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SOME THEOREMS ON RECURRENT MANIFOLDS AND CONFORMALLY RECURRENT MANIFOLDS

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Abstract. In this paper, we show that a recurrent manifold with harmonic curvature tensor is locally symmetric and that an Einstein and conformally recurrent manifold is locally symmetric. As a consequence, Einstein and recurrent manifolds must be locally symmetric. On the other hand, we have obtained some results for a (conformally) recurrent manifold with parallel vector field and also investigated some results for a (conformally) recurrent manifold with concircular vector field.

1. Introduction

Let M be an $n(\geq 4)$-dimensional Riemannian manifold and let g_{ij}, R^i_{jkl} and W^i_{jkl} be the Riemannian metric tensor, the Riemannian curvature tensor and the Weyl curvature tensor on M respectively. Also ∇_s denotes covariant differentiation with respect to g_{ij}. A Riemannian manifold M is said to be recurrent (resp. conformally recurrent) [1,3,6] if $\nabla_s R^i_{jkl} = \theta_s R^i_{jkl}$ (resp. $\nabla_s W^i_{jkl} = \theta_s W^i_{jkl}$), where θ_s is a 1-form. In this paper, we shall study (conformally) recurrent manifolds satisfying various conditions. More precisely, we prove the followings:

Theorem 1.1. *Let M be a manifold either recurrent and has harmonic curvature tensor, or Einstein and conformally recurrent. Then M is locally symmetric.*

It is obvious that a recurrent manifold is conformally recurrent. Consequently, we have

Corollary 1.2. *Let M be an Einstein and recurrent manifold. Then M is locally symmetric.*

Concerning a (conformally) recurrent manifold with parallel vector field, we prove

Theorem 1.3. *If M is a recurrent manifold with parallel vector field V^j, then either M is flat or V^j is orthogonal to θ^i.*

We also obtain

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Theorem 1.4. *If M is a conformally recurrent manifold with parallel vector field V^j, then either M is conformally flat or V^j is orthogonal to θ^i.*

Consequently, we have

Corollary 1.5. *If M is a recurrent manifold with parallel vector field V^j such that V^j is not orthogonal to θ^i, then M is flat.*

Corollary 1.6. *If M is a conformally recurrent manifold with parallel vector field V^j such that V^j is not orthogonal to θ^i, then M is conformally flat.*

A vector field V^j is called to be concircular [8] if it satisfies $\nabla_s V^j = \rho\delta^j_s + \phi_s V^j$, where ρ is a scalar function and ϕ_s is a gradient vector. Concerning a recurrent manifold with concircular vector field, it is known [7] that every one with concircular vector field is flat. On the other hand, we show

Theorem 1.7. *Let M be a conformally recurrent manifold with concircular vector field V^j. Then the following cases occur: (1) M is conformally flat, (2) $\rho+\frac{1}{2}V^j\theta_j = 0$.*

As a consequence, we obtain

Corollary 1.8. *If M is a conformally recurrent manifold with concircular vector field V^j such that $\rho + \frac{1}{2}V^j\theta_j \neq 0$, then M is conformally flat.*

2. Preliminaries

Let M be an $n(\geq 4)$-dimensional Riemannian manifold with Riemannian metric tensor g_{ij} and let R^i_{jkl}, R_{ij}, R and W^i_{jkl} be the Riemannian curvature tensor, the Ricci tensor, the scalar curvature and the Weyl curvature tensor respectively. From now on the components of tensors shall be considered under orthonormal frame and we adopt the summation convention of Einstein, but, as we work with orthonormal frame, there is no need to raise and lower the indices. For instance, using our notation, we have $R_{ij} = R_{aija}, R = R_{abba}$. The curvature tensor R_{ijkl} is called harmonic provided $\nabla_i R_{ijkl} = 0$. Note that the second Bianchi identity implies $\nabla_i R_{ijkl} = \nabla_l R_{jk} - \nabla_k R_{jl}$. M is called Einstein if and only if there exists a real-valued function λ on M such that $R_{ij} = \lambda g_{ij}$. If dim $M \geq 3$, then λ must be a constant [2,4,5]. Hence every $n(\geq 4)$-dimensional Einstein manifold has harmonic curvature tensor. We shall consider the Weyl curvature tensor W_{ijkl} on M given by

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{il}\delta_{jk} - R_{ik}\delta_{jl} + R_{jk}\delta_{il} - R_{jl}\delta_{ik}) + \frac{R}{(n-1)(n-2)}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}).$$

In particular, if M is Einstein, then we have

$$W_{ijkl} = R_{ijkl} - \frac{R}{n(n-1)}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}).$$

As is well known, the Weyl curvature tensor satisfies

$$W_{ijkl} = -W_{jikl} = -W_{ijlk} = W_{klij}, \tag{1}$$

$$W_{iikl} = W_{ijil} = W_{ijki} = 0, W_{ijkl} + W_{iljk} + W_{iklj} = 0. \tag{2}$$

A Riemannian manifold M is said to be recurrent (resp. conformally recurrent) if $\nabla_s R_{ijkl} = \theta_s R_{ijkl}$ (resp. $\nabla_s W_{ijkl} = \theta_s W_{ijkl}$). In particular, it is said to be locally symmetric (resp. conformally symmetric) if $\nabla_s R_{ijkl} = 0$ (resp. $\nabla_s W_{ijkl} = 0$). A vector field V_j defined by the following equation is called to be concircular:$\nabla_s V_j = \rho\delta_{js} + \phi_s V_j$, where ρ is a scalar function and ϕ_s is a gradient vector. The concircular vector field, for the first time, was considered by Yano [8] in the theory of his concircular geometry. He has obtain many interesting results. Concerning about a recurrent manifold with concircular vector field, Okumura [7] proved that a recurrent manifold with concircular vector field is flat.

3. Proof of Theorem 1.1

We assume that the Riemannian curvature tensor R_{ijkl} of M satisfies

$$\nabla_s R_{ijkl} = \theta_s R_{ijkl}, \nabla_s R_{sijk} = 0.$$

By virtue of the above equations, we obtain

$$\theta_s R_{sijk} = 0.$$

From the above equation and the second Bianchi identity, we get

$$\theta_s\theta_s R_{ijkl}R_{ijkl} = \nabla_s R_{ijkl}\nabla_s R_{ijkl} = (\nabla_l R_{ijks} - \nabla_k R_{ijls})\theta_s R_{ijkl} =$$

$$= (\theta_l R_{ijks} - \theta_k R_{ijls})\theta_s R_{ijkl} = \theta_l\theta_s R_{ijks}R_{ijkl} + \theta_k\theta_s R_{ijsl}R_{ijkl} = 2\theta_l\theta_s R_{ijks}R_{ijkl} = 0,$$

from which follows that either $\theta_s = 0$ or $R_{ijkl} = 0$. Therefore, in any case, we see that $\nabla_s R_{ijkl} = 0$. Hence M is locally symmetric. Now we suppose that M is both conformally recurrent and Einstein. A function f on M is defined as follows; $f = W_{ijkl}W_{ijkl}$. Let U' be the subset of M consisting of points x in M such that $f(x) = 0$. Then we have

$$\nabla_s f = (\nabla_s W_{ijkl})W_{ijkl} + W_{ijkl}\nabla_s W_{ijkl} = 2\theta_s f$$

on the open subset $M - U'$ and hence we have $2\theta_s = \frac{\nabla_s f}{f}$, from which it follows that $2\theta_s = \nabla_s log|f|$. This implies that $\nabla_m\theta_n = \nabla_n\theta_m$ on $M - U'$. Therefore, since $\nabla_m\nabla_n W_{ijkl} = (\theta_m\theta_n + \nabla_m\theta_n)W_{ijkl}$ on M, we have

$$\nabla_m\nabla_n W_{ijkl} = \nabla_n\nabla_m W_{ijkl}.$$

Accordingly, by the Ricci identity, we get

$$R_{mnir}W_{rjkl} + R_{mnjr}W_{irkl} + R_{mnkr}W_{ijrl} + R_{mnlr}W_{ijkr} = 0. \tag{3}$$

Differentiating (3) covariantly and taking into account of $\nabla_p W_{ijkl} = \theta_p W_{ijkl}$ and (3), we have

$$(\nabla_p R_{mnir})W_{rjkl} + (\nabla_p R_{mnjr})W_{irkl} + (\nabla_p R_{mnkr})W_{ijrl} + (\nabla_p R_{mnlr})W_{ijkr} = 0. \tag{4}$$

Since M is both Einstein and conformally recurrent, we obtain

$$\nabla_p R_{ijkl} = \nabla_p W_{ijkl} = \nabla_p[R_{ijkl} - \frac{R}{n(n-1)}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})] =$$

$$= \theta_p[R_{ijkl} - \frac{R}{n(n-1)}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})].$$

By virtue of the above equation, the equation (4) yields

$$(\theta_p R_{mnir})W_{rjkl} + (\theta_p R_{mnjr})W_{irkl} + (\theta_p R_{mnkr})W_{ijrl} + (\theta_p R_{mnlr})W_{ijkr} -$$

$$-\frac{\theta_p R}{n(n-1)}[(\delta_{mr}\delta_{ni} - \delta_{mi}\delta_{nr})W_{rjkl} + (\delta_{mr}\delta_{nj} - \delta_{mj}\delta_{nr})W_{irkl} +$$

$$+(\delta_{mr}\delta_{nk}-\delta_{mk}\delta_{nr})W_{ijrl}+(\delta_{mr}\delta_{nl}-\delta_{ml}\delta_{nr})W_{ijkr}]=0.$$

Now putting $n=i$ in the above equation and taking account of (3) , we have

$$\frac{\theta_p R}{n(n-1)}[(n-2)W_{mjkl}+W_{kjml}+W_{ljkm}]=0.$$

By virtue of (1) and (2), the above equation implies that

$$\theta_p R W_{mjkl}=0,$$

from which follows

$$R\nabla_p W_{mjkl}=0. \tag{5}$$

Since M is Einstein, M has either $R=0$ identically or $R\neq 0$ everywhere. In any case, by virtue of (5), M is locally symmetric. This completes the proof of Theorem 1.1. Since an Einstein and recurrent manifold is conformally recurrent and has harmonic curvature tensor, Theorem 1.1 implies that Corollary 1.2 holds.

4. Proof of Theorem 1.3 and Theorem 1.4

Let V_j be a parallel vector field. Since V_j satisfies $\nabla_m V_j=0$, we have

$$\nabla_n\nabla_m V_j-\nabla_m\nabla_n V_j=0.$$

Consequently, making use of the Ricci identity and the second Bianchi identity, we have

$$V_j R_{ijmn}=0, V_j R_{jm}=0, V_j\nabla_s R_{ijmn}=0, V_j\nabla_s R_{jm}=0,$$

$$V_j\nabla_j R_{ikmn}=0, V_j\nabla_j R_{km}=0, V_j\nabla_j R=0.$$

From the above equations and the definition of Weyl curvature tensor W_{ikmn}, we obtain

$$V_j\nabla_j R_{ikmn}=0, V_j\nabla_j W_{ikmn}=0. \tag{6}$$

Let us assume that M is a recurrent manifold with parallel vector field V_j. Then (6) implies that

$$V_j\theta_j R_{ikmn}=0,$$

from which follows that either M is flat or V_j is orthogonal to θ_i. This completes the proof of Theorem 1.3. Now suppose that M is a conformally recurrent manifold with parallel vector field V_j. Then (6) implies that

$$V_j\theta_j W_{ikmn}=0.$$

Therefore, we obtain that either M is conformally flat or V_j is orthogonal to θ_i. This completes the proof of Theorem 1.4. Corollary 1.5 and Corollary 1.6 are immediate consequences of Theorem 1.3 and Theorem 1.4 respectively.

5. Conformally recurrent manifold and concircular vector field

Let us assume that M is a conformally recurrent manifold with concircular vector field V_j. Then V_j satisfies

$$\nabla_l V_i = \rho\delta_{il} + \phi_l V_i,$$

where ρ is a scalar function and ϕ_l is a gradient vector. Differentiating the above equation covariantly, we get

$$\nabla_m\nabla_l V_i = (\nabla_m\rho)\delta_{il} + \rho\phi_l\delta_{im} + \phi_l\phi_m V_i + (\nabla_m\phi_l)V_i.$$

Making use of the Ricci identity and taking account of the above equation, we have

$$V_j R_{ijlm} = (\rho\phi_l - \nabla_l\rho)\delta_{im} - (\rho\phi_m - \nabla_m\rho)\delta_{il}.$$

Differentiating the above equation covariantly, we obtain

$$(\nabla_s V_j)R_{ijlm} + V_j\nabla_s R_{ijlm} = [(\nabla_s\rho)\phi_l + \rho\nabla_s\phi_l - \nabla_s\nabla_l\rho]\delta_{im} -$$

$$-[(\nabla_s\rho)\phi_m + \rho\nabla_s\phi_m - \nabla_s\nabla_m\rho]\delta_{il},$$

from which follows by virtue of the above equation and the definition of concircular vector field V_j

$$V_j\nabla_s R_{lmij} = -\rho R_{islm} + \psi_{ls}\delta_{mi} - \psi_{ms}\delta_{il},$$

where $\psi_{ls} = (\nabla_s\rho)\phi_l + \rho\nabla_s\phi_l + (\nabla_l\rho)\phi_s - \rho\phi_l\phi_s - \nabla_s\nabla_l\rho$. From the second Bianchi identity and the above equation, we have

$$V_j\nabla_j R_{lmsi} = -2\rho R_{lmsi} - \psi_{ls}\delta_{mi} + \psi_{ms}\delta_{il} + \psi_{li}\delta_{ms} - \psi_{mi}\delta_{sl}. \tag{7}$$

Consequently

$$V_j\nabla_j R_{ms} = -2\rho R_{ms} + (n-2)\psi_{ms} + \psi\delta_{ms}, \tag{8}$$

$$V_j\nabla_j R = -2\rho R + 2(n-1)\psi, \tag{9}$$

where $\psi = \psi_{mm}$. Now, from the definition of Weyl curvature tensor W_{lmsi}, we obtain

$$V_j\nabla_j W_{lmsi} = V_j\nabla_j R_{lmsi} - \frac{V_j}{n-2}[(\nabla_j R_{li})\delta_{ms} - (\nabla_j R_{ls})\delta_{mi} + (\nabla_j R_{ms})\delta_{li} -$$

$$-(\nabla_j R_{mi})\delta_{ls}] + \frac{V_j\nabla_j R}{(n-1)(n-2)}(\delta_{li}\delta_{ms} - \delta_{ls}\delta_{mi}).$$

By virtue of $(7), (8)$ and (9), the above equation implies

$$V_j\theta_j W_{lmsi} = -2\rho W_{lmsi}.$$

Hence we conclude that either M is conformally flat or $\rho = -\frac{1}{2}V_j\theta_j$. This completes the proof of Theorem 1.7. Corollary 1.8 is an immediate consequence of Theorem 1.7.

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