $\{\zeta_n\}_{n\geq 1}$ is said to be $\mathbb{D}-$ Cauchy sequence $\zeta \in \mathbb{D}$ if for $\varepsilon \in \mathbb{D}^+ \setminus \{0\}$ $\exists\, N \in \mathbb{N}$ such that

$$|\zeta_{N+m} - \zeta_N|_{\mathbb{D}} \prec_{\mathbb{D}} \varepsilon$$

for all $m = 1, 2, 3, \ldots$.

Note that a sequence of hyperbolic numbers $\{\zeta_n\}_{n\geq 1}$ is convergent if and only if it is a $\mathbb{D}-$ Cauchy sequence.

Definition 2.6. [11] A hyperbolic series $\sum\limits_{n=1}^{\infty} \zeta_n$ is convergent if and only if its partial sum is a $\mathbb{D}-$ Cauchy sequence, i.e., for any $\varepsilon \in \mathbb{D}^+ \setminus \{0\}$ $\exists\, N \in \mathbb{N}$ such that

$$\left| \sum_{k=1}^{m} \zeta_{N+k} \right|_{\mathbb{D}} \prec_{\mathbb{D}} \varepsilon$$

for any $m \in \mathbb{N}$.

Definition 2.7. [11] A hyperbolic series $\sum\limits_{n=1}^{\infty} \zeta_n$ is $\mathbb{D}-$absolutely convergent if the series $\sum\limits_{n=1}^{\infty} |\zeta_n|_{\mathbb{D}}$ is convergent.

Every $\mathbb{D}-$ absolutely convergent series is convergent.

Definition 2.8. [2] Let $\alpha = a_1\mathbf{e}_1 + a_2\mathbf{e}_2, \beta = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 \in \mathbb{D}$ with $\alpha \preceq_{\mathbb{D}} \beta$. The closed hyperbolic interval ($\mathbb{D}-$interval) is defined by

$$[\alpha, \beta]_{\mathbb{D}} = \{\zeta \in \mathbb{D} : \alpha \preceq_{\mathbb{D}} \zeta \preceq_{\mathbb{D}} \beta\}.$$

Similarly the open hyperbolic interval ($\mathbb{D}-$interval) is defined by

$$(\alpha, \beta)_{\mathbb{D}} = \{\zeta \in \mathbb{D} : \alpha \prec_{\mathbb{D}} \zeta \prec_{\mathbb{D}} \beta\}.$$

The length of the $\mathbb{D}-$interval $[\alpha, \beta]_{\mathbb{D}}$ or $(\alpha, \beta)_{\mathbb{D}}$ is defined by

$$l_{\mathbb{D}}\left([\alpha, \beta]_{\mathbb{D}}\right) = \beta - \alpha.$$

$[\alpha, \beta]_{\mathbb{D}}$ is called a degenerate closed $\mathbb{D}-$interval if $\beta - \alpha$ is a nonnegative zero divisor hyperbolic number and $[\alpha, \beta]_{\mathbb{D}}$ is called a nondegenerate closed $\mathbb{D}-$interval if $\beta - \alpha \in \mathbb{D}^{+} \backslash \mathbb{O}_0$.

DEFINITION 2.9. [3] A set $A(\subset \mathbb{D})$ is called product-type set if $A{=}A_1\mathbf{e}_1 + A_2 e_2$ for some real sets A_1, A_2.

DEFINITION 2.10. [10] Let $[\alpha, \beta]_{\mathbb{D}}$ be a nondegenerate closed $\mathbb{D}-$interval. The partition P of $[\alpha, \beta]_{\mathbb{D}}$ is the set $\{\zeta_0, \zeta_1, \zeta_2, ..., \zeta_n\} \subset [\alpha, \beta]_{\mathbb{D}}$ such that

$$\alpha = \zeta_0 \prec_{\mathbb{D}} \zeta_1 \prec_{\mathbb{D}} \zeta_2 \prec_{\mathbb{D}} ... \prec_{\mathbb{D}} \zeta_n = \beta$$

and

$$\zeta_k - \zeta_{k-1} \in \mathbb{D}^{+} \backslash \mathbb{O}_0, k = 1, 2, ..., n.$$

A definition of infinity in the hyperbolic case is given in [8] as $\infty_{\mathbb{D}} = \infty \mathbf{e}_1 + \infty_{\mathbb{D}} \mathbf{e}_2$.

DEFINITION 2.11. [11] A hyperbolic path or $\mathbb{D}-$path is a $\mathbb{D}-$continuous function $\Gamma : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$, for some $\mathbb{D}-$interval $[\alpha, \beta]_{\mathbb{D}}$.

In that case we get for $\tau = t\mathbf{e}_1 + s\mathbf{e}_2 \in [\alpha, \beta]_{\mathbb{D}}$,

$$\Gamma(\tau) = \gamma_1(t)\,\mathbf{e}_1 + \gamma_2(s)\,\mathbf{e}_2$$

where $\gamma_1 : [a_1, b_1] \to \mathbb{C}$, $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ are two paths in $\mathbb{C}$ for $\alpha = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 \in \mathbb{D}, \beta = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 \in \mathbb{D}$.

DEFINITION 2.12. [6] A function $f : A{=}A_1\mathbf{e}_1{+}A_2\mathbf{e}_2 \subset \mathbb{BC} \to \mathbb{BC}$ is called product-type if there exist $f_i : A_i \to \mathbb{C}$ for $i = 1, 2$ such that $f(\alpha_1 e_1 + \alpha_2 e_2) = f_1(\alpha_1)\mathbf{e}_1 + f_2(\alpha_2)\mathbf{e}_2$ for all $\alpha_1 e_1 + \alpha_2 e_2 \in A$.

EXAMPLE 2.13. A $\mathbb{D}-$path $\Gamma(= \gamma_1 e_1 + \gamma_2 e_2) : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ is a product-type function.

3. Main Results

In this section we prove our main results.

DEFINITION 3.1. A function $\Gamma : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$, for $[\alpha, \beta]_{\mathbb{D}} \subset \mathbb{D}$, is of $\mathbb{D}-$bounded variation if there exists $M \in \mathbb{D}^{+}$ such that for any partition $P = \{\zeta_0, \zeta_1, \zeta_2, ..., \zeta_n\}$ of $[\alpha, \beta]_{\mathbb{D}}$

$$v(\Gamma; P) = \sum_{k=1}^{n} |\Gamma(\zeta_k) - \Gamma(\zeta_{k-1})|_{\mathbb{D}} \preceq_{\mathbb{D}} M.$$

The total $\mathbb{D}-$variation of Γ, $V(\Gamma)$, is defined by

$$V(\Gamma) = \sup\nolimits_{\mathbb{D}} \{v(\Gamma; P) : P \text{ is a partition of } [\alpha, \beta]_{\mathbb{D}}\}.$$

Obviously

$$V(\Gamma) \preceq_{\mathbb{D}} M \prec_{\mathbb{D}} \infty_{\mathbb{D}}.$$

PROPOSITION 3.2. *Let $\Gamma : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ be of $\mathbb{D}-$bounded variation.Then*
(a) If P and Q are partitions of $[\alpha, \beta]_{\mathbb{D}}$ and $P \subset Q$ then

$$v(\Gamma; P) \preceq_{\mathbb{D}} v(\Gamma; Q).$$

(*b*) *If* $\Lambda : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ *is also of* $\mathbb{D}-$*bounded variation and* $a, b \in \mathbb{BC}$ *then* $a\Gamma + b\Lambda$ *is of* $\mathbb{D}-$*bounded variation and*

$$V(a\Gamma + b\Lambda) \preceq_{\mathbb{D}} |a|_{\mathbb{D}} V(\Gamma) + |b|_{\mathbb{D}} V(\Lambda).$$

Proof. (*a*) Let $P = \{\zeta_0, \zeta_1, \zeta_2, ..., \zeta_n\}$ be a partition of $[\alpha, \beta]_{\mathbb{D}}$.

First we examine the effect of adjoining one additional point η to P.

The subinterval $[\zeta_{k-1}, \zeta_k]_{\mathbb{D}}$ is divided into two smaller subintervals $[\zeta_{k-1}, \eta]_{\mathbb{D}}$ and $[\eta, \zeta_k]_{\mathbb{D}}$ such that

$$\alpha = \zeta_0 \prec_{\mathbb{D}} \zeta_1 \prec_{\mathbb{D}} ... \prec_{\mathbb{D}} \zeta_{k-1} \prec_{\mathbb{D}} \eta \prec_{\mathbb{D}} \zeta_k \prec_{\mathbb{D}} ... \prec_{\mathbb{D}} \zeta_n = \beta,$$

and

$$\eta - \zeta_{k-1}; \zeta_k - \eta \in \mathbb{D}^+ \backslash \mathbb{O}_0.$$

Then the set $P_1 = \{\zeta_0, \zeta_1, \zeta_2, ..., \zeta_{k-1}, \eta, \zeta_k, ..., \zeta_n\}$ is a partition of $[\alpha, \beta]_{\mathbb{D}}$ such that $P \subset P_1$.

Now,

$$v(\Gamma; P) = |\Gamma(\zeta_1) - \Gamma(\zeta_0)|_{\mathbb{D}} + ... + |\Gamma(\zeta_k) - \Gamma(\zeta_{k-1})|_{\mathbb{D}} + ... + |\Gamma(\zeta_n) - \Gamma(\zeta_{n-1})|_{\mathbb{D}},$$

and

$$\begin{aligned} v(\Gamma; P_1) &= |\Gamma(\zeta_1) - \Gamma(\zeta_0)|_{\mathbb{D}} + ... + |\Gamma(\eta) - \Gamma(\zeta_{k-1})|_{\mathbb{D}} + |\Gamma(\zeta_k) - \Gamma(\eta)|_{\mathbb{D}} + ... \\ &\quad + |\Gamma(\zeta_n) - \Gamma(\zeta_{n-1})|_{\mathbb{D}}. \end{aligned}$$

Since

$$\begin{aligned} |\Gamma(\zeta_k) - \Gamma(\zeta_{k-1})|_{\mathbb{D}} &= |\Gamma(\zeta_k) - \Gamma(\eta) + \Gamma(\eta) - \Gamma(\zeta_{k-1})|_{\mathbb{D}} \\ &\preceq_{\mathbb{D}} |\Gamma(\zeta_k) - \Gamma(\eta)|_{\mathbb{D}} + |\Gamma(\eta) - \Gamma(\zeta_{k-1})|_{\mathbb{D}}, \end{aligned}$$

it follows that

$$v(\Gamma; P) \preceq_{\mathbb{D}} v(\Gamma; P_1).$$

Since Q can be obtained from P by adjoining a finite number of additional points to P, one at a time, by repeating the arguments a finite number of times, we have

$$v(\Gamma; P) \preceq_{\mathbb{D}} v(\Gamma; Q).$$

(*b*) Let $\Omega(x) = a\Gamma(x) + b\Lambda(x)$, $x \in [\alpha, \beta]_{\mathbb{D}}$.

Let $P = \{\zeta_0, \zeta_1, \zeta_2, ..., \zeta_n\}$ be a partition of $[\alpha, \beta]_{\mathbb{D}}$. Then

$$v(\Gamma; P) = |\Gamma(\zeta_1) - \Gamma(\zeta_0)|_{\mathbb{D}} + ... + |\Gamma(\zeta_k) - \Gamma(\zeta_{k-1})|_{\mathbb{D}} + ... + |\Gamma(\zeta_n) - \Gamma(\zeta_{n-1})|_{\mathbb{D}},$$

$$v(\Lambda; P) = |\Lambda(\zeta_1) - \Lambda(\zeta_0)|_{\mathbb{D}} + ... + |\Lambda(\zeta_k) - \Lambda(\zeta_{k-1})|_{\mathbb{D}} + ... + |\Lambda(\zeta_n) - \Lambda(\zeta_{n-1})|_{\mathbb{D}},$$

$$v(\Omega; P) = |\Omega(\zeta_1) - \Omega(\zeta_0)|_{\mathbb{D}} + ... + |\Omega(\zeta_k) - \Omega(\zeta_{k-1})|_{\mathbb{D}} + ... + |\Omega(\zeta_n) - \Omega(\zeta_{n-1})|_{\mathbb{D}}.$$

Now,

$$\begin{aligned} |\Omega(\zeta_r) - \Omega(\zeta_{r-1})|_{\mathbb{D}} &= |a\Gamma(\zeta_r) + b\Lambda(\zeta_r) - a\Gamma(\zeta_{r-1}) - b\Lambda(\zeta_{r-1})|_{\mathbb{D}} \\ &\preceq_{\mathbb{D}} |a|_{\mathbb{D}} |\Gamma(\zeta_r) - \Gamma(\zeta_{r-1})|_{\mathbb{D}} + |b|_{\mathbb{D}} |\Lambda(\zeta_r) - \Lambda(\zeta_{r-1})|_{\mathbb{D}} \end{aligned}$$

Therefore

$$v(\Omega; P) \preceq_{\mathbb{D}} |a|_{\mathbb{D}} v(\Gamma; P) + |b|_{\mathbb{D}} v(\Lambda; P).$$

Since Γ and Λ are functions of $\mathbb{D}-$bounded variations on $[\alpha, \beta]_{\mathbb{D}}$, we have

$$v(\Gamma; P) \preceq_{\mathbb{D}} V(\Gamma),$$

and

$$v(\Lambda; P) \preceq_{\mathbb{D}} V(\Lambda),$$

for all partitions P of $[\alpha,\beta]_{\mathbb{D}}$.
Therefore

$$v(\Omega;P)\preceq_{\mathbb{D}}|a|_{\mathbb{D}}V(\Gamma)+|b|_{\mathbb{D}}V(\Lambda),$$

for all partitions P of $[\alpha,\beta]_{\mathbb{D}}$.
This shows that

$$\begin{aligned}\sup_{\mathbb{D}}\{v(\Omega;P):P\text{ is a partition of }[\alpha,\beta]_{\mathbb{D}}\} &\preceq_{\mathbb{D}}|a|_{\mathbb{D}}V(\Gamma)+|b|_{\mathbb{D}}V(\Lambda)\\ &\preceq_{\mathbb{D}}M,\text{ for some }M\in\mathbb{D}^{+}.\end{aligned}$$

Hence $\Omega(=a\Gamma+b\Lambda)$ is a function of $\mathbb{D}-$bounded variation on $[\alpha,\beta]_{\mathbb{D}}$ and

$$V(a\Gamma+b\Lambda)\preceq_{\mathbb{D}}|a|_{\mathbb{D}}V(\Gamma)+|b|_{\mathbb{D}}V(\Lambda).$$

□

Definition 3.3. A function $\Gamma:[\alpha,\beta]_{\mathbb{D}}\to\mathbb{BC}$ is $\mathbb{D}-$differentiable at $\tau=t\mathbf{e}_1+s\mathbf{e}_2\in[\alpha,\beta]_{\mathbb{D}}$ if

$$\lim_{h\to0,h\notin\mathbb{O}}\frac{\Gamma(\tau+h)-\Gamma(\tau)}{h}$$

exists in $\mathbb{D}$. Then we say

$$\Gamma'(\tau)=\lim_{h\to0,h\notin\mathbb{O}}\frac{\Gamma(\tau+h)-\Gamma(\tau)}{h},$$

the $\mathbb{D}-$derivative of Γ at τ.

A $\mathbb{D}-$path $\Gamma:[\alpha,\beta]_{\mathbb{D}}\to\mathbb{BC}$ is called $\mathbb{D}-$smooth if $\Gamma'(\tau)$ exists for each $\tau\in[\alpha,\beta]_{\mathbb{D}}$. Also Γ is piecewise $\mathbb{D}-$smooth if there is a partition $P=\{\zeta_0,\zeta_1,\zeta_2,...,\zeta_n\}$ of $[\alpha,\beta]_{\mathbb{D}}$ such that Γ is $\mathbb{D}-$smooth on each subinterval $[\zeta_{k-1},\zeta_k]$.

One can easily check that if γ_1,γ_2 are (piecewise) smooth the $\Gamma=\gamma_1\mathbf{e}_1+\gamma_2\mathbf{e}_2$ is (piecewise) $\mathbb{D}-$smooth.

Proposition 3.4. *Let* $\Gamma(=\gamma_1\mathbf{e}_1+\gamma_2\mathbf{e}_2):[\alpha,\beta]_{\mathbb{D}}\to\mathbb{BC}$, *for* $[\alpha,\beta]_{\mathbb{D}}\subset\mathbb{D}$, *is a* $\mathbb{D}-$*path. Then* Γ *is of* $\mathbb{D}-$*bounded variation if and only if* γ_1,γ_2 *are of bounded variation. Also*

$$V(\Gamma)=V(\gamma_1)\mathbf{e}_1+V(\gamma_2)\mathbf{e}_2.$$

Proof. Since $\Gamma(=\gamma_1\mathbf{e}_1+\gamma_2\mathbf{e}_2):[\alpha,\beta]_{\mathbb{D}}\to\mathbb{BC}$ is a $\mathbb{D}-$path, for $i=1,2$, $\gamma_i:[\alpha_i,\beta_i]\to\mathbb{C}$ are paths in $\mathbb{C}$ where $\alpha=\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2$ and $\beta=\beta_1\mathbf{e}_1+\beta_2\mathbf{e}_2$.

Let $P=\{\zeta_0,\zeta_1,\zeta_2,...,\zeta_n\}$ be a partition of $[\alpha,\beta]_{\mathbb{D}}$. Taking $\zeta_i=\zeta_i^1\mathbf{e}_1+\zeta_i^2\mathbf{e}_2$, we get two partitions $P_1=\{\zeta_0^1,\zeta_1^1,\zeta_2^1,...,\zeta_n^1\}$ and $P_2=\{\zeta_0^2,\zeta_1^2,\zeta_2^2,...,\zeta_n^2\}$ of $[\alpha_1,\beta_1]$ and $[\alpha_2,\beta_2]$ respectively. Again for any two partitions P_1 of $[\alpha_1,\beta_1]$ and P_2 of $[\alpha_2,\beta_2]$, we can get a partition P of $[\alpha,\beta]_{\mathbb{D}}$.

Now

$$\begin{aligned}v(\Gamma;P) &= \sum_{k=1}^{n}|\Gamma(\zeta_k)-\Gamma(\zeta_{k-1})|_{\mathbb{D}}\\ &= \sum_{k=1}^{n}\left|\gamma_1(\zeta_k^1)-\gamma_1(\zeta_{k-1}^1)\right|\mathbf{e}_1+\sum_{k=1}^{n}\left|\gamma_2(\zeta_k^2)-\gamma_2(\zeta_{k-1}^2)\right|\mathbf{e}_2.\end{aligned}$$

Therefore

$$v(\Gamma;P)=v(\gamma_1;P_1)\mathbf{e}_1+v(\gamma_2;P_2)\mathbf{e}_2. \tag{1}$$

Now it is clear from (1) that for $M = M_1\mathbf{e}_1 + M_2\mathbf{e}_2 \in \mathbb{D}^+$

$$v(\Gamma; P) \preceq_{\mathbb{D}} M \Leftrightarrow v(\gamma_1; P_1) \le M_1 \text{ and } v(\gamma_2; P_2) \le M_2.$$

So, Γ is of $\mathbb{D}-$bounded variation iff γ_1 and γ_2 are of bounded variation.
Also,

$$\begin{aligned} V(\Gamma) &= \sup_{\mathbb{D}} \{v(\Gamma; P) : P \text{ is a partition of } [\alpha, \beta]_{\mathbb{D}}\} \\ &= \sup_{\mathbb{D}}\{v(\gamma_1; P_1)\mathbf{e}_1 + v(\gamma_2; P_2)\mathbf{e}_2 : P_i \text{ are partitions of } [\alpha_i, \beta_i], i = 1, 2\} \\ &= V(\gamma_1)\mathbf{e}_1 + V(\gamma_2)\mathbf{e}_2, \text{ using Definition 2.3.} \end{aligned}$$

□

Proposition 3.5. *If $\Gamma : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ is piecewise $\mathbb{D}-$smooth then Γ is of $\mathbb{D}-$bounded variation and*

$$V(\Gamma) = \int\limits_{[\alpha,\beta]_{\mathbb{D}}} |\Gamma'(\tau)|_{\mathbb{D}} \, d\tau.$$

Proof. Let $\Gamma = \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2$, $\alpha = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2$ and $\beta = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2$.
Since $\Gamma : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ is piecewise $\mathbb{D}-$smooth, then $\gamma_i : [\alpha_i, \beta_i] \to \mathbb{C}$, for $i = 1, 2$ are piecewise smooth.
Then by Proposition 1.3([5], Chapter IV), γ_i are of bounded variation and

$$V(\gamma_i) = \int\limits_{\alpha_i}^{\beta_i} \left|\gamma_i'(\tau_i)\right| d\tau_i, \text{ for } i = 1, 2.$$

Then by Proposition 3.4, Γ is of $\mathbb{D}-$bounded variation and

$$V(\Gamma) = V(\gamma_1)\mathbf{e}_1 + V(\gamma_2)\mathbf{e}_2.$$

Since $\Gamma' : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ is a $\mathbb{BC}-$function, then $\left|\Gamma'\right|_{\mathbb{D}} : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{D}$ is a natural hyperbolic function (see [16]) defined by $\left|\Gamma'\right|_{\mathbb{D}}(\tau) = \left|\Gamma'(\tau)\right|_{\mathbb{D}}$ for $\tau = \tau_1\mathbf{e}_1 + \tau_2\mathbf{e}_2$.
Now

$$\begin{aligned} \int\limits_{[\alpha,\beta]_{\mathbb{D}}} \left|\Gamma'(\tau)\right|_{\mathbb{D}} d\tau &= \left(\int\limits_{\alpha_1}^{\beta_1} \left|\gamma_1'(\tau_1)\right| d\tau_1\right)\mathbf{e}_1 + \left(\int\limits_{\alpha_2}^{\beta_2} \left|\gamma_2'(\tau_2)\right| d\tau_2\right)\mathbf{e}_2 \\ &= V(\gamma_1)\mathbf{e}_1 + V(\gamma_2)\mathbf{e}_2 \\ &= V(\Gamma). \end{aligned}$$

So, we have

$$V(\Gamma) = \int\limits_{[\alpha,\beta]_{\mathbb{D}}} |\Gamma'(\tau)|_{\mathbb{D}} \, d\tau.$$

□

Theorem 3.6. *Let $\Gamma(= \gamma_1 e_1 + \gamma_2 e_2) : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ be product-type function of $\mathbb{D}-$bounded variation and suppose that the product-type function $f(= f_1 e_1 + f_2 e_2) : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ is $\mathbb{D}-$continuous. Then there is $I \in \mathbb{BC}$ such that for every $\epsilon_{\mathbb{D}} \succ_{\mathbb{D}} 0$*

there is a $\delta_{\mathbb{D}} \succ_{\mathbb{D}} 0$ *such that when* $P = \{\zeta_0, \zeta_1, \zeta_2, ..., \zeta_n\}$ *be a partition of* $[\alpha, \beta]_{\mathbb{D}}$ *with* $\|P\|_{\mathbb{D}} = \sup_{\mathbb{D}}\{l_{\mathbb{D}}([\zeta_{k-1}, \zeta_k]_{\mathbb{D}}) : 1 \leq k \leq n\} \prec_{\mathbb{D}} \delta_{\mathbb{D}}$ *then*

$$\left| I - \sum_{k=1}^{n} f(\tau_k)[\Gamma(\zeta_k) - \Gamma(\zeta_{k-1})] \right|_{\mathbb{D}} \prec_{\mathbb{D}} \epsilon_{\mathbb{D}}$$

for whatever choice of points τ_k, $\zeta_{k-1} \preceq_{\mathbb{D}} \tau_k \preceq_{\mathbb{D}} \zeta_k$.

Proof. Since $\Gamma(= \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2) : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ is a product-type function of $\mathbb{D}-$bounded variation, by Proposition 3.4 for $i = 1, 2$ $\gamma_i : [\alpha_i, \beta_i] \to \mathbb{C}$ are of bounded variation, where $\alpha = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2$ and $\beta = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2$.

Also since $f(= f_1\mathbf{e}_1 + f_2\mathbf{e}_2) : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ is $\mathbb{D}-$continuous and product-type function, for $i = 1, 2$ $f_i : [\alpha_i, \beta_i] \to \mathbb{C}$ are continuous functions.

Then by Theorem 1.4 [5], for $i = 1, 2$ there exist $I_i \in \mathbb{C}$ such that for every $\epsilon_i > 0$ there is a $\delta_i > 0$ such that when $P_i = \{\zeta_0^i, \zeta_1^i, \zeta_2^i, ..., \zeta_n^i\}$ are partitions of $[\alpha_i, \beta_i]$ with $\|P_i\| = \max\{(\zeta_k^i - \zeta_{k-1}^i) : 1 \leq k \leq n\} < \delta_i$ then

$$\left| I_i - \sum_{k=1}^{n} f_i(\tau_k^i)[\gamma_i(\zeta_k^i) - \gamma_i(\zeta_{k-1}^i)] \right| < \epsilon_i$$

for whatever choice of points τ_k^i, $\zeta_{k-1}^i \leq \tau_k^i \leq \zeta_k^i$.

Let $I = I_1\mathbf{e}_1 + I_2\mathbf{e}_2$, $\epsilon_{\mathbb{D}} = \epsilon_i\mathbf{e}_1 + \epsilon_2\mathbf{e}_2$, $\delta_{\mathbb{D}} = \delta_1\mathbf{e}_1 + \delta_2\mathbf{e}_2$, $\tau_k = \tau_k^1\mathbf{e}_1 + \tau_k^2\mathbf{e}_2$ and $\zeta_k = \zeta_0^i\mathbf{e}_1 + \zeta_0^i\mathbf{e}_2$ for $k = 1, 2, ..., n$.

Then $P = \{\zeta_0, \zeta_1, \zeta_2, ..., \zeta_n\}$ be a partition of $[\alpha, \beta]_{\mathbb{D}}$ with $\|P\|_{\mathbb{D}} = \|P_1\|\,\mathbf{e}_1 + \|P_2\|\,\mathbf{e}_2 \prec_{\mathbb{D}} \delta_{\mathbb{D}}$ and $\zeta_{k-1} \preceq_{\mathbb{D}} \tau_k \preceq_{\mathbb{D}} \zeta_k$.

Now

$$\begin{aligned} \left| I - \sum_{k=1}^{n} f(\tau_k)[\Gamma(\zeta_k) - \Gamma(\zeta_{k-1})] \right|_{\mathbb{D}} &= \left| I_1 - \sum_{k=1}^{n} f_1(\tau_k^1)[\gamma_1(\zeta_k^1) - \gamma_i(\zeta_{k-1}^1)] \right| \mathbf{e}_1 \\ &\quad + \left| I_2 - \sum_{k=1}^{n} f_2(\tau_k^2)[\gamma_2(\zeta_k^2) - \gamma_2(\zeta_{k-1}^2)] \right| \mathbf{e}_2 \\ &\prec_{\mathbb{D}} \epsilon_i\mathbf{e}_1 + \epsilon_i\mathbf{e}_2 = \epsilon_{\mathbb{D}}. \end{aligned}$$

□

Remark 3.7. The number $I \in \mathbb{BC}$ of Theorem 3.6 is called the Riemann-Stieljes $\mathbb{D}-$integral of f with respect to Γ over $[\alpha, \beta]_{\mathbb{D}}$ and is designated by

$$I = \int_{[\alpha,\beta]_{\mathbb{D}}} f d\Gamma = \int_{[\alpha,\beta]_{\mathbb{D}}} f(\tau) d\Gamma(\tau).$$

Remark 3.8. From the proof of Theorem 3.6 and Theorem 1.4 [5], we can write

$$I = \int_{[\alpha,\beta]_{\mathbb{D}}} f d\Gamma = \left(\int_{\alpha_1}^{\beta_1} f_1(t) d\gamma_1(t) \right) \mathbf{e}_1 + \left(\int_{\alpha_2}^{\beta_2} f_2(s) d\gamma_2(s) \right) \mathbf{e}_2.$$

The next result is very easy to prove, so we only state the result.

PROPOSITION 3.9. *Let f and g be two product-type bicomplex functions defined on $[\alpha,\beta]_{\mathbb{D}}$ and $\Gamma,\Lambda : [\alpha,\beta]_{\mathbb{D}} \to \mathbb{BC}$ be product-type functions of $\mathbb{D}-$bounded variation. Then for any $a, b \in \mathbb{BC}$*

$$(i) \int\limits_{[\alpha,\beta]_{\mathbb{D}}} (af+bg)d\Gamma = a \int\limits_{[\alpha,\beta]_{\mathbb{D}}} fd\Gamma + b \int\limits_{[\alpha,\beta]_{\mathbb{D}}} gd\Gamma$$

$$(ii) \int\limits_{[\alpha,\beta]_{\mathbb{D}}} fd(a\Gamma+b\Lambda) = a \int\limits_{[\alpha,\beta]_{\mathbb{D}}} fd\Gamma + b \int\limits_{[\alpha,\beta]_{\mathbb{D}}} fd\Lambda.$$

PROPOSITION 3.10. *Let $\Gamma(=\gamma_1\mathbf{e}_1+\gamma_2\mathbf{e}_2) : [\alpha,\beta]_{\mathbb{D}} \to \mathbb{BC}$ be product-type function of $\mathbb{D}-$bounded variation and $f(= f_1\mathbf{e}_1 + f_2\mathbf{e}_2) : [\alpha,\beta]_{\mathbb{D}} \to \mathbb{BC}$ be $\mathbb{D}-$continuous product-type function. If $\alpha = \zeta_0 \prec_{\mathbb{D}} \zeta_1 \prec_{\mathbb{D}} ... \prec_{\mathbb{D}} \zeta_{k-1} \prec_{\mathbb{D}} \zeta_k \prec_{\mathbb{D}} ... \prec_{\mathbb{D}} \zeta_n = \beta$, then*

$$\int\limits_{[\alpha,\beta]_{\mathbb{D}}} fd\Gamma = \sum_{k=1}^{n} \int\limits_{[\zeta_{k-1},\zeta_k]_{\mathbb{D}}} fd\Gamma.$$

Proof. Let $\alpha = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2$, $\beta = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2$ and $\zeta_k = \zeta_k^1\mathbf{e}_1 + \zeta_k^2\mathbf{e}_2$ for $k = 0, 1, 2, ..., n$.

Now for $i = 1, 2$ $\gamma_i : [\alpha_i, \beta_i] \to \mathbb{C}$ are of bounded variation and $f_i : [\alpha_i, \beta_i] \to \mathbb{C}$ are continuous and also

$$\alpha_i = \zeta_0^i < \zeta_1^i < ... < \zeta_n^i = \beta_i.$$

Then by Proposition 1.8 [5], we have for $i = 1, 2$

$$\int_{\alpha_i}^{\beta_i} f_i d\gamma_i = \sum_{k=1}^{n} \int_{\zeta_{k-1}^i}^{\zeta_k^i} f_i d\gamma_i.$$

By Remark 3.8 we have

$$\begin{aligned}
\int\limits_{[\alpha,\beta]_{\mathbb{D}}} fd\Gamma &= \left(\int_{\alpha_1}^{\beta_1} f_1(t)d\gamma_1(t)\right)\mathbf{e}_1 + \left(\int_{\alpha_2}^{\beta_2} f_2(s)d\gamma_2(s)\right)\mathbf{e}_2 \\
&= \left(\sum_{k=1}^{n}\int_{\zeta_{k-1}^1}^{\zeta_k^1} f_1(t)d\gamma_1(t)\right)\mathbf{e}_1 + \left(\sum_{k=1}^{n}\int_{\zeta_{k-1}^2}^{\zeta_k^2} f_2(s)d\gamma_2(s)\right)\mathbf{e}_2 \\
&= \sum_{k=1}^{n} \int\limits_{[\zeta_{k-1},\zeta_k]_{\mathbb{D}}} fd\Gamma.
\end{aligned}$$

□

THEOREM 3.11. *If $\Gamma(= \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2) : [\alpha,\beta]_{\mathbb{D}} \to \mathbb{BC}$ is piecewise $\mathbb{D}-$smooth and $f(= f_1\mathbf{e}_1 + f_2\mathbf{e}_2) : [\alpha,\beta]_{\mathbb{D}} \to \mathbb{BC}$ be $\mathbb{D}-$continuous product-type function, then*

$$\int\limits_{[\alpha,\beta]_{\mathbb{D}}} fd\Gamma = \int\limits_{[\alpha,\beta]_{\mathbb{D}}} f(\tau)\Gamma'(\tau)d\tau.$$

Proof. Since Γ is piecewise $\mathbb{D}-$smooth, for $i = 1, 2$ $\gamma_i : [\alpha_i, \beta_i] \to \mathbb{C}$ are piecewise smooth, where $\alpha = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2$, $\beta = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2$.

Also since f is $\mathbb{D}-$continuous product-type function, $i = 1, 2$ $f_i : [\alpha_i, \beta_i] \to \mathbb{C}$ are continuous function.

Then by Theorem 1.9 [5], for $i = 1, 2$

$$\int_{\alpha_i}^{\beta_i} f_i d\gamma_i = \int_{\alpha_i}^{\beta_i} f_i(t_i)\gamma_i'(t_i)dt_i. \tag{2}$$

Let $\tau = t_1\mathbf{e}_1 + t_2\mathbf{e}_2$. Then $\tau \in [\alpha, \beta]_{\mathbb{D}}$ and $\Gamma'(\tau) = \gamma_1'(t_1)\mathbf{e}_1 + \gamma_2'(t_2)\mathbf{e}_2$.
By Remark 3.8

$$\begin{aligned}
\int_{[\alpha,\beta]_{\mathbb{D}}} f d\Gamma &= \left(\int_{\alpha_1}^{\beta_1} f_1 d\gamma_1\right)\mathbf{e}_1 + \left(\int_{\alpha_2}^{\beta_2} f_2 d\gamma_2\right)\mathbf{e}_2 \\
&= \left(\int_{\alpha_1}^{\beta_1} f_1(t_1)\gamma_1'(t_1)dt_1\right)\mathbf{e}_1 + \left(\int_{\alpha_2}^{\beta_2} f_2(t_2)\gamma_2'(t_2)dt_2\right)\mathbf{e}_2, \text{ by 2} \\
&= \int_{[\alpha,\beta]_{\mathbb{D}}} f(\tau)\Gamma'(\tau)d\tau.
\end{aligned}$$

□

If $\Gamma(= \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2) : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ is a $\mathbb{D}-$path, then the set $\{\Gamma(\tau) : \alpha \preceq_{\mathbb{D}} \tau \preceq_{\mathbb{D}} \beta\}$ is called the trace of Γ and is denoted by $\{\Gamma\}$. Γ is a rectifiable $\mathbb{D}-$path if Γ is a function of $\mathbb{D}-$bounded variation. For a partition P of $[\alpha, \beta]_{\mathbb{D}}$, $v(\Gamma; P)$ is the sum of hyperbolic lengths of the line segment connecting points on the trace of Γ. So Γ is rectifiable if it has finite hyperbolic length and its length is $V(\Gamma)$. If Γ is piecewise $\mathbb{D}-$smooth, then Γ is rectifiable and by Proposition 3.5, its legth is $\int_{[\alpha,\beta]_{\mathbb{D}}} |\Gamma'(\tau)|_{\mathbb{D}}\, d\tau$.

If $\Gamma : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ is a rectifiable $\mathbb{D}-$path with $\{\Gamma\} \subset \mathbb{E} \subset \mathbb{BC}$ and $f : \mathbb{E} \to \mathbb{BC}$ is $\mathbb{D}-$continuous product-type function, then $f \circ \Gamma : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ is a $\mathbb{D}-$continuous product-type function.

Remark 3.12. If $\Gamma(= \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2) : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ is a $\mathbb{D}-$path, then $\{\Gamma\} = \{\gamma_1\}\mathbf{e}_1 + \{\gamma_2\}\mathbf{e}_2$.

Definition 3.13. If $\Gamma : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ is a rectifiable $\mathbb{D}-$path and f is a product-type function defined and $\mathbb{D}-$continuous on the trace of Γ then the (line) integral of f along Γ is

$$\int_{[\alpha,\beta]_{\mathbb{D}}} f(\Gamma(\tau))d\Gamma(\tau).$$

This line integral is also denoted by

$$\int_{\Gamma} f = \int_{\Gamma} f(z)dz.$$

REMARK 3.14. If $\Gamma(=\gamma_1\mathbf{e}_1+\gamma_2\mathbf{e}_2):[\alpha,\beta]_{\mathbb{D}}\to\mathbb{BC}$ is a rectifiable $\mathbb{D}-$path and $f=(f_1\mathbf{e}_1+f_2\mathbf{e}_2)$ is a product-type function defined and $\mathbb{D}-$continuous on $\{\Gamma\}$, then it is easy to verify that

$$\int_{\Gamma} f=\left(\int_{\gamma_1} f_1\right)\mathbf{e}_1+\left(\int_{\gamma_2} f_2\right)\mathbf{e}_2.$$

DEFINITION 3.15. A function $\Phi:[\lambda,\mu]_{\mathbb{D}}\to[\alpha,\beta]_{\mathbb{D}}$ is said to be $\mathbb{D}-$monotone function if any one of the following hold

$i)$ $\Phi(\xi)\preceq_{\mathbb{D}}\Phi(\tau)$ for any $\xi,\tau\in[\lambda,\mu]_{\mathbb{D}}$ with $\xi\preceq_{\mathbb{D}}\tau$;

$ii)$ $\Phi(\xi)\preceq_{\mathbb{D}}\Phi(\tau)$ for any $\xi,\tau\in[\lambda,\mu]_{\mathbb{D}}$ with $\xi\succeq_{\mathbb{D}}\tau$.

REMARK 3.16. In the above definition if (i) holds then Φ is said to be $\mathbb{D}-$monotone increasing function on $[\lambda,\mu]_{\mathbb{D}}$ and if (ii) holds then Φ is said to be $\mathbb{D}-$monotone decreasing function on $[\lambda,\mu]_{\mathbb{D}}$.

REMARK 3.17. If $\Phi=(\Phi_1\mathbf{e}_1+\Phi_2\mathbf{e}_2):[\lambda,\mu]_{\mathbb{D}}\to[\alpha,\beta]_{\mathbb{D}}$ is a $\mathbb{D}-$monotone increasing product-type function then for each $i=1,2$ $\Phi_i:[\lambda_i,\mu_i]\to[\alpha_i,\beta_i]$ is monotone increasing function on $[\lambda_i,\mu_i]$, where $\lambda=\lambda_1\mathbf{e}_1+\lambda_2\mathbf{e}_2,\mu=\mu_1\mathbf{e}_1+\mu_2\mathbf{e}_2,\alpha=\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2,\beta=\beta_1\mathbf{e}_1+\beta_2\mathbf{e}_2$.

If $\Gamma:[\alpha,\beta]_{\mathbb{D}}\to\mathbb{BC}$ is a rectifiable $\mathbb{D}-$path and $\Phi:[\lambda,\mu]_{\mathbb{D}}\to[\alpha,\beta]_{\mathbb{D}}$ is a $\mathbb{D}-$continuous, $\mathbb{D}-$monotone increasing function with $\Phi([\lambda,\mu]_{\mathbb{D}})=[\alpha,\beta]_{\mathbb{D}}$ (i.e., $\Phi(\lambda)=\alpha$, $\Phi(\mu)=\beta$) then $\Gamma\circ\Phi:[\lambda,\mu]_{\mathbb{D}}\to\mathbb{BC}$ is a $\mathbb{D}-$path such that $\{\Gamma\circ\Phi\}=\{\Gamma\}$. Also, if $\Phi(z)\notin\mathbb{O}$ for all $z\in[\lambda,\mu]_{\mathbb{D}}$, then $\Gamma\circ\Phi$ is rectifiable because if $P=\{\zeta_0,\zeta_1,\zeta_2,...,\zeta_n\}$ be a partition of $[\lambda,\mu]_{\mathbb{D}}$ then $P_1=\{\Phi(\zeta_0),\Phi(\zeta_1),\Phi(\zeta_2),...,\Phi(\zeta_n)\}$ is a partition of $[\alpha,\beta]_{\mathbb{D}}$. Therefore

$$\sum_{k=1}^{n}|\Gamma(\Phi(\zeta_k))-\Gamma(\Phi(\zeta_{k-1}))|_{\mathbb{D}}\preceq_{\mathbb{D}}V(\Gamma)$$

so that $V(\Gamma\circ\Phi)\preceq_{\mathbb{D}}V(\Gamma)\prec_{\mathbb{D}}\infty_{\mathbb{D}}$. So if f is product-type $\mathbb{D}-$continuous on $\{\Gamma\}=\{\Gamma\circ\Phi\}$ then $\int_{\Gamma\circ\Phi} f$ is well defined.

PROPOSITION 3.18. *If* $\Gamma(=\gamma_1\mathbf{e}_1+\gamma_2\mathbf{e}_2):[\alpha,\beta]_{\mathbb{D}}\to\mathbb{BC}$ *is a rectifiable* $\mathbb{D}-$*path and* $\Phi=(\Phi_1\mathbf{e}_1+\Phi_2\mathbf{e}_2):[\lambda,\mu]_{\mathbb{D}}\to[\alpha,\beta]_{\mathbb{D}}$ *is a* $\mathbb{D}-$*monotone increasing product-type function with* $\Phi(\lambda)=\alpha$, $\Phi(\mu)=\beta$ *and* $\Phi(z)\notin\mathbb{O}$ *for all* $z\in[\lambda,\mu]_{\mathbb{D}}$; *then for any product-type* $\mathbb{D}-$*continuous function* $f(=f_1\mathbf{e}_1+f_2\mathbf{e}_2)$ *on* $\{\Gamma\}$

$$\int_{\Gamma} f=\int_{\Gamma\circ\Phi} f.$$

Proof. Let $\lambda=\lambda_1\mathbf{e}_1+\lambda_2\mathbf{e}_2,\mu=\mu_1\mathbf{e}_1+\mu_2\mathbf{e}_2,\alpha=\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2,\beta=\beta_1\mathbf{e}_1+\beta_2\mathbf{e}_2$.

Since $\Gamma(=\gamma_1\mathbf{e}_1+\gamma_2\mathbf{e}_2):[\alpha,\beta]_{\mathbb{D}}\to\mathbb{BC}$ is a rectifiable $\mathbb{D}-$path and $\Phi=(\Phi_1\mathbf{e}_1+\Phi_2\mathbf{e}_2):[\lambda,\mu]_{\mathbb{D}}\to[\alpha,\beta]_{\mathbb{D}}$ is a $\mathbb{D}-$monotone increasing product-type function with $\Phi(\lambda)=\alpha$, $\Phi(\mu)=\beta$, then for $i=1,2$ $\gamma_i:[\alpha_i,\beta_i]\to\mathbb{C}$ are rectifiable path and $\Phi_i:[\lambda_i,\mu_i]\to[\alpha_i,\beta_i]$ are continuous increasing functions with $\Phi_i(\lambda_i)=\alpha_i$ and $\Phi_i(\mu_i)=\beta_i$.

Also since $f(=f_1\mathbf{e}_1+f_2\mathbf{e}_2)$ is $\mathbb{D}-$continuous on $\{\Gamma\}$, then by Remark 3.12 we have f_i are continuous on $\{\gamma_i\}$ for $i=1,2$.

Then by Proposition 1.13 ([5], Chapter IV), we have

$$\int_{\gamma_i} f = \int_{\gamma_i \circ \Phi_i} f, \text{ for } i = 1, 2.$$

Then by Remark 3.14, we have

$$\begin{aligned} \int_{\Gamma} f &= \left(\int_{\gamma_1} f_1 \right) \mathbf{e}_1 + \left(\int_{\gamma_2} f_2 \right) \mathbf{e}_2 \\ &= \left(\int_{\gamma_1 \circ \Phi_1} f_1 \right) \mathbf{e}_1 + \left(\int_{\gamma_2 \circ \Phi_2} f_2 \right) \mathbf{e}_2 \\ &= \int_{\Gamma \circ \Phi} f. \end{aligned}$$

□

Let $\Gamma(= \gamma_1 \mathbf{e}_1 + \gamma_2 \mathbf{e}_2) : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ is a rectifiable $\mathbb{D}-$path and for $\alpha \preceq_{\mathbb{D}} \tau \preceq_{\mathbb{D}} \beta$, let $(\Gamma)_\tau$ be $V(\Gamma; [\alpha, \tau]_{\mathbb{D}})$. That is

(3) $$(\Gamma)_\tau = \sup_{\mathbb{D}} \left\{ \sum_{k=1}^{n} |\Gamma(\tau_k) - \Gamma(\tau_{k-1})|_{\mathbb{D}} : \{\tau_0, \tau_1, ..., \tau_n\} \text{ is a partition of } [\alpha, \tau]_{\mathbb{D}} \right\}.$$

Let $\alpha = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2, \beta = \beta_1 \mathbf{e}_1 + \beta_2 \mathbf{e}_2, \tau = t\mathbf{e}_1 + s\mathbf{e}_2$ and $\tau_k = t_k \mathbf{e}_1 + s_k \mathbf{e}_2$ for $k = 0, 1, ...n$.

Since Γ is a rectifiable $\mathbb{D}-$path, for $i = 1, 2$ $\gamma_i : [\alpha_i, \beta_i] \to \mathbb{C}$ are rectifiable path.

Let $\alpha_1 \leq t \leq \beta_1$, $\alpha_2 \leq s \leq \beta_2$ and also

$$(\gamma_1)_t = \sup \left\{ \sum_{k=1}^{n} |\gamma_1(t_k) - \gamma_1(t_{k-1})| : \{t_0, t_1, ..., t_n\} \text{ is a partition of } [\alpha_1, t] \right\},$$

$$(\gamma_2)_s = \sup \left\{ \sum_{k=1}^{n} |\gamma_2(s_k) - \gamma_2(s_{k-1})| : \{s_0, s_1, ..., s_n\} \text{ is a partition of } [\alpha_1, s] \right\}.$$

Then from (3) we have

$$(\Gamma)_\tau = (\gamma_1)_t \mathbf{e}_1 + (\gamma_2)_s \mathbf{e}_2.$$

Since $(\gamma_1)_t$ and $(\gamma_2)_s$ are increasing, $(\gamma_1)_t : [\alpha_i, \beta_i] \to \mathbb{R}$ and $(\gamma_2)_s : [\alpha_i, \beta_i] \to \mathbb{R}$ are bounded variation. So, by Proposition 3.4, $(\Gamma)_\tau : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{D}$ is of $\mathbb{D}-$bounded variation.

If $f(= f_1 \mathbf{e}_1 + f_2 \mathbf{e}_2)$ is product-type $\mathbb{D}-$continuous function on $\{\Gamma\}$ define

(4) $$\int_{\Gamma} f \, |dz|_{\mathbb{D}} = \int_{[\alpha, \beta]_{\mathbb{D}}} f(\Gamma(\tau)) d(\Gamma)_\tau.$$

Clearly f_1 is continuous on $\{\gamma_1\}$ and f_2 is continuous on $\{\gamma_2\}$. If we define

$$\int_{\gamma_1} f_1 \, |dz_1| = \int_{\alpha_1}^{\beta_1} f_1(\gamma_1(t)) d(\gamma_1)_t,$$

$$\int_{\gamma_2} f_2 \, |dz_2| = \int_{\alpha_2}^{\beta_2} f_2(\gamma_2(s)) d(\gamma_2)_s,$$

then for $dz = dz_1\mathbf{e}_1 + dz_2\mathbf{e}_2$, from (4) we have

$$\int_{\Gamma} f \, |dz|_{\mathbb{D}} = \left(\int_{\gamma_1} f_1 \, |dz_1| \right) \mathbf{e}_1 + \left(\int_{\gamma_2} f_2 \, |dz_2| \right) \mathbf{e}_2. \tag{5}$$

If Γ is rectifiable $\mathbb{D}-$curve in $\mathbb{BC}$ then denote by $-\Gamma$ the $\mathbb{D}-$curve defined by $(-\Gamma)(\tau) = \Gamma(-\tau)$ for $-\beta \preceq_{\mathbb{D}} \tau \preceq_{\mathbb{D}} -\alpha$. Also if $c \in \mathbb{BC}$ let $\Gamma + c$ denote the curve defined by $(\Gamma + c)(\tau) = \Gamma(\tau) + c$ for $\tau \in [\alpha, \beta]_{\mathbb{D}}$.

Proposition 3.19. *Let* $\Gamma (= \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2) : [\alpha, \beta]_{\mathbb{D}} \to \mathbb{BC}$ *is a rectifiable* $\mathbb{D}-$*path and suppose that* $f (= f_1\mathbf{e}_1 + f_2\mathbf{e}_2)$ *is a product-type* $\mathbb{D}-$*continuous function on* $\{\Gamma\}$. *Then*

a) $\int_{\Gamma} f = - \int_{-\Gamma} f$;

b) $\left| \int_{\Gamma} f \right|_{\mathbb{D}} \preceq_{\mathbb{D}} \int_{\Gamma} |f|_{\mathbb{D}} \, |dz|_{\mathbb{D}} \preceq_{\mathbb{D}} V(\Gamma) \sup_{\mathbb{D}} [|f(z)|_{\mathbb{D}} : z \in \{\Gamma\}]$;

c) If $c \in \mathbb{BC}$ *then* $\int_{\Gamma} f(z)dz = \int_{\Gamma + c} f(z - c)dz$.

Proof. Since Γ is rectifiable $\mathbb{D}-$path, for $i = 1, 2$ γ_i are rectifiable curve in $\mathbb{C}$ and f_i are continuous on $\{\gamma_i\}$.

Then by Proposition 1.17 ([5], Chapter IV) we have for $i = 1, 2$

i) $\int_{\gamma_i} f_i = - \int_{-\gamma_i} f_i$;

ii) $\left| \int_{\gamma_i} f_i \right| \leq \int_{\gamma_i} |f_i| \, |dz_i| \leq V(\gamma_i) \sup [|f_i(z_i)| : z_i \in \{\gamma_i\}]$;

iii) If $c_i \in \mathbb{C}$ then $\int_{\gamma_i} f_i(z_i)dz_i = \int_{\gamma_i + c_i} f_i(z_i - c_i)dz_i$.

Let $dz = dz_1\mathbf{e}_1 + dz_2\mathbf{e}_2$ and $c = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$. Then using Remark 3.14, equation 5, Definition 2.3, Proposition 3.4 and properties of $\mathbb{D}-$modulus we have the required results. □

It is easy to verify that ($\mathbb{BC}$,$d_{\mathbb{D}}$) is a Hyperbolic Valued Metric Space [7], where $d_{\mathbb{D}}(x, y) = d_1(x_1, y_1)\mathbf{e}_1 + d_2(x_2, y_2)\mathbf{e}_2$ for $x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2, y = y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \in \mathbb{BC}$ and d_1, d_2 are usual metric in $\mathbb{C}$. Let $G = G_1\mathbf{e}_1 + G_2\mathbf{e}_2$ be product-type open set in ($\mathbb{BC}$,$d_{\mathbb{D}}$), then G_1 and G_2 are open sets in complex metric space.

Definition 3.20. A product-type function F is called product-type primitive of a product-type $\mathbb{D}-$continuous function f on a product-type open set G if $F'(x) = f(x)$ for all $x \in G$.

Remark 3.21. In the above definition if we take $F = F_1\mathbf{e}_1 + F_2\mathbf{e}_2, f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$ and $G = G_1\mathbf{e}_1 + G_2\mathbf{e}_2$, then F_1, F_2 are primitives of f_1, f_2 on G_1, G_2 respectively.

The next theorem is the bicomplex analogue of the Fundamental Theorem of Calculus for line integrals.

Theorem 3.22. *Let G be product-type open set in the hyperbolic metric space $(\mathbb{BC}, d_{\mathbb{D}})$ and let Γ be a rectifiable $\mathbb{D}-$path in G with initial and end points α and β respectively. If $f : G \to \mathbb{BC}$ is a product-type $\mathbb{D}-$continuous function with a product-tye primitive $F : G \to \mathbb{BC}$, then*

$$\int_{\Gamma} f = F(\beta) - F(\alpha).$$

Proof. Let $G = G_1\mathbf{e}_1 + G_2\mathbf{e}_2$, $F = F_1\mathbf{e}_1 + F_2\mathbf{e}_2$, $f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$, $\Gamma = \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2$, $\alpha = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2$, and $\beta = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2$.

Then for $i = 1, 2$ G_i are open sets in $\mathbb{C}$ and γ_i are rectifiable path in G_i with initial and end points α_i and β_i respectively.

Therefore by Remark 3.21 and by Theorem 1.18 ([5], Chapter IV) we have

$$\int_{\gamma_i} f_i = F_i(\beta_i) - F_i(\alpha_i) \quad \text{for } i = 1, 2. \tag{6}$$

Then by Remark 3.14 we have

$$\begin{aligned} \int_{\Gamma} f &= \left(\int_{\gamma_1} f_1 \right) \mathbf{e}_1 + \left(\int_{\gamma_2} f_2 \right) \mathbf{e}_2 \\ &= (F_1(\beta_1) - F_1(\alpha_1))\mathbf{e}_1 + (F_2(\beta_2) - F_2(\alpha_2))\mathbf{e}_2, \text{ by (6)} \\ &= F(\beta) - F(\alpha). \end{aligned}$$

□

Corollary 3.23. *Let G, Γ and f satisfy the same hypothesis as in Theorem 3.22. If Γ is closed curve then*

$$\int_{\Gamma} f = 0.$$

Proof. Let $G = G_1\mathbf{e}_1 + G_2\mathbf{e}_2$, $f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$ and $\Gamma = \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2$.

Then by Corollary 1.22 ([5], Chapter IV) and using Remark 3.14 we have

$$\int_{\Gamma} f = 0.$$

□

4. Declarations

Conflict of interest: The authors declare that they have no conflict of interest.

References

[1] D. Alpay, M.E. Luna-Elizarrarás, M. Shapiro and D.C. Struppa, *Basics of Functional Analysis with Bicomplex Scalars, and Bicomplex Schur Analysis*, Springer International Publishing, (2014).

[2] A.S. Balankin, J. Bory-Reyes, M.E. Luna-Elizarrarás and M. Shapiro, *Cantor-type sets in hyperbolic numbers*, Fractals, **24** (04) (2016).

[3] J. Bory-Reyes, C.O. P´erez-Regalado and M. Shapiro, *Cauchy type integral in bicomplex setting and its properties*, Complex Anal. Oper. Theory, **13** (6) (2019), 2541–2573.

[4] J. Cockle, *On certain functions resembling quaternions and on a new imaginary in algebra*, Lond-Dublin-Edinb. Philos. Mag., **3** (33) (1848), 435–439.

[5] J.B. Conway, *Functions of One Complex Variable I*, Second Edition, Graduate Texts in Mathematics, Springer-Verlag, New York-Berlin, (1978).

[6] C.M. Davenport, R. Ablamowicz, J.M. Parra, M. Josep and P. Lounesto, *A Commutative Hypercomplex Algebra with Associated Function Theory*, Clifford Algebras with Numeric and Symbolic Computations, Birkhäuser, Boston, (1996), 213–227.

[7] C. Ghosh, A. Bandyopadhyay and S. Mondal, *Hyperbolic Valued Metric Space*, ArXiv [math.CV] (2021), arXiv:2108.07100.

[8] C. Ghosh, S. Biswas and T. Yasin, *Hyperbolic valued signed measures*, Int. J. Math. Trends Technol., **55** (7) (2018), 515–522.

[9] W.R. Hamilton, *On a new species of imaginary quantities connected with a theory of quaternions*, Proc. R. Ir. Acad., **2** (1844), 424–434.

[10] M.E. Luna-Elizarrarás, *Integration of Functions of a Hyperbolic Variable*, Complex Anal. Oper. Theory, **16**, 35 (2022). `https://doi.org/10.1007/s11785-022-01197-9`.

[11] M.E. Luna-Elizarraras, M. Shapiro, D.C. Struppa and A. Vajiac, *Bicomplex holomorphic functions: The algebra, geometry and analysis of bicomplex numbers*, Frontiers in Mathematics, Birkhäuser Basel, (2015).

[12] G.B. Price, *An Introduction to Multicomplex Spaces and Functions*, Marcel Dekker Inc., New York, (1991).

[13] H. Saini, A. Sharma, R. Kumar, *Some Fundamental Theorems of Functional Analysis with Bicomplex and Hyperbolic Scalars*, Adv. Appl. Clifford Algebras, **30** (66) (2020), 01–23.

[14] C. Segre, *Le rappresentazioni reali delle forme complessee Gli Enti Iperalgebrici*, Math. Ann., **40** (1892), 413–467.

[15] G. Sobczyk, *The hyperbolic number plane*, Coll. Maths. Jour., **26** (4) (1995), 268–280.

[16] G.Y. Tellez-Sanchez and J.B Rayes, *Hyperbolic Functions of Bounded Variations and Riemann-Stieltjes Integral involving strong partitions of Hyperbolic Intervals*, arXiv:2111.14019v1[math.CV], 1–12.

Chinmay Ghosh
Department of Mathematics, Kazi Nazrul University,
Nazrul Road, P.O.- Kalla C.H. Asansol-713340, West Bengal, India
E-mail: `chinmayarp@gmail.com`

Soumen Mondal
28, Dolua Dakshinpara Haridas Primary School, Beldanga, Murshidabad,
Pin-742133, West Bengal, India
E-mail: `mondalsoumen79@gmail.com`